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Statistical Entropy of Schwarzschild Black Holes

F. Englert[1]†* and E. Rabinovici[2]‡*

†*Service de Physique Théorique*
Université Libre de Bruxelles, Campus Plaine, C.P.225
Boulevard du Triomphe, B-1050 Bruxelles, Belgium

‡*Racah institute of Physics*
Hebrew University, Jerusalem, Israel

* *Theory Division, CERN*
CH-1211 Geneva 23, Switzerland

Abstract

The entropy of a seven dimensional Schwarzschild black hole of arbitrary large radius is obtained by a mapping onto a near extremal self-dual three-brane whose partition function can be evaluated. The three-brane arises from duality after submitting a neutral blackbrane, from which the Schwarzschild black hole can be obtained by compactification, to an infinite boost in non compact eleven dimensional space-time and then to a Kaluza-Klein compactification. This limit can be defined in precise terms and yields the Bekenstein-Hawking value up to a factor of order one which can be set to be exactly one with the extra assumption of keeping only transverse brane excitations. The method can be generalized to five and four dimensional black holes.

[1]E-mail: fenglert@ulb.ac.be
[2]E-mail: eliezer@vms.huji.ac.il

1. GENERAL CONSIDERATIONS

A major justification for indulging in the study of string theory, brane configurations and M-theory is that it may offer a useful point of view on quantum gravity. Indeed discoveries in D-brane physics have enabled the study of problems which lay outside the realm of classical general relativity, and in particular the evaluation of the black hole entropy from a counting of quantum states. Such computations were successfully performed for extreme supersymmetric black holes in cases where the dilaton field is finite on the horizon [1]. This is the case for charged black holes formed from marginally bound intersecting BPS branes in four and five space-time dimensions [2, 3, 4, 5, 6]. The correct Bekenstein-Hawking value $A/4$ for the entropy was also obtained for near extremal black three-branes [7], which also has a finite (and in fact constant) dilaton field although its area goes to zero at extremality. However in this case the factor $1/4$ was not obtained by direct calculation. It was suggested [7] to recover it by disregarding the longitudinal excitations of the brane. This recipe has not been clarified; it might arise from some peculiarity of the gauge field dynamics or perhaps just provide a phenomenological way to take into account the departure from supersymmetry in a case where the entropy vanishes with the area in the BPS extremal limit. Recently, microscopic considerations based on the Matrix model have also been applied to Schwarzschild black holes [8, 9, 10, 11, 12] but hitherto in a qualitative way. Other considerations [13] involve a connection between the Schwarzschild black hole and the 2+1 dimensional BTZ black hole [14], which has been given a microscopic description in [15, 16, 17]. In this note we suggest a quantitative analysis applying some M-theory ideas.

We propose to evaluate the entropy of Schwarzschild black holes of arbitrary large radius by mapping them through boosts and duality to nearly extremal charged black holes. We use the idea, first proposed in reference [18], to study properties of a Schwarzschild black hole by viewing it as a compactification of a blackbrane of eleven dimensional supergravity and relating it to a charged black hole. The latter is obtained by subjecting the blackbrane to a boost in uncompactified space-time followed by a Kaluza-Klein compactification on a different radius. By tuning the ratio of the two compactifications radii, it is possible to ensure that the classical thermodynamic entropy of the two holes are the same and we shall argue that the equivalence extends to the counting at the quantum level. The extreme limit corresponds to infinite boosts and vanishing Schwarzschild radius. We shall examine here a different limit, which we call the near extremal limit, for which the Schwarzschild radius remains arbitrarily large for infinite boost. Departure in energy from extremality are made vanishingly small although the mass of the charged black hole obtained after compactification of the boosted neutral blackbrane increases indefinitely with the boost. The entropic equivalence of the neutral and the charged black hole remains valid for infinite boosts as a consequence of an infinite concomitant rescaling of the Newton constant and of the charge quantum in the near extremal limit.

We shall test our proposal on the seven dimensional Schwarzschild black hole, which will be related to a near-extremal three-brane, and give a detailed analysis of our limiting procedure. Counting the number of the three-brane excitations, one recovers the Bekenstein-Hawking value for the entropy up to a factor of order unity which becomes exactly one if, for example, only the transverse excitations of the brane are counted. The reason of this incomplete success is rooted in the vanishing of the area in the extreme limit of the three-brane which necessitates a counting of non BPS-states to get a non vanishing entropy close to extremality. The use of supersymmetry to relate the semi-classical entropy to a string theory counting of states in flat space, is then, as in reference [7], not a priori fully justified. This is not the case for black holes in four and five space-time dimensions generated by intersecting branes which have a finite area in the extremal limit. One therefore expects that, using the mapping procedure, complete agreement may be reached for Schwarzschild black holes in five and four dimensions. However to generate intersecting branes from Kaluza-Klein charges requires several boosts [18]. The analysis of the limiting procedure is then more involved and is differed to a separate publication [19].

In the next section we review the relation between the neutral Schwarzschild black hole and the charged one and give a precise definition of the near extremal limit. In section 3 this limiting procedure is analyzed and the computation of the entropy of the seven dimensional Schwarzschild black hole is performed.

2. FROM NEUTRAL TO CHARGED BLACK HOLES.

The line element of a neutral black $p+1$ brane in eleven dimensional space-time is

$$-[1-(\frac{r_0}{r})^{D-3}]dt^2+[1-(\frac{r_0}{r})^{D-3}]^{-1}dr^2+r^2d\Omega_{D-2}+dz^2+(dx_1^2+dx_2^2+\ldots+dx_p^2) \quad (1)$$

where $D=10-p$. When compactified on $S^1\times T^p$, Eq.(1) describes a Schwarzschild black hole in D space-time dimensions. Its mass M and entropy S are

$$M = \frac{1}{16\pi G_D}\Omega_{D-2}(D-2)r_0^{D-3}, \quad (2)$$

$$S = \frac{1}{4G_D}\Omega_{D-2}r_0^{D-2}. \quad (3)$$

Here r_0 is the Schwarzschild radius and the D-dimensional Newton constant G_D is related to the eleven dimensional Planck length l_p by

$$G_D = \frac{l_p}{2\pi R L^p} \tag{4}$$

where $2\pi R$ is the length of the S^1-circle in the z-direction and L^p the volume of the T^p-torus.

Following reference [18], let us perform a Lorentz boost with rapidity α on the black-brane in the uncompactified eleven dimensional space-time along the z-direction:

$$z' = z\cosh\alpha + t\sinh\alpha \tag{5}$$

$$t' = t\cosh\alpha + z\sinh\alpha. \tag{6}$$

This boost is not a symmetry of the solution but it will be useful to view it as a coordinate transformation. When compactified on $S^{1'} \times T^p$ where the radius of the $S^{1'}$ circle in the z' direction is R', one obtains a charged black hole with metric

$$ds_D^2 = f^{\frac{1}{D-2}}(r)\left[-f^{-1}(r)[1-(\frac{r_0}{r})^{D-3}]dt'^2 + [1-(\frac{r_0}{r})^{D-3}]^{-1}dr^2 + r^2 d\Omega_{D-2}\right] \tag{7}$$

where

$$f(r) = 1 + (\frac{r_0}{r})^{D-3}\sinh^2\alpha. \tag{8}$$

The mass and the entropy of the charged black hole, expressed in terms of the new Newton constant $G'_D = l_p/2\pi R' L^p$, follow from the metric Eq.(7). One gets

$$M_{charged} = \frac{1}{16\pi G'_D}\Omega_{D-2}(D-3)(\sqrt{Q^2 + \frac{1}{4}r_0^{2(D-3)}} + \frac{D-1}{2(D-3)}r_0^{D-3}) \tag{9}$$

where

$$Q = r_0^{D-3}\cosh\alpha\sinh\alpha. \tag{10}$$

Here Q is a Kaluza-Klein charge and r_0 plays also the role of a parameter measuring the departure from extremality. The entropy of the charged black hole is

$$S_{charged} = \frac{1}{4G'_D}\Omega_{D-2}r_0^{D-2}\cosh\alpha. \tag{11}$$

If one relates the compactification radius R and R' by

$$R = R'\cosh\alpha, \tag{12}$$

the expansion of the horizon area due to the boost is compensated by the increase of the Newton constant

$$G'_D = G_D \cosh\alpha \tag{13}$$

and the entropies of the neutral and charged black hole become identical [11, 18]. This can be understood in the following terms. In the non compact eleven dimensional space-time the points to be identified by the Kaluka-Klein compactification form, in the boosted frame, an array separated by equal space intervals $2\pi R'$ at fixed time t'. These points are viewed from the rest frame as dilated according to Eq.(12) but they are separated by time intervals $\Delta t = 2\pi R' \sinh\alpha$. For the neutral blackbrane at rest such time intervals become on the horizon instantaneous events because of the infinite redshift. Horizons of the two compactified solutions in ten or lower dimensions are thus related simply by a coordinate transformation in eleven dimensions. If the horizons store the relevant degrees of freedom responsible for the black hole entropy, the degrees of freedom of the neutral and of the charged hole should be equivalent to one another under the eleven dimensional diffeomorphism group. This suggests that the equivalence of the entropy of the neutral and charged black hole is not merely a consequence of matching the classical formulas Eqs.(3) and (11) but should remain valid at the level of a quantum mechanical counting of states. We shall take as a working hypothesis that this is indeed the case.

The inner and outer horizons of the charged black hole occur respectively at $r = 0$ and $r = r_0$. It follows from Eq.(10) that at fixed charge Q the charged black hole becomes extremal in the limits $r_0 \to 0$, $\alpha \to \infty$, with $r_0^{D-3} \sinh\alpha \cosh\alpha$ kept fixed. We shall consider a different limit. Labeling by ΔM the difference in mass between $M_{charged}$ and the extremal value M_{ext} for the same charge one gets, to first order in r_0^{D-3}/Q,

$$\begin{aligned} \Delta M = \frac{1}{16\pi G'_D}\Omega_{D-2}\frac{1}{2}(D-1)r_0^{D-3} &= \frac{D-1}{(D-3)} M_{ext} \sinh^{-1} 2\alpha \\ &= \frac{D-1}{2(D-2)} M \cosh^{-1}\alpha. \end{aligned} \tag{14}$$

We see that $\Delta M / M_{ext}$ tends exponentially to zero as $\exp(-2\alpha)$ when $\alpha \to \infty$ independently of the value of r_0. This suggests to let α tend to infinity keeping r_0 finite. In this way we can approach the extremal limit for charges Q growing indefinitely and arbitrary large Schwarzschild radii. More precisely, we shall consider the limit as $\alpha \to \infty$ of a sequence of charged black holes with arbitrary large but fixed r_0 obtained by first boosting with rapidity α in non compact eleven dimensions and then compactifying on the radius R' defined in Eq.(12). This we call the near extremal limit. This limit may be different from the one which would result from considering a sequence of extremal black holes with increasing charges. Indeed, when α tends to infinity at fixed r_0, the entropy remains always finite because of the compensation of the divergence in the horizon area against the divergence of the Newton constant. However in some cases, as in the one considered below, the entropy vanishes at extremality with the area for all finite values of the charges. In such cases, the near extremal limit is still well defined and keeps track of perturbative departures from extremality.

3. COUNTING STATES OF SEVEN DIMENSIONAL BLACK HOLES

The near extremal limit was defined in classical terms only. However quantum theory imposes the quantization of the Kaluza-Klein charges. When ΔM is strictly zero, the black hole is extremal and charge quantization is equivalent to mass quantization. The minimal mass of a Kaluza-Klein excitation being $1/R'$, the number of charge quanta at extremality is $N = M_{ext}R'$. As ΔM vanishes in the near extremal limit this relation must remain true in this limit provided N remains finite. This is indeed the case because, as α goes to infinity, the $\exp(2\alpha)$ divergence of the charge Q in Eq.(10) cancels against the product of the divergences of Newton's constant G'_D in Eq.(9) and of the mass quantum R'^{-1}. We get

$$N = \frac{1}{16\pi G_D}\Omega_{D-2}(D-3)r_0^{D-3}R = \frac{D-3}{D-2}MR. \tag{15}$$

For extremal charged black holes with zero horizon area, the only way entropy could then arise in the near extremal limit would thus be through a finite contribution per charge quantum for infinitesimal ΔM! Remarkably, as we shall now see, this is exactly what M-theory predicts.

We use the assumed existence of M-theory to reduce, under some conditions, the computation of a Schwarzschild black hole entropy to a counting of string theory states in flat space. The strategy is to keep the eleven dimensional Planck length l_p fixed. The charged and neutral black holes which were compactified on different eleven dimensional radii are described by different ten dimensional string theories and particles carrying Kaluza-Klein charges become D0-branes. We use T-duality to interpret the charged black hole as generated from Dp-branes. We then extrapolate to zero curvature and prove that only massless excitations are present in the near extremal limit. Finally, we evaluate the D-brane degeneracy from the effective action of the zero-mass excitations.

Such a counting of states for branes close to extremality can only be performed when the dilaton field is not singular on the horizon [20]. For the case of non intersecting branes considered here, this selects three-branes. We shall therefore consider here only the three-brane case which is related to the seven dimensional Schwarzschild black hole.

The computation in the limit $\alpha \to \infty$ will reproduce the results of reference [7]. But this limit requires a detailed analysis which we now perform.

Keeping the eleven dimensional Planck length l_p fixed, the relation Eq.(12) means that the charged black hole is described by a ten dimensional string theory with string length l'_s and string coupling constant g' related to the unboosted string parameters l_s and g by

$$l_s = l'_s \cosh^{-1/2} \alpha, \qquad g = g' \cosh^{3/2} \alpha. \tag{16}$$

The seven dimensional Schwarzschild black hole is related through the boost to the near extremal charged black hole described by the metric Eq.(7) with $D = 7$. This metric can also be viewed as resulting, after compactification, from charged near extremal self-dual three-branes in ten dimensions wrapped over a three-torus [20, 21]. This follows from T-dualities. The three-brane is formed from N coincident D-branes, where N tends, as $\alpha \to \infty$, to the value Eq.(15) for $D = 7$, wrapped over a torus dual to the original one whose volume is L^3. Massless excitations are encoded in a $U(N)$ super Yang-Mills field theory on the dual torus. The volume of the dual torus L_d^3 and the string dual coupling constant g_d are given by

$$L_d = \frac{4\pi^2 l_s'^2}{L}, \tag{17}$$

$$g_d = g' \frac{8\pi^3 l_s'^3}{L^3}. \tag{18}$$

Taking into account Eqs.(16), the string coupling g_d equals $g 8\pi^3 l_s^3 / L^3$; it is independent of the boost and thus stays finite when the boost parameter α tends to infinity. The size of the dual torus diverges as $\exp \alpha$ as seen in Eq.(17).

Extrapolating to the weak coupling limit at arbitrary large but fixed value of α, we use perturbative string theory to count states in flat space. As pointed out in section one, this step is not a priori justified because we are departing from BPS states; moreover the physics of the conformal super Yang-Mills theory in $3 + 1$ dimensions in the phase of unbroken scale invariance is not well understood, except at zero gauge coupling. We shall however assume, following common practice, that the problem is reduced to a counting of only the transverse free massless excitations. Our point is indeed not to debate about the full justification of this procedure but rather to use it as a successful phenomenology.

The massless states can then be counted: there are $6N^2$ bosons and fermion massless transverse excitations present. If the thermal limit were valid, the entropy would be

$$S_{gas} = 2^{5/4} 3^{-(3/4)} \pi^{1/2} N^{1/2} \Delta M^{3/4} L_d^{3/4}. \tag{19}$$

We now see why this can lead to a finite answer in the near extremal limit. As announced, the contribution to the entropy per charge quantum is finite: the vanishing of ΔM is exactly compensated as $\alpha \to \infty$ by the divergence of L_d. The near extremal limit differs from a limit leading to the same black hole charge through a sequence of extremal black holes: it keeps track of brane excitations which, if ΔM were put to zero at the outset, would be absent. This results in a finite limiting entropy.

The fact that the size of the dual torus increases without bound means that the density of excitations goes to zero and hence ensures stability against decays which are neglected in the super Yang-Mills description (see however [22]). The validity of the estimate Eq.(19) requires that the inverse temperature be smaller than the characteristic size of the dual torus, namely $T^{-1} \ll L_d$. The Hawking temperature T_H of the charged black hole is

$$T_H = (\pi r_0 \cosh \alpha)^{-1}, \tag{20}$$

and therefore Eq.(17) would require $r_0 \ll l_s^2/L$, which would invalidate the classical description of both the Schwarzschild black hole and the charged one in the framework of general relativity. To be valid, this description requires indeed that the curvature at the horizon in the string frame be much smaller than the inverse string length squared. This condition is simply $r_0 \gg l_s$ for the Schwarzschild black hole and remains unaltered for the charged one: in the string frame r_0 gets multiplied by $\cosh^{1/2} \alpha$ [23], which cancels the rescaling of the string length. The validity of the classical description is reinstalled by taking instead of N coincident branes, the entropically favored solution consisting of branes wrapping N times over the dual torus[1], thus breaking $U(N)$ to $U(1)^N$ [25]. Eq.(19) must now to be interpreted as describing N (instead of N^2) modes on a torus of volume NL_d^3. Using the estimate Eq.(15), we express the seven dimensional Newton constant in terms of string parameters through $G_7 = G_{10}/L^3$ with

$$G_{10} = 8\pi^6 g^2 l_s^8 \tag{21}$$

and the M-theory relation $gl_s = R$. We now get $l_s^4 \ll (L^2/R) r_0^3$ which is consistent with the classical description and in fact justifies the use of an exact thermal distribution in the limit of asymptotically large Schwarzschild black holes. However a potentially dramatic consequence of the large boost would be the onset of massive string states, which in the limit $\alpha \to \infty$ become massless. This does not happen because, comparing the scaling properties of the temperature Eq.(20) and of the string length Eq.(16), we see that massive string states are exponentially suppressed in the tensionless limit $\alpha \to \infty$!

[1] this trick was used in references [24] [8] in a different but related context.

The computation of the gas entropy is now straightforward. Using Eq.(21) and $gl_s = R$, we express L_d given by Eq.(17) in terms of G_7:

$$L_d = 2G_7^{1/3} R^{-(2/3)} \cosh \alpha. \tag{22}$$

From Eqs.(19),(14),(15),(22),(2), and the value π^3 of the five dimensional unit sphere, one gets, in the limit $\alpha \to \infty$,

$$S_{gas} = \frac{1}{4G_7} \Omega_5 r_0^5, \tag{23}$$

in perfect agreement with the Bekenstein Hawking value Eq.(3) for $D = 7$.

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