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D-BRANE CONFORMAL FIELD THEORY

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Abstract

We outline the structure of boundary conditions in conformal field theory. A boundary condition is specified by a consistent collection of reflection coefficients for bulk fields on the disk together with a choice of an automorphism ω of the fusion rules that preserves conformal weights. Non-trivial automorphisms ω correspond to D-brane configurations for arbitrary conformal field theories.

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String theory and conformal field theory.

A complete understanding of string theory certainly requires many more ingredients than just conformal field theory, e.g. when it comes to finding a guiding principle that would tell what solitonic sectors (and with which multiplicities) must be included to arrive at a consistent theory. On the other hand, both at a conceptual and at a computational level, conformal field theory does lead very far indeed. While at the level of string perturbation theory this is more or less accepted knowledge in the case of closed strings, it is a prevailing prejudice that some of the more recently discovered structures that are tied to the presence of open strings with non-trivial boundary conditions are inaccessible to conformal field theory. This is of course a logical possibility, but before making a decision on this issue one should better inspect the tools that are summarized under the name ‘conformal field theory’ with sufficient care. In the course of these investigations it may well turn out that present day knowledge about these matters is as yet incomplete and that the uses of conformal field theory can be largely expanded by further efforts.

Indeed we claim that the basic new features of open as compared to closed strings, such as e.g. D-branes (possibly with field strength, or multiply wrapped) are well accessible to conformal field theory. Moreover, once a suitable framework for conformal field theory on closed orientable Riemann surfaces (*closed* conformal field theory, for short) is formulated [1], establishing the theory also on the open and/or non-orientable surfaces (*open* conformal field theory) that arise as world sheets of open strings does not pose any major conceptual problems any more, though there are several new ingredients which considerably complicate matters at a more technical level.

Building blocks.

Let us first recall a few facts about the world sheet picture of closed strings. The guiding principle for the construction of a string theory is to start with some given conformal field theory (supposed to be consistently defined on all closed orientable Riemann surfaces C) and then to discard the dependence on the properties of the world sheet C while still keeping information about the field theory on C . This is achieved by eliminating first the (super-) Virasoro algebra via the relevant semi-infinite cohomology, then the choice of a conformal structure on C via integration over the moduli space of complex structures, and finally the choice of topology of C by a summation over topologies. The latter sum is weighted by the power $\gamma^{-\chi}$ of the string coupling constant γ , with $\chi = 2 - 2g$ the Euler number of C . In particular, string scattering amplitudes are obtained from the n -point correlation functions $\mathcal{F}_{g,n} \equiv \mathcal{F}_{g,n}(\vec{\lambda}; \vec{z}, \vec{\tau})$ of the conformal field theory by integrating over the moduli $\vec{\tau}$ of the genus- g surface C and (modulo Möbius transformations) over the insertion points $\vec{z} \equiv (z_1, z_2, \dots, z_n)$, and afterwards multiplying with $\gamma^{-\chi}$ and summing over χ .

For a conformal field theory to be consistently defined on all surfaces C , the correlation functions $\mathcal{F}_{g,n}$ have to satisfy various locality and factorization constraints. The former require that the $\mathcal{F}_{g,n}$ are ordinary functions of the insertion points \vec{z} and (up to the Weyl anomaly) of the moduli $\vec{\tau}$, while the latter implement compatibility with singular limits in the moduli spaces. These constraints are formulated in terms of the conformal field theory on C (which is *orientable*, but does *not* come naturally as an *oriented* surface), to which we

refer as the stage of *full* conformal field theory. This stage must be carefully distinguished from the stage of *chiral* conformal field theory, where in place of the correlation functions one is dealing with chiral blocks. Usually this stage is introduced by a somewhat heuristic recipe for ‘splitting the theory into two chiral halves’. A more appropriate, and for the present purposes more convenient, description of the chiral theory is as a conformal field theory on an oriented covering surface \hat{C} which has the structure of a complex curve and from which the original surface C can be recovered by dividing out an anti-conformal involution [1].

For large classes of conformal field theories, in particular for WZW models, all correlation functions $\mathcal{F}_{g,n}$ can in principle be computed exactly (i.e., fully non-perturbatively in terms of the field theory on the world sheet). Moreover, in many interesting cases – including, but by no means exhausted by, free field theories – at least at string tree level this can also be achieved in actual practice. The reason is that the chiral blocks can be obtained as the solutions to the Ward identities of the theory. Let us note that even though conformal field theory is typically formulated in an operator picture, for establishing the Ward identities (and also for many other purposes) the existence of an operator formalism is not needed. Namely, the Ward identities constitute identities for chiral blocks that can be formulated solely in terms of the representation theory of the relevant chiral algebra \mathfrak{W} , without making use of an operator formalism. Also, once the chiral blocks are known, the correlation functions are determined by the locality and factorization constraints, also known as sewing constraints, which (are believed to) possess a unique solution. Of course, in string theory one usually interprets the scattering amplitudes as expectation values for products of suitable vertex operators for the string modes. In conformal field theory terms this amounts to working with an operator formalism, in which the string modes are realized as (Virasoro-primary) chiral vertex operators in the chiral, respectively as corresponding fields in the full theory. The locality and factorization properties constitute a necessary prerequisite for the existence of operator product expansions of the full theory.

Via factorization, one can reduce many issues of interest to statements about only a small number of building blocks, namely the chiral 3-point blocks on \mathbb{P}^1 , and these building blocks can be studied in terms of the representation theory of the chiral algebra \mathfrak{W} . For instance, the index set $\{\lambda\}$ (an n -tuple of which labels the correlation functions $\mathcal{F}_{g,n}$, and which in the operator picture indicate the allowed fields) corresponds to a suitable set $\{\mathcal{H}_\lambda\}$ of irreducible modules of the algebra \mathfrak{W} , and in rational theories the numbers $\mathcal{N}_{\lambda_1 \lambda_2}^{\lambda_3}$ of independent 3-point blocks of type $(\lambda_1, \lambda_2, \lambda_3)$ are related, via the Verlinde formula, to the modular behavior of the characters χ_λ of these modules \mathcal{H}_λ .

For *open* strings, including *D-branes*, the situation is more complicated technically, but not conceptually. Some of the concepts mentioned above are now realized in a somewhat different manner, but still they can be applied in much the same way as before. For instance, we have:

- The Euler characteristic χ still counts the order in the string perturbation theory. But now χ is given by $\chi = 2 - 2g - b - c$, where g , b and c are the numbers of handles, boundary components, and crosscaps of the surface C , respectively.
- One must still distinguish between the two stages of the chiral and the full conformal field theory. The full theory on C can again be expressed in terms of a chiral theory on some surface \hat{C} by imposing locality and factorization constraints.

- Again \hat{C} is an oriented cover of C from which one recovers C by modding out an anti-conformal involution I . But now \hat{C} is connected, whereas in the closed case it consists of two connected components each of which is isomorphic to C as a real manifold [1]. Also, the involution I may now possess fixed points, giving rise to boundaries of C .
- Again factorization allows to formulate the theory in terms of a few building blocks. But besides the 3-point blocks on \mathbb{P}^1 , one now also needs the 1-point blocks on the disk $D = \mathbb{P}^1/z \rightarrow 1/z^*$ as well as the 1-point blocks on the crosscap $\mathbb{P}\mathbb{R}^2 = \mathbb{P}^1/z \rightarrow -1/z^*$.

Boundary states and boundary conditions.

In contrast to the closed case, in open conformal field theory [2–4, 1] the locality and factorization constraints typically admit more than one solution, e.g. the 1-point correlation functions $\langle \phi_{\lambda, \bar{\lambda}} \rangle_A$ of bulk fields $\phi_{\lambda, \bar{\lambda}}$ on the disk D depend on some additional label A . These correlators are simply proportional to the corresponding 1-point blocks; the constant of proportionality is the product of two factors N_0^{AA} and $R_{\lambda \bar{\lambda}; 0}^A$. The number N_0^{AA} is interpreted as the expectation value $\langle \Psi_0^{AA} \rangle$ of a ‘boundary vacuum field’ [2] Ψ_0^{AA} ; roughly, the role of the boundary field is to make a geometric boundary component into a ‘field theoretic boundary’ that carries the boundary label A . Similarly, $R_{\lambda \bar{\lambda}; 0}^A$ is a *reflection coefficient*, defined via the expansion [3, 4]

$$\phi_{\lambda, \bar{\lambda}}(re^{i\sigma}) \sim \sum_{\mu} \sum_a (r^2 - 1)^{-\Delta_{\lambda} - \Delta_{\bar{\lambda}} + \Delta_{\mu}} R_{\lambda \bar{\lambda}; \mu}^A \Psi_{\mu}^{AA}(e^{i\sigma}) \quad \text{for } r \rightarrow 1 \quad (1)$$

of $\phi_{\lambda, \bar{\lambda}}$ in terms of boundary fields. Every consistent collection of 1-point correlators for all bulk fields, or equivalently, every consistent collection of reflection coefficients $R_{\lambda \bar{\lambda}; 0}^A$, is referred to [2] as a *boundary condition* A . For free fields these amount to boundary conditions in the ordinary geometric sense, but in the general case such an interpretation is not available. Roughly, one can interpret the relation (1) by imagining that to every bulk field there is associated a kind of mirror charge on $\mathbb{P}^1 \setminus D$, which in turn corresponds to some charge distribution on the boundary.

In the literature it is common to denote the 1-point chiral blocks on the disk by $|B_{\lambda}\rangle$ and to refer to them, as well as to their linear combinations

$$|\mathcal{B}^A\rangle := \sum_{\lambda} N_0^{AA} R_{\lambda \bar{\lambda}; 0}^A |B_{\lambda}\rangle, \quad (2)$$

as *boundary states*. Such an object is, however, not a state in the usual sense. While formally it satisfies relations of the form

$$(W_n \otimes \mathbf{1} - (-1)^{\Delta(W)} \mathbf{1} \otimes W_{-n}) |B_{\lambda}\rangle = 0, \quad (3)$$

and in concrete examples can be written¹ as an (infinite) sum of basis elements of the tensor product space $\mathcal{H}_{\lambda} \otimes \mathcal{H}_{\bar{\lambda}}$ of the relevant \mathfrak{W} -modules, it is *not* an element of that space, nor even of the completion of the tensor product space with respect to its standard scalar product. Rather, the correct interpretation is indeed as a 1-point block on the disk. At a more technical level, this can be described as a so-called *co-invariant* of the space

¹ The formulæ in the literature actually describe the specific situation that the insertion point is at $z = 0$ and that standard local coordinates on the covering surface $\hat{C} = \mathbb{P}^1$ of the disk are chosen.

$\mathcal{H}_\lambda \otimes \mathcal{H}_{\tilde{\lambda}}$ with respect to the action $W_n \otimes \mathbf{1} - (-1)^{\Delta(W)} \mathbf{1} \otimes W_{-n}$ of the chiral algebra [1]. In place of these somewhat unfamiliar objects one may equivalently consider the singlets in the dual space $(\mathcal{H}_\lambda \otimes \mathcal{H}_{\tilde{\lambda}})^*$; thus roughly, the boundary states may also be regarded as genuine vectors in this dual space. (Briefly, the notion of a co-invariant generalizes the concept of a singlet-submodule to the case of non-fully reducible modules, and the co-invariants of a module \mathcal{H} form a vector space isomorphic to the singlets in \mathcal{H}^* .)

In string theory, one often regards the boundary state $|\mathcal{B}^A\rangle$ as a synonym for the boundary condition A ; its proper interpretation is that by saturating one leg of a multi-reggeon vertex with $|\mathcal{B}^A\rangle$ amounts to introducing a boundary of type A on the world sheet. The quantities $|\mathcal{B}^A\rangle$ also appear naturally in the vacuum amplitude for the annulus, which can be evaluated with the help of the formula

$$\langle \mathbf{B}_\lambda | e^{2\pi i \tau (L_0 + \tilde{L}_0 - c/12)} | \mathbf{B}_\lambda \rangle = \chi_\lambda(2\tau), \quad (4)$$

where $\chi_\lambda(\tau) \equiv \chi_\lambda(\tau, 0, 0)$ is the Virasoro-specialized character of the \mathfrak{W} -module \mathcal{H}_λ (normalized, for convenience, to the quantum dimensions).

Twisted actions of the chiral algebra.

A basic task in open conformal field theory is to determine all possible boundary conditions. The properties to be imposed depend on the application that one has in mind. In the context of two-dimensional critical phenomena typically the boundary condition need to preserve just the Virasoro algebra; in special situations it may even be sufficient to respect only part of it. In string theory, one commonly requires to preserve the symmetry that is gauged, i.e. the Virasoro algebra respectively its relevant super extension in the case of superstrings; but boundary conditions for which the (super-)Virasoro algebra is preserved only up to BRST-exact terms seem to be perfectly admissible as well. Boundary conditions that violate part of the bulk symmetries can be roughly imagined as describing boundaries that carry some charge already in the absence of any fields.

The boundary blocks $|\mathbf{B}_\lambda\rangle$ introduced above do preserve the full chiral algebra \mathfrak{W} . Here the precise sense of the term ‘preservation’ is that \mathfrak{W} acts on $\mathcal{H}_\lambda \otimes \mathcal{H}_{\tilde{\lambda}}$ as prescribed in the formula (3), i.e. the action on the second factor $\mathcal{H}_{\tilde{\lambda}}$ is twisted by the automorphism

$$\sigma_0 : W_n \mapsto (-1)^{\Delta(W)+1} W_{-n} \quad (5)$$

of \mathfrak{W} . It is then natural to look for other chiral blocks that constitute co-invariants for some differently twisted action of \mathfrak{W} . One way to achieve this is to replace σ_0 by the product $\sigma \circ \sigma_0$, with σ some other automorphism of \mathfrak{W} . One can check that (formal) solutions to

$$(W_n \otimes \mathbf{1} - (-1)^{\Delta(W)} \mathbf{1} \otimes \sigma(W_{-n})) |\mathbf{B}_\lambda\rangle_{(\sigma)} = 0 \quad (6)$$

(which replaces the condition (3)) are given by $|\mathbf{B}_\lambda\rangle_{(\sigma)} = (\mathbf{1} \otimes \theta_\sigma) |\mathbf{B}_\lambda\rangle$, where the map θ_σ which acts on $\mathcal{H}_{\tilde{\lambda}}$ is characterized by its ‘ σ -twining’ property $\theta_\sigma \circ W_n = \sigma(W_n) \circ \theta_\sigma$.

Note that for non-trivial σ , such boundary conditions typically do *not* preserve the Virasoro algebra, and accordingly they shouldn’t play a role in applications to strings.

As a side remark, we mention that a large class of examples for Virasoro non-preserving automorphisms σ , for which the induced map θ_σ still has reasonable properties, is provided by the automorphisms $\sigma = \sigma_J$ that implement [5] the action of simple currents J of WZW

models. When such an automorphism σ_J has order two, then e.g. analogues of the formula (4) are given by

$$\begin{aligned} {}_{(\sigma)}\langle B_\lambda | e^{2\pi i \tau(L_0 + \tilde{L}_0 - c/12)} | B_\lambda \rangle_{(\sigma)} &= \chi_\lambda(2\tau, -\bar{\omega}_J \tau, (\bar{\omega}_J, \bar{\omega}_J) \tau / 2), \\ {}_{(\sigma)}\langle B_\lambda | e^{2\pi i \tau(L_0 + \tilde{L}_0 - c/12)} | B_\lambda \rangle &= \begin{cases} 0 & \text{for } J \star \lambda \neq \lambda, \\ \check{\chi}_\lambda(2\tau, 0, 0) & \text{for } J \star \lambda = \lambda. \end{cases} \end{aligned} \quad (7)$$

Here $\bar{\omega}_J$ is the horizontal part of the fundamental weight of the relevant affine Lie algebra that characterizes the simple current J and $\check{\chi}_\lambda$ is a so-called twining character [5], a generalized character-valued index. Similar formulæ hold when one twists in addition by an inner automorphism.

D-branes.

We now focus our attention on boundary conditions which are relevant to strings and D-branes. To this end we consider boundaries that respect the full chiral algebra. The natural structure underlying such boundary conditions turns out to be the one of *automorphisms ω of the fusion rules* that preserve conformal weights [1]. The origin of these automorphisms is the freedom that is present in relating the two labels λ and $\tilde{\lambda}$ of a bulk field $\phi_{\lambda, \tilde{\lambda}}$, and thus is quite similar to the origin of the appearance of fusion rule automorphisms in the classification of consistent torus partition functions. But in distinction to the case of closed conformal field theory, the factorization constraints do not require that this freedom is fixed in one and the same manner on all surfaces. Specifically, given a definite torus partition function, which (by taking the chiral algebra \mathfrak{W} sufficiently large) can be assumed to correspond to some fusion rule automorphism π , the pairing of λ and $\tilde{\lambda}$ is as prescribed by π on all closed orientable surfaces, but on the disk any other allowed fusion rule automorphism ω can appear as well. When $\omega = \pi$ one is dealing with an analogue of *Neumann* boundary conditions for free bosons, while the counterpart of *Dirichlet* boundary conditions of free bosons is given by $\omega = \pi \circ \omega_C$, where $\omega_C: \lambda \mapsto \lambda^+$ denotes charge conjugation.

Note that the choice of ω not only influences the values of the constants N_0^{AA} and $R_{\lambda\tilde{\lambda};0}^A$ in the relation (2), but also the explicit form of the 1-point block $|B_\lambda\rangle$, which therefore should more precisely be denoted by $|B_\lambda\rangle_\omega$. Adopting the terminology from the free boson case, one should refer to the co-invariants $|B_\lambda\rangle_\omega$ as *D-brane states*, or better as *D-brane blocks*. In the specific case of the theory of d uncompactified free bosons X^i with diagonal torus partition function and $\omega = \text{diag}((+1)^{p+1}, (-1)^{d-p-1}) \in O(d)$ (acting on the X^i), $|B_{\lambda=0}\rangle_\omega$ is indeed nothing but the usual Dirichlet p -brane with vanishing field strength on the $p+1$ -dimensional world volume. The automorphisms ω form a group (which in some cases is a Lie group, e.g. $O(d)$ for d free bosons). In a space-time interpretation, the choice of a connected component of that group looks like a topological information; thus the automorphism ω encodes global topological features of the D-brane.

The choice of a fusion rule automorphism ω does not refer to a boundary of C at all. Therefore this freedom is already present in the absence of boundaries, e.g. for $C = \mathbb{P}\mathbb{R}^2$. In contrast, as soon as boundaries are present there is an additional freedom, namely the (in general non-unique) choice of a consistent collection of reflection coefficients $R_{\lambda\tilde{\lambda};0}^A$. Thus a boundary condition A should be regarded as a *pair* $A \equiv (\omega, a)$, where ω is a fusion

rule automorphism respecting conformal weights, while the label a is tied to the existence of the boundary. In a space-time interpretation, a characterizes local properties of the D-brane, such as its position or a field strength on it [1]. In [1], ω is called the *automorphism type* of the boundary condition, while a is referred to as the *Chan–Paton type* because in string theory one must attach a distinct Chan–Paton multiplicity N_a to each allowed value of a . (The numbers N_a are to be determined by string theoretic arguments, e.g. tadpole cancellation.) Note that the summation in (1) is over all possible Chan–Paton types a such that $A = (\omega, a)$ with fixed automorphism type ω .

So far we did not say much about the possible values of the label a . According to [2] in the Neumann case $\omega = \pi = \omega_C$ the allowed index set is equal to the set $\{\lambda\}$ and the associated reflection coefficients $R_{\lambda\bar{\lambda};0}^A$ furnish one-dimensional representations of the fusion rule algebra. In [1] evidence was collected for the fact that (for all rational theories, and similarly for certain non-rational ones), for fixed automorphism type ω the number of labels a equals the dimension of some commutative associative algebra \mathfrak{C}_ω that generalizes the fusion rule algebra, and that the reflection coefficients $R_{\lambda\bar{\lambda};0}^A$ furnish one-dimensional \mathfrak{C}_ω -representations. The structure constants of \mathfrak{C}_ω are expected to satisfy some analogue of the Verlinde formula, related to structures similar to those uncovered in [5]. One particular class of examples for such classifying algebras had already been obtained before in [4] (for WZW models) and [6] (for arbitrary conformal field theories); several other examples are listed in [1].

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