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A Note on the Supersymmetries of the Self-Dual Supermembrane

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Abstract

In this letter we discuss the supersymmetry issue of the self-dual supermembranes in $(8+1)$ and $(4+1)$-dimensions. We find that all genuine solutions of the $(8+1)$-dimensional supermembrane, based on the exceptional group G_2, preserve one of the sixteen supersymmetries while all solutions in $(4+1)$-dimensions preserve eight of them.

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Recently, a new duality for fundamental membranes [1] in $(4+1)$-dimensions, has been extended to $(8+1)$-dimensions using the structure constants of the octonionic algebra [2, 3, 4]. Explicit solutions have been constructed in various dimensions and connections with string instantons have been found [5].

The fundamental supermembranes as extended objects were first described in [6] by a manifestly space-time supersymmetric Green-Schwarz (GS)-action. It was further shown [7] that they emerge as a solution of the eleven dimensional supergravity field equations with their zero modes corresponding to the physical degrees of freedom of the GS-action. It is now known that one of the fundamental problems of supermembrane theory is the existence of a convenient perturbative expansion and the derivation of effective low energy Lagrangian (which is expected to be the 11-dimensional $N=1$ supergravity theory). For this problem, the existence of a self-dual sector of BPS-states for the supermembrane, preserving a number of supersymmetries which would guarantee the absence of perturbative corrections could be a way out. An interesting property of the self-duality equations for supermembrane is that in three dimensions the system is an integrable one and in principle all the spectrum of the corresponding BPS-states could be determined. In this case, after the light-cone gauge fixing, one restricts the membrane to the three of the nine dimensions in order to formulate the self-duality equations. The corresponding integrability of the seven-dimensional case is still under investigation.

In this letter, we study the supersymmetry transformations for the octonionic self-dual membranes and we determine the number of supersymmetries left in seven and three dimensions. We find that the seven-dimensional case preserves one supersymmetry, while the three-dimensional solutions preserve eight of them. The G_2 symmetry of the seven-dimensional case can be used to embed the $N=8$, $d=3$ BPS-states into $N=8$, $d=7$ superalgebra.

We start by recalling the light-cone gauge formulation of the supermembrane, where half of the rigid space-time supersymmetry as well as the local κ-symmetry is fixed. We then provide the supersymmetries left intact.

It is known that in $d = 8$ dimensions there is a connection of the Clifford algebra with the octonionic algebra and this is the information needed to study the behaviour of the octonionic self-duality equations under supersymmetry transformations. The relation with the octonions has been noticed in the 80's during the studies of the S^7-compactifications of the 11-d supergravity as well as for the $N = 8$ gauged supergravities [8, 9, 10]. Recently, the embedding of octonionic Yang-Mills (YM) instantons in the ten-dimensional effective supergravity theories of strings has been constructed [11, 12, 13] where it was found that one supersymmetry survives. More generally, wrapped membrane compactifications have been recently discussed in the literature [14].

In the light-cone gauge, after the elimination of the X_- variable from the reparametrization constraints, the supersymmetric Hamiltonian [6, 15] is defined as

$$\mathcal{H} = \frac{1}{P_0^+} \int d^2\sigma \left(\frac{1}{2} P^I P_I + \frac{1}{4} \{X^I, X^J\}^2 - P_0^+ \bar{\theta} \Gamma_- \Gamma_I \{X^I, \theta\} \right) \tag{1}$$

where $P_I = \dot{X}_I$ and the indices $I, J = 1, ..., 9$ while we have fixed the area preserving parameters so that $w = 1$ [6]. The compatibility condition for the uniqueness of X_-, is the Gauss law

$$\{\dot{X}_I, X_I\} + \{\bar{\theta}\Gamma_-, \theta\} = 0, \;\; I = 1, ..., 9. \tag{2}$$

where summation over repeated incides is assumed. The Clifford generators Γ_I, in (1) are represented by real 32×32 matrices which can be chosen in the following form

$$\Gamma_I = \sigma_3 \otimes \gamma_I \tag{3}$$

where σ_3 is the Pauli matrix, γ_I represent the 16×16 matrices and $\gamma_9 = \gamma_1 \cdots \gamma_8$. Further, Γ_- (and Γ_+) correspond to the light-cone coordinates $(X_{10} \pm X_0)/\sqrt{2}$, thus they are given by a similar decomposition

$$\Gamma_\pm = \frac{1}{\sqrt{2}} \left(\Gamma_{10} \pm \Gamma_0 \right). \tag{4}$$

Thus, we have

$$\Gamma_- = \imath\sqrt{2}\begin{pmatrix} 0 & 1_{16} \\ 0 & 0 \end{pmatrix}, \quad \Gamma_+ = \imath\sqrt{2}\begin{pmatrix} 0 & 0 \\ 1_{16} & 0 \end{pmatrix} \tag{5}$$

The Hamiltonian (1) is invariant under area-preserving transformations of the membrane (which for non-trivial topologies of the membrane contain also global elements $2g$ in number, where g is the genus of the membrane [15]). The local area-preserving transformations are generated by the Gauss law (2). Here the canonical variables satisfy Dirac brackets

$$\left(X^I(\sigma), \dot{X}^J(\sigma')\right)_{DB} = \delta^{IJ}\delta^2(\sigma - \sigma') \tag{6}$$

$$\left(\theta^I(\sigma), \bar{\theta}^J(\sigma')\right)_{DB} = \frac{1}{4}(\Gamma_+)^{IJ}\delta^2(\sigma - \sigma') \tag{7}$$

(where we have chosen $P_+ = 1$). It can be verified that in the light-cone gauge there are two independent spinor supersymmetry charges

$$Q = Q^+ + Q^- = \int d^2\sigma J^0 \tag{8}$$

where $Q^\pm = \frac{1}{2}\Gamma_\pm\Gamma_\mp Q$, and

$$Q^+ = \int d^2\sigma(2\dot{X}^I\Gamma_I + \{X^I, X^J\}\Gamma_{IJ})\theta \tag{9}$$

$$Q^- = 2\int d^2\sigma S = 2\Gamma_-\theta_0 \tag{10}$$

which are constants of motion and θ_0 is the momentum conjugate to the center-of-mass coordinate of the fermionic degrees of freedom of the membrane. The corresponding supersymmetry transformations which leave the Hamiltonian invariant, are given by

$$\delta X^I = -2\bar{\epsilon}\Gamma^I\theta \tag{11}$$

$$\delta\theta = \frac{1}{2}\Gamma_+\left(\dot{X}\Gamma_I + \Gamma_-\right)\epsilon + \frac{1}{4}\{X^I, X^J\}\Gamma_+\Gamma^{IJ}\epsilon \tag{12}$$

On the other hand, the local fermionic κ-symmetry has been fixed by imposing the condition

$$\Gamma_+\theta = 0 \tag{13}$$

Due to this gauge condition, the fermionic coordinates are restricted to $SO(9)$ spinors, satisfying

$$\Gamma_1 \cdots \Gamma_9 \theta = \theta \tag{14}$$

while the $SO(9)$ Γ-matrices satisfy $\Gamma_I^T = \Gamma^I$.

The self-duality equations for the bosonic part of the supermembrane have been initially introduced in the light-cone gauge fixing $X_4, \ldots X_9$ to be constants [1],

$$\dot{X}_i = \frac{1}{2}\epsilon_{ijk}\{X_j, X_k\}, \quad i, j = 1, 2, 3 \tag{15}$$

These equations have been proposed as an analogue of the electric-magnetic duality where the local velocity of the membrane corresponds to the electric field while the RHS which is the normal to the membrane surface, corresponds to the magnetic field. They imply the Gauss law and the Euclidean-time equations of motion with fermionic degrees of freedom (dof) set to zero [1, 16]. This system has been shown to be integrable and a Lax pair was found. In order to go to higher dimensions one should have the notion of cross product of two vectors and this is provided as the unique other possibility by the structure constants of the algebra of octonions (Cayley numbers) [17]. The octonionic units o_i satisfy the algebra

$$o_i o_j = -\delta_{ij} + \Psi_{ijk} o_k. \tag{16}$$

where $i = 1, \ldots, 7$ are the 7 octonionic imaginary units with the property

$$\{o_i, o_j\} = -2\delta_{ij} \tag{17}$$

The totally antisymmetric symbol Ψ_{ijk} appearing in (16) is defined to be equal to 1 when the indices are [17]

$$\Psi_{ijk} = \left\{ \begin{array}{ccccccc} 1 & 2 & 4 & 3 & 6 & 5 & 7 \\ 2 & 4 & 3 & 6 & 5 & 7 & 1 \\ 3 & 6 & 5 & 7 & 1 & 2 & 4 \end{array} \right. \tag{18}$$

and zero for all other cases. With this multiplication table, Ψ_{ijk} provides for every two seven-dimensional vectors a third one, normal to the first two. Thus, it is possible to extend the three-dimesional self-duality equations to seven dimensions, fixing only the values of X_8, X_9 membrane coordinates. Then, the self-duality equations [2] become

$$\dot{X}_i = \frac{1}{2}\Psi_{ijk}\{X_j, X_k\} \tag{19}$$

The Gauss law results automatically by making use of the Ψ_{ijk} cyclic symmetry

$$\{\dot{X}_i, X_i\} = 0 \tag{20}$$

The Euclidean equations of motion are obtained easily from (19)

$$\ddot{X}_i \quad = \quad \{X_k, \{X_i, X_k\}\} \tag{21}$$

where use has been made of the identity

$$\Psi_{ijk}\Psi_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} + \phi_{ijlm} \tag{22}$$

and of the cyclic property of the symbol ϕ_{ijlm} [17] which is defined to be equal to 1 when its indices take values of the following table

$$\phi^{ij}{}_{kl} = \begin{cases} 4 & 3 & 6 & 5 & 7 & 1 & 2 \\ 5 & 7 & 1 & 2 & 4 & 3 & 6 \\ 6 & 5 & 7 & 1 & 2 & 4 & 3 \\ 7 & 1 & 2 & 4 & 3 & 6 & 5 \end{cases} \tag{23}$$

whilst it is zero for any other combination of indices. In terms of these units an octonion can be written as follows

$$X = x_0 o_0 + \sum_{i=1}^{7} x_i o_i \tag{24}$$

with o_0 the identity element. The conjugate is

$$\bar{X} = x_0 o_0 - \sum_{i=1}^{7} x_i o_i \tag{25}$$

The octonions over the real numbers can also be defined as pairs of quaternions

$$X = (x_1, x_2) \tag{26}$$

where $x_1 = x_1^\mu \sigma_\mu$, $x_2 = x_2^\mu \sigma_\mu$ and the indices μ run from 0 to 3, while $x_{1,2}^0$ are real numbers and $x_{1,2}^i$, $(i = 1, 2, 3)$ are imaginary numbers. Finally, σ_0 is the identity 2×2 matrix and σ_i are the three standard Pauli matrices.

If $q = (q_1, q_2)$ and $r = (r_1, r_2)$ are two octonions, the multiplication law is defined as

$$q * r \equiv (q_1, q_2) * (r_1, r_2) = (q_1 r_1 - \bar{r}_2 q_2, r_2 q_1 + q_2 \bar{r}_1), \tag{27}$$

where $q_1 = q_1^0 + q_1^i \sigma_i$ and $\bar{q}_1 = q_1^0 - q_1^i \sigma_i$. One can also define a conjugate operation for an octonion as

$$\bar{q} \equiv \overline{(q_1, q_2)} = (\bar{q}_1, -q_2) \tag{28}$$

and we get the possibility to define the norm and the scalar product q and r

$$\begin{aligned} q\bar{q} &= (q_1 \bar{q}_1 + \bar{q}_2 q_2, 0) \\ &= \sum_{\mu=0}^{3} \left(q_1^{\mu 2} + q_2^{\mu 2} \right) \end{aligned} \tag{29}$$

$$\langle q | r \rangle = \frac{1}{2} (q\bar{r} + \bar{q} r) \tag{30}$$

In terms of the above formalism, the self-duality equations can be written as follows

$$\dot{X} = \frac{1}{2} \{X, X\}, \tag{31}$$

where $X = X^i o_i$ with $i = 1, \cdots, 7$ and the Poisson bracket for two octonions is defined as

$$\{X, Y\} = \frac{\partial X}{\partial \xi_1} \frac{\partial Y}{\partial \xi_2} - \frac{\partial X}{\partial \xi_2} \frac{\partial Y}{\partial \xi_1}. \tag{32}$$

After these preliminaries, we come now to the question regarding the number of supersymmetries preserved by the self-duality equations. In our analysis we will explore the number of supersymmetries preserved by (3+1)- and (7+1)-dimensional solutions. We will see that 3-d solutions preserve as many as eight out of the sixteen supersymmetries while the 7-d self-duality equations preserve only one supersymmetry. The supersymmetry transformation is defined [6]

$$\delta\theta \;=\; \frac{1}{2}\left(\Gamma_+(\Gamma_I\dot{X}^I+\Gamma_-)+\frac{1}{2}\Gamma_+\Gamma^{IJ}\{X_I,X_J\}\right)\begin{pmatrix}\imath\epsilon_A\\ \epsilon_B\end{pmatrix}$$

In terms of the 16×16 γ-matrices, the above is written

$$\delta\theta \;=\; \begin{pmatrix} 0 & 0\\ \imath\sqrt{2}\left(\gamma^I\dot{X}_I+\frac{1}{2}\gamma^{IJ}\{X_I,X_J\}\right) & -2\cdot 1_{16}\end{pmatrix}\begin{pmatrix}\imath\epsilon_A\\ \epsilon_B\end{pmatrix} \tag{33}$$

which implies that

$$\sqrt{2}\left(\gamma^I\dot{X}_I+\frac{1}{2}\gamma^{IJ}\{X_I,X_J\}\right)\epsilon_A+2\cdot 1_{16}\epsilon_B=0 \tag{34}$$

where ϵ_A,ϵ_B are 16-dimensional spinors. From the form of eq.(34), we observe that if self-duality equations are going to play a role in the preservation of a number of supersymmetries, we should necessarily impose the condition $\epsilon_B=0$. Thus, at least half of the supersymmetries are broken. Now, the last term in (34) is zero and eq.(34) simply becomes

$$\left(\gamma^I\dot{X}_I+\frac{1}{2}\gamma^{IJ}\{X_I,X_J\}\right)\epsilon_A=0 \tag{35}$$

Under the assumption that $\dot{X}_{8,9}=0$, it can be shown that the above reduces to a simpler –8×8– matrix equation. In order to find a convenient explicit form, we first express the 16×16 matrices in terms of the octonionic structure constants Ψ_{ijk} as follows: let the index n run from 1 to 7; then we define

$$\gamma_8=\begin{pmatrix}0 & 1_8\\ -1_8 & 0\end{pmatrix},\quad \gamma_n=\begin{pmatrix}0 & \beta_n\\ -\beta_n & 0\end{pmatrix} \tag{36}$$

where 1_8 is the 8×8-identity matrix and β_n are seven 8×8 γ-matrices with elements [10]

$$(\beta_n)^i_j=\Psi_{imj},\quad (\beta_n)^i_8=\delta^i_j,\quad (\beta_n)^8_j=-\delta^i_j \tag{37}$$

while it can be easily checked that $\beta_1\cdots\beta_7=-1_8$ and

$$\gamma_9 \;=\; \begin{pmatrix}1_8 & 0\\ 0 & -1_8\end{pmatrix} \tag{38}$$

The commutation relations of β_m give:

$$([\beta_m,\beta_n])^8_j = +2\Psi_{nmj} \tag{39}$$

$$([\beta_m,\beta_n])^j_8 = -2\Psi_{nmj} \tag{40}$$

$$([\beta_m,\beta_n])^i_j = -2\mathcal{X}^{mn}{}_{ij}(-4) \tag{41}$$

where the tensors $\mathcal{X}^{mn}{}_{ij}(u)$ are defined as follows [3]

$$\mathcal{X}^{ij}{}_{kl}(u) = \Delta^{ij}{}_{kl} + \frac{u}{4}\phi^{ij}{}_{kl} \tag{42}$$

where $\Delta^{ij}{}_{kl} = \frac{1}{2}(\delta^i_k\delta^j_l - \delta^i_l\delta^j_k)$. Next, we impose the following condition on the components of the 16-spinor ϵ_A

$$\epsilon_A = \begin{pmatrix} 1 \\ -\imath \end{pmatrix} \otimes \varepsilon \tag{43}$$

where $\otimes$ stands for the direct product and ε is an eight-component spinor whose components are left unspecified. Clearly, condition (43) reduces further the sixteen supersymmetry charges to eight. Separating the eight components of $\varepsilon = (\varepsilon_7, \varepsilon_1)$ where $\varepsilon_{7(1)}$ is a seven- (one-) dimensional vector, we find that eq.(35) reduces to the matrix equation

$$\mathcal{O}\,\varepsilon \equiv \begin{pmatrix} \Psi_{imj}\dot{X}_m + \frac{\imath}{2}\mathcal{X}^{mn}{}_{ij}(-4)\{X_m,X_n\} & \dot{X}_i + \frac{\imath}{2}\Psi_{imn}\{X_m,X_n\} \\ -\left(\dot{X}_i + \frac{\imath}{2}\Psi_{imn}\{X_m,X_n\}\right) & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_7 \\ \varepsilon_1 \end{pmatrix} = 0 \tag{44}$$

The rather interesting fact here is that the matrix elements $\mathcal{O}_{8j}$ and $\mathcal{O}_{j8}$, $(j = 1, ..., 7)$ multiplying the ε_1-component are the self-duality equations (15) in eight dimensions when the Euclidean time-parameter t is replaced with $\imath t$ (Minkowski). Thus, ε_1-component remains unspecified and there is always one supersymmetry unbroken for any eight-dimensional solution of the self-duality equations.

Let us now turn our discussion to the upper 7×7 part of the matrix equation (44). In general, the quantity specifying these elements, namely

$$\Psi_{imj}\dot{X}_m + \frac{\imath}{2}\mathcal{X}^{mn}{}_{ij}(-4)\{X_m,X_n\} \tag{45}$$

is *not* automatically zero. However, there is a particular case –which turns out to be the most interesting one– where the above quantity is the self-duality equation itself. In fact, if we consider only three-dimensional solutions of the equations, the 'curvature' factor ${\phi^{ij}}_{kl}$ is automatically zero while the tensor ${\mathcal{X}^{ij}}_{kl}$ simply becomes

$$ {\mathcal{X}^{ij}}_{kl} = {\Delta^{ij}}_{kl} = \frac{1}{2}(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) \quad \text{for} \quad {\phi^{ij}}_{kl} = 0. \tag{46} $$

In this case, it can be easily seen that (45) reduces to the self-duality equations in three-dimensions. In this latter case, all eight supersymmetries survive.

We summarize this note discussing also the importance of the supersymmetric self-duality configurations in three and seven dimensions. The absence of a natural perturbative expansion for the 11-d fundamental supermembrane prohibits so far the derivation of its low energy effective Lagrangian which is expected to contain 11-d, N=1 supergravity interacting with solitonic two- and five-branes in a duality symmetric way. The Euclidean self-dual membrane configurations in three and seven dimensions, after light-cone gauge fixing, provide non-perturbative minima of the action, which could survive perturbative corrections if enough supersymmetries are left intact. The quantum mechanical amplitudes calculated in supermembrane theory could be then determined by transforming the path-integral integration around these minima into the infinite moduli-space integration of the self-dual configurations of supermembranes. The best candidate for these seem to be the three-dimensional integrable self-dual sector where eight supersymmetries survive. The problem then is reduced to find the moduli space and its integration measure of the minimum action 3-d configurations. We hope to come back to this problem in a future work.

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