PAR–LPTHE 98–01

CERN–TH 98–04

ON GENERALIZED SELF-DUALITY EQUATIONS TOWARDS SUPERSYMMETRIC QUANTUM FIELD THEORIES OF FORMS

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ABSTRACT

We classify possible "self-duality" equations for p-form gauge fields in space-time dimension up to $D = 16$, generalizing the pioneering work of Corrigan *et al.* (1982) on Yang-Mills fields ($p = 1$) in $4 < D \leq 8$. We impose two crucial requirements. First, there should exist a $2(p+1)$-form T invariant under a sub-group H of SO_D. Second, the representation for the SO_D curvature of the gauge field must decompose under H in a relevant way. When these criteria are fulfilled, the "self-duality" equations can be candidates as gauge functions for SO_D-covariant and H-invariant topological quantum field theories. Intriguing possibilities occur for $D \geq 10$ for various p-form gauge fields.

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1 Introduction

A large class of topological quantum field theories (denoted as $TQFT$s) for a p-form gauge field $A_{(p)}$ with $(p+1)$-form curvature $F_{(p+1)}$ can be understood as theories which explore the moduli space of (anti-)self-duality equations

$$\pm F_{\mu_1\cdots\mu_{p+1}} = \frac{1}{(D/2)!}\epsilon_{\mu_1\cdots\mu_{p+1}}{}^{\mu_{p+2}\cdots\mu_D}F_{\mu_{p+2}\cdots\mu_D}. \tag{1.1}$$

Assuming that the gauge symmetry of the topological theory is arbitrary, local redefinitions of the classical fields, the self-duality equations (1.1) are relevant as topological gauge-fixing functions. With the method of $BRST$ quantization, one determines the $BRST$-invariant Lagrangian to be used in the path integral formulation of the $TQFT$ as well as the observables [1], [2]. Moreover, the topological $BRST$ symmetry can often be (un)twisted to recover an ordinary space-time supersymmetry [3], [4].

The self-duality equations (1.1) rely on the existence of the totally antisymmetric tensor $\epsilon_{\mu_1\cdots\mu_D}$. Thus, one expects that Yang-Mills-like $TQFT$'s can only be constructed for p-form gauge fields satisfying the condition $2(p+1) = D$. However, relaxing some constraints, one foresees the possibility for new classes of $TQFT$'s involving p-forms in various dimensions D. This seems an interesting perspective to us: it could provide effective field-theories on the world-volume of branes (see for instance [4], [5], [6], [7]); or a 12-dimensional $TQFT$ for a 3-form gauge field, which could be candidate for the elusive "F-theory". Furthermore, new instanton solutions could be of interest, as for $p = 1$, $D = 8$ [8].

Such theories reduce the D-dimensional Euclidean *invariance* down to invariance under a sub-group $H \subset SO_D$ (with Lie algebra h), while preserving the SO_D *covariance*. From a geometrical point of view, in curved space, H would be the holonomy group of the D-dimensional manifold over which the supersymmetric $TQFT$ is constructed. This generalization is done by changing $\epsilon_{\mu_1\cdots\mu_D}$ in (1.1) into a more general $2(p+1)$-tensor T. For $D > 2(p+1)$, T cannot be invariant (*i.e.* covariantly constant in curved space) under SO_D, but rather under one of its sub-groups $H \subset SO_D$. The determination of such invariant tensors T was first worked out by Corrigan *et al.* in [9], for Yang-Mills fields in dimensions up to eight.

In the same spirit of [4], our purpose here is thus to exhibit equations of the type

$$\lambda F_{\mu_1\cdots\mu_{p+1}} \quad = \quad T_{\mu_1\cdots\mu_{p+1}}{}^{\nu_1\cdots\nu_{p+1}} F_{\nu_1\cdots\nu_{p+1}} \tag{1.2}$$

where λ is to be understood as one of T's eigenvalues. Keeping in mind that we wish (1.2) to be relevant as a gauge-fixing function, we expect that the field equations remain a consequence of the Bianchi identity together with (1.2). This implies that we ought to restrict ourselves to totally antisymmetric tensors T. Therefore, equation (1.2) can be called a "generalized self-duality equation" by analogy with the generic case (1.1) where $2(p+1) = D$. Furthermore, since it only involves one gauge field, we call (1.2) of "pure" type, as opposed to self-duality equations that would mix the curvatures for various gauge fields.

If a D-dimensional $TQFT$ can be constructed from (1.2), its Lagrangian will only be SO_D-*covariant* and H-*invariant*. Its supersymmetry will be a topological $BRST$-symmetry, where all the fermions are ghosts. However, it may happen that the full SO_D-invariance be recovered from a twist of the various fields involved in the theory, and the ordinary supersymmetry from untwisting the topological $BRST$ symmetry (a further dimensional reduction can be necessary). An example of this scenario is the one in [4], where a covariant eight-dimensional $TQFT$ is built for a Yang-Mills field, with the SO_8 invariance broken down to a $G_2 \subset SO_7 \subset SO_8$ invariance: the action explicitly depends on an SO_8 4-form T, invariant under G_2 only. Nevertheless, untwisting the corresponding Lagrangian enables to absorb this dependence into field redefinitions, giving rise to the ordinary Lorentz invariant supersymmetric action in eight dimensions. The latter is the one that is obtained by dimensional reduction of the usual $N = 1$, $D = 10$ super-Yang-Mills theory [4], [5].

Determining H-invariant tensors T can be done using the branching rules for irreducible representations of semi-simple Lie algebras (we used the tables in the book from W.G.McKay, J.Patera [10]: they provide the information as to semi-simple algebras, while abelian factors can be found in [11]). Furthermore, we are interested in a situation where a TQFT can be constructed, with the self duality equation (1.2) as gauge function. In this perspective, (1.2) must ideally give a number of independent equations on the curvature, matching the number N_{eq} of gauge invariant components of $A_{(p)}$. As we shall see, the realization of the requirements towards a $TQFT$ is non trivial, as soon as it comes to cases other than the generic $2(p+1) = D$ for even D. It even came as a surprise to us that we found possibilities for other $(p,\, D)$.

Our paper has two aspects. On one hand, we give the methodology to probe the possible relevance of such theories, enlightening the selection of h and T for given $(p,\, D)$'s. On another hand, we implement the method and explicitly determine the eligible sub-algebras h for each possible value of p and D, using the tables in [10] and unpublished results concerning higher dimensional cases.[1]

There are limitations in our analysis. First, the existing tables give the decompositions of the SO_D-representations into maximal semi-simple sub-algebras. We have thus restricted our analysis to maximal sub-algebras h (except for $su_{D/2} \subset so_D$ cases that we obtain explicitly by complexification). This could lead us to miss possible solutions. For instance, the interesting octonionic solution of non-maximal $g_2 \subset so_7 \subset so_8$ for $(p = 1,\, D = 8)$ can be extracted from the solutions of the non-maximal (and thus not recorded in [10]) sub-algebra $su_4 \subset so_8$. However, g_2 is maximal within so_7, and the octonionic solution can nevertheless be found from [10], through a slightly different and more refined analysis (see subsection 2.1).

Second, we shall not consider in this paper the possible mixing of curvatures in the self-duality equation (1.2). Mixed self-duality equations involving several gauge fields occur, for instance, from dimensional reduction of a "pure" case like (1.2). An example was observed in [4]: the "pure" $(p = 1,\, D = 8)$ equation gives, when reduced to 4 dimensions, a non abelian version of the Seiberg-Witten equations, involving a spinor and a Yang-Mills field. We emphasize that other mixed self-duality equations may exist (that do not come from dimensional reduction of a "pure" case), but we shall not consider them here.

Despite these points, and despite the fact that we only exhibit the possible h's without any certainty that the corresponding $TQFT$ can be given a sense, we nevertheless believe that the results presented here are already quite interesting by themselves. In particular, they point out the diversity of the possibilities, depending on the various values of D. They also indicate the specificity of $D = 8$ which is the only case where an exceptional group, G_2, arises in our analysis (the only orthogonal groups that have G_2 as a maximal sub-group are SO_7 and SO_{14}, but in the later case, the requirements we impose cannot be realized).

The organization is as follows: in section 2 are given the arguments for the selection of the invariance algebra h and for the conditions on T together with its λ's. Section 3 contains our results: we display the possible invariance sub-algebra h for each pair $(p,\, D)$ and comment on the peculiarities of each case we have explored.

[1] We gratefully thank J. Patera for private communication of those.

2 Analysis of the Generalized Self-Duality Equation

Our aim for this section is to probe the various criteria to be fulfilled for the self-duality equation (1.2) to count for N_{eq} gauge fixing conditions, and to be appropriate for building $TQFT$'s of various (p, D). Let us first consider equation (1.2) from a general point of view, keeping in mind that we seek a systematic check of the following requirements:

i) the SO_D form T should be an $H \subset SO_D$-invariant;

ii) the SO_D irreducible representation for F should contain an N_{eq}-dimensional representation in its decomposition under H;

iii) the eigenvalues λ should have a suitable degeneracy, allowing (1.2) to count for N_{eq} independent equations on F;

iv) the properties of T should enable to put the action under a Yang-Mills-like form.

We shall detail the points *i), ii)* in the following subsection. Criteria *iii)* will be the object of subsection 2.2, and the remaining criteria *iv)* of subsection 2.3.

2.1 On the selection of the invariance sub-algebra h

Before describing the method, let us set up our notations: we consider a p-form gauge field in D space-time dimensions. Its curvature is a ($p_F = p+1$)-form, and has $d_F = \binom{D}{p_F}$ independent components. Consequently, the invariant form T, relevant to write (1.2), is a p_T-form with $p_T = 2p_F = 2(p+1)$. Since it is chosen totally antisymmetric, T has $d_T = \binom{D}{p_T}$ independent components. Thus, for a given p, we seek a $2(p+1)$-form T that is invariant under a sub-group H of SO_D, while T (resp. F) lies in a representation of SO_D of dimension d_T (resp. d_F).

The first check *i)* is to examine whether an SO_D p_T-form can be left invariant by the transformations of some sub-group $H \subset SO_D$. It consists in inspecting, in [10], the decomposition of the so_D-representation $\underline{d_T}$ upon irreducible representations of all the possible sub-algebras h. We then select the h's that leave the p_T-form invariant, *i.e.* the sub-algebra under which the decomposition of $\underline{d_T}$ produces a singlet $\underline{1}$. Notice that we restrict ourselves to maximal sub-algebras, except for the particular case of a sub-algebra h leaving $\underline{d_T}$ irreducible. In this case, we pick up the sub-algebra $\tilde{h} \subset h$ that leaves T invariant. This is the situation of the ($p = 1$, $D = 8$) case, where the 4-form T can be decomposed into self and anti-self-dual parts, irreducible under so_7. More generally, this occurs for $D = 4(p+1)$.

Once such a sub-algebra h (or $\tilde{h}$) has been selected, we impose that it satisfies criteria *ii)*: to end up with a proper $TQFT$ for the p-form gauge field, the self-duality equations (1.2) must count for as many equations as there are of gauge-invariant components of $A_{(p)}$ (the longitudinal gauge degrees of freedom can be fixed with the conventional $BRST$ formalism for forms [2]). A p-form gauge field has a total of $\binom{D}{p}$ components, but is defined up to an anticommuting gauge symmetry parameter $\xi_{(p-1)}$. The latter has a total of $\binom{D}{p-1}$ components, but is also defined up to a commuting gauge symmetry parameter $\xi_{(p-2)}$ with $\binom{D}{p-2}$ components, and so on. Thus, the number N_{eq} of gauge-invariant components of $A_{(p)}$ is the number of components of a p-form in $(D-1)$ dimensions, that is

$$N_{eq} = \binom{D}{p-1} - \binom{D}{p-2} + \cdots \pm \binom{D}{0} = \binom{D-1}{p}. \tag{2.1}$$

Therefore, criteria *ii)* consists in checking whether the so_D-representation $\underline{d_F}$ contains, in its h-decomposition $\underline{d_F} = \bigoplus \underline{d_F}^i$ $(i = 1 \cdots n)$, a representation $\underline{N_{eq}} = \bigoplus \underline{d_F}^j$ with total dimension N_{eq}. The right number of constraints on the gauge field $A_{(p)}$ can then be obtained by imposing (1.2) if the latter allows to project F on the space corresponding to $\underline{(d_F - N_{eq})}$. In this final sorting of the sub-algebras, we do not impose that $\underline{N_{eq}}$, nor $\underline{(d_F - N_{eq})}$, be irreducible representations of h. Allowing these representations to be a sum of irreducible representations does not lead to inconsistencies: (1.2) is then expressed as independant sets of equations that involve independant combinations of F's components.

T can be understood as a linear operator, mapping the space of $(p+1)$-forms into itself; and since it has been chosen a singlet of h, Schur's lemma implies that T acts block diagonally in each of the $\underline{d_F}^i$'s (T being an invariant of H, its action on an irreducible representation of H remains in the same representation). Each $\underline{d_F}^i$ is associated with an eigenvalue λ^i, and one can always choose an appropriate basis to diagonalize T in each $\underline{d_F}^i$ so that it reads:

$$T_{diag} = \begin{pmatrix} [\lambda^1 \mathbb{I}_{(1)}] & & & & \\ & [\lambda^2 \mathbb{I}_{(2)}] & & & \\ & & [\ddots] & & \\ & & & [\lambda^i \mathbb{I}_{(i)}] & \\ & & & & [\ddots] \end{pmatrix} \tag{2.2}$$

where $\mathbb{I}_{(i)}$ denotes the relevant $(d_F{}^i)$-dimensional identity operator involved in the associated sub-space.[2]

[2]This actually holds as long as each $\underline{d_F}^i$ appears only once in the decomposition $\underline{d_F} = \bigoplus \underline{d_F}^i$. If a given

Taking into account the possible degeneracy of the eigenvalues, the $(\lambda^1, \cdots, \lambda^i, \cdots, \lambda^n)$ take m distinct values $(\mu^1, \cdots, \mu^a, \cdots, \mu^m)_{(m\leq n)}$. One can then re-write T in terms of the μ's, and define the h-invariant operator P_{μ^a}, key to the gauge fixing, as $P_{\mu^a} = T - \mu^a \mathbb{I}_{(a)}$. According to the degeneracy of the λ's, each μ^a is now associated to one or a sum of representations (with the same λ^j: $\underline{d_a} = \bigoplus \underline{d_F{}^j}$), of total dimension d_a. Imposing $P_{\mu^a} F = 0$ projects F on the sub-space corresponding to $\underline{d_a}$. It amounts to cancelling the projection of F onto all the other sub-spaces corresponding to $\mu^b \neq \mu^a$. This gives a number of independent equations on F equal to the total dimension $\sum d_b = (d_F - d_a)$ of the $(\bigoplus \underline{d_b})$ representation. Hence, $P_{\mu^a} F = 0$ may be suitable as a gauge fixing function in the sense described above, provided that it counts for N_{eq} equations, *i.e.* provided that $(d_F - d_a) = N_{eq}$.

Before taking this any further (we shall come back on this point in subsection 2.3), let us first probe the conditions that one has on the λ^i's, and in particular the issue of criteria *iii)*.

2.2 On the degeneracy of the eigenvalues λ

Suppose that one has found a sub-algebra h and a form T satisfying criteria *i), ii)*; and consider only one of the sub-spaces upon which F decomposes. To compute the corresponding eigenvalue λ^i, one can apply T on (1.2) once more to get:

$$(\lambda^i)^2 F_{\mu_1\cdots\mu_{p+1}} = T_{\mu_1\cdots\mu_{p+1}}{}^{\nu_1\cdots\nu_{p+1}} T_{\nu_1\cdots\nu_{p+1}}{}^{\sigma_1\cdots\sigma_{p+1}} F_{\sigma_1\cdots\sigma_{p+1}}. \tag{2.3}$$

One obtains an equation for λ^i if one knows how $(T^2)_{\mu_1\cdots\mu_{p+1}}{}^{\sigma_1\cdots\sigma_{p+1}}$ expands upon h-invariant tensors, and how these act on F. Unlike T, T^2 lies in a reducible representation $\underline{d_{T^2}}$ of so_D: it transforms like the product of two p_F-forms, $\underline{d_{T^2}} \equiv \underline{d_F} \otimes \underline{d_F}$, that one first expands as a sum of irreducible representations of so_D. Second, T^2 is a tensor of so_D, necessarily h-invariant because T is, so it can only expand as a linear combination of h-invariant tensors. One is sure to find, in T^2, at least the trace part $\delta_{[\mu_1}{}^{[\sigma_1} \cdots \delta_{\mu_{p+1}]}{}^{\sigma_{p+1}]}$, as well as a $T_{\mu_1\cdots\mu_{p+1}}{}^{\sigma_1\cdots\sigma_{p+1}}$ term, but there might be others. Generically, let us write the decomposition of $\underline{d_{T^2}}$ upon h-invariant irreducible representations of so_D as

$$\underline{d_{T^2}} \equiv \underline{d_F} \otimes \underline{d_F} = \underline{1} \oplus \underline{d_T} \oplus \left(\bigoplus_m \underline{d_{T_m}} \right). \tag{2.4}$$

representation has multiplicity m higher than one, then T's action on these m identical representations is one single block (instead of m), that is not necessarily diagonal.

Whenever T and the trace term are the only tensors to appear in (2.4), one has

$$(T^2)_{\mu_1\cdots\mu_{p+1}}{}^{\nu_1\cdots\nu_{p+1}} = A\delta_{[\mu_1}{}^{[\nu_1}\cdots\delta_{\mu_{p+1}]}{}^{\nu_{p+1}]} + BT_{\mu_1\cdots\mu_{p+1}}{}^{\nu_1\cdots\nu_{p+1}} \tag{2.5}$$

and the λ^i's are solutions of

$$(\lambda^i)^2 = A(p+1)! + B\lambda^i, \tag{2.6}$$

where the trace part appears with a non-vanishing coefficient A.[3] The statement that $A \neq 0$ ensures that $\lambda^i \neq 0$. Equation (2.6) tells us a lot more: as long as the hypothesis of the unicity of T in (2.4) holds, the λ^i's can only take two possible values, whatever number of h-irreducible representations is involved in the decomposition of F. One can thus expect some quite restrictive degeneracy of the λ's.

The only possibility for the λ's to be allowed more than two values is when other invariant tensors than T appear in (2.5). To each of those invariant tensors T_m is associated a set of eigenvalues. Reproducing the same string of arguments we have used for T^2, one gets the decomposition of each $(T_m)^2$ upon the set of invariant tensors. These lead to a set of equations for the eigenvalues of T together with the eigenvalues of the T_m's.

In general, we expect a high level of degeneracy for the λ's. Unfortunately, to be sure that (1.2) eventually counts for N_{eq} constraints on F, one would need to know, case by case, the detailed spectrum of the λ^i's. Since the explicit construction of the T tensor is not our goal here, we shall not take this any further, but rather turn to requirement *iv)*.

2.3 On the relevance of the self-duality equation towards a $TQFT$

As to the relevance of the self-duality equation, there are two things to be kept in mind: in all the cases we expose in the coming section, the check *ii)* ensures that one can always write N_{eq} H-invariant equations. However, it is not at all systematic that these equations can be imposed with the sole constraint (1.2). Even if it is the case, it is not guaranted that the form T is suitable to recover a Yang-Mills like action, plus topological terms, when the gauge function is squared. Notice that one can always find an h-diagonal matrix with the spectrum

[3] Indeed, if one contracts the remaining indices to get $(T^2)_{\mu_1\cdots\mu_{p+1}}{}^{\mu_1\cdots\mu_{p+1}}$, the only term that survives is the trace term. Furthermore, and since we are considering Euclidean spaces, its coefficient A cannot be zero unless T itself is zero.

of eigenvalues suitable to project the curvature on the relevant sub-spaces. However, the later does not necessarily correspond to an antisymmetric tensor of SO_D and thus is not relevant in view of *iv)*.

Criteria *iv)* involves non trivial properties of T. The most favorable case is when T is the only tensor to appear in the right hand side of (2.4), and when $P_{\mu^a} F = 0$ projects on the relevant sub-space. The N_{eq} equations coming from (1.2) can then be written as duality relations between the "electric" and "magnetic" parts of the curvature:

$$F_{Di_1 \cdots i_p} = c_{i_1 \cdots i_p}{}^{j_1 \cdots j_{p+1}} F_{j_1 \cdots j_{p+1}}, \tag{2.7}$$
$$c_{i_1 \cdots i_p j_1 \cdots j_{p+1}} \equiv T_{Di_1 \cdots i_p j_1 \cdots j_{p+1}},$$

with $(i_1, i_2, \cdots, j_1, \cdots)$ being $(D-1)$-dimensional indices. Therefore, when the gauge function is squared, using (2.5), one gets the appropriate terms $F \wedge \star F$ and $(\star T) \wedge F \wedge F$ in the action for the $TQFT$:

$$\begin{aligned} & \left|\lambda F_{\mu_1 \cdots \mu_{p+1}} + T_{\mu_1 \cdots \mu_{p+1}}{}^{\nu_1 \cdots \nu_{p+1}} F_{\nu_1 \cdots \nu_{p+1}}\right|^2 & (2.8)\\ = \quad & \alpha \left|F_{\mu_1 \cdots \mu_{p+1}}\right|^2 + \beta \left(T_{\mu_1 \cdots \mu_{p+1}}{}^{\nu_1 \cdots \nu_{p+1}} F_{\mu_1 \cdots \mu_{p+1}} F_{\nu_1 \cdots \nu_{p+1}}\right) & (2.9)\\ \mapsto \quad & \alpha \left(F \wedge \star F\right) + \beta \left(\star T \wedge F \wedge F\right) & \end{aligned}$$

with the $[(D-2(p+1)) \equiv (D-p_T)]$-form $(\star T)$ defined as $\epsilon_{\mu_1 \cdots \mu_D} T^{\mu_1 \cdots \mu_{p_T}} = (\star T)_{\mu_{p_T+1} \cdots \mu_D}$. The way supersymmetric terms are added to (2.9), giving a $BRST$-invariant action, is a straightforward generalization of the work detailed in [4].

For less ideal cases, the degeneracy of the λ's can have drastic consequences on the relevance of (1.2) as a gauge function. Very often, the self-duality equations amount to a number of independent conditions exceeding N_{eq}. These are the most common cases, for which the N_{eq}-dimensional sub-space shares its eigenvalue λ with other eigenspaces. Another situation is when the N_{eq}-dimensional sub-space contains different eigenspaces which are associated with different eigenvalues $(\lambda^1 \cdots \lambda^k)$. Here, one can tentatively use as gauge function

$$\left(\prod_{i=1}^{k} P_{\lambda^i}\right) F \equiv \left(\prod_{i=1}^{k} \left(T_{\mu_1 \cdots \mu_{p+1}}{}^{\nu_1 \cdots \nu_{p+1}} - \lambda^i \delta_{[\mu_1}{}^{[\nu_1} \cdots \delta_{\mu_{p+1}]}{}^{\nu_{p+1}]}\right)\right) F_{\nu_1 \cdots \nu_{p+1}}. \tag{2.10}$$

Whatever the degeneracy of the λ's may be, *i.e.* with either (2.8) or (2.10) as gauge function, if the expansion for T^2 involves other H-invariant tensors T_m, the resulting $TQFT$ shall have an unusual Lagrangian, that possibly depends on these other invariant tensors. Notice that the additional T_m-terms are not topological, since the T_m's do not correspond to totally antisymmetric tensors of SO_D; these terms can nevertheless be of interest.

For a given sub-group H satisfying our criteria, we often found only one singlet in the decomposition of $\underline{d_T}$. To the later corresponds a given spectrum of the eigenvalues λ^i (*i.e.* a given set of μ^a), often such that the number of equations obtained on the components of F is greater than N_{eq}. An example is the case of $h \equiv so_{D-1}$, with $2(p+1) = (D-1)$, in odd space-time dimension D. Here, (1.2) amounts to canceling $F_{Di_1 \cdots i_p}$ (N_{eq} equations) *and* the $\frac{1}{2}\binom{D-1}{p+1}$ anti-self-dual components of $F_{i_1 \cdots i_{p+1}}$ ($i_1, i_2, \cdots$ are $(D-1)$ dimensional indices). Another example is the $h \equiv so_d \times so_{D-d} \subset so_D$ case. Here, the only possible T again has eigenvalues ± 1 and leads to more than N_{eq} constraints on the curvature.

However, there are cases where $\underline{d_T}$ gives several singlets in its decomposition under h, giving more flexibility as to the spectrum of the eigenvalues. The general solution for T is then expressed as a linear combination of the corresponding tensors: $T = \sum_k \alpha_k T^{(k)}$, with $\lambda = \sum_k \alpha_k \lambda^{(k)}$. Thus, one has the opportunity of adjusting the coefficients α_k, such that the eventual spectrum allows to project F precisely on the relevant sub-spaces. The situation of having several T's occurs in few and privileged cases, but there are some examples.

Notice that relaxing the condition that h be maximal would increase the probability to have several independent invariant tensors T. Indeed, let h' be a sub-algebra of h, itself being maximally embedded in so_D. In the chain of decompositions of $\underline{d_T}$, singlet pieces can come from the various "upper level" components of T: under $h \subset so_D$, $\underline{d_T}$ decomposes into $\underline{1}$+$\underline{d_T}$', and under $h' \subset h$, $\underline{d_T}$' can give rise to additional singlet pieces. We shall point out such a privileged situations in the next section, for instance in the eight-dimensional case.

Following the steps described above, we have obtained non trivial solutions, that we are now going to expose.

3 Results and Comments

We present our various results under the form of a table exploring the possibilities for $p_T = 2(p+1) \leq D$ up to $D = 16$. This table gathers the invariance sub-algebras h which fulfill our conditions *i), ii)*. Each possibly relevant case will be separately detailed afterwards (though, again, T is not explicitly constructed here, so that we are not in a position to check *iv)*.

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$
$D=4$	$su_2 \times su_2$						
$D=5$	$su_2 \times su_2$						
$D=6$		su_4					
$D=7$	$su_2 \times su_2 \times su_2$	su_4					
$D=8$	$g_2 \subset so_7{}^\star$ su_4		so_8				
$D=9$		$su_2 \times su_4$	so_8				
$D=10$		$su_2 \times su_2 \times su_4$	su_5	so_{10}			
$D=11$	su_2		su_2	so_{10}			
$D=12$		$su_4 \times sp_4 \subset so_{11}{}^\star$	su_6 $su_2 \times sp_6$		so_{12}		
$D=13$			su_2		so_{12}		
$D=14$					su_7	so_{14}	
$D=15$	su_2		su_2 su_4 $su_2 \times sp_4$	su_4	su_2 su_4	so_{14}	
$D=16$			$sp_4 \times sp_4$ $su_2 \subset so_{15}{}^\star$ $su_4 \subset so_{15}{}^\star$ $su_2 \times sp_4 \subset so_{15}{}^\star$ $so_7 \times so_8 \subset so_{15}{}^\star$		su_8 $sp_4 \times sp_4$ $su_2 \times sp_8$		so_{16}

The index $(^\star)$ indicates the particular cases $4(p+1) = D$, where T can be decomposed into self- and anti-self dual parts, so that $\underline{d_T}$ is not irreducible under so_D, but rather under $so_{(D-1)}$.

Let us remind the isomorphisms between simple Lie algebras: $su_2 \sim so_3 \sim sp_2$, $sp_4 \sim so_5$, $su_4 \sim so_6$.

In the above table, the generic self-duality (1.1) systematically appears on the diagonal for even $D = 2(p+1)$. Similarly, one has the odd dimensional $(D-1) = 2(p+1)$ briefly discussed in sub-section 2.3.

In 5, 7 and 9 dimensions, the possibilities are of the aforementioned type $h \equiv so_{D-1}$ or $so_d \times so_{D-d}$, for which $\lambda = \pm 1$ and T does not project F properly: one gets more than N_{eq} constraints on the gauge field. Details for the solutions $p = 1$ of $D = 5, 7$ can be found in [9]. The other cases with $h \equiv so_d \times so_{D-d}$ or so_{D-1} occurring in higher dimensions are analogous. Therefore, we only comment on the cases ($D = 8$, $p = 1$) and $D \geq 10$ where more peculiar sub-algebras h occur. Furthermore, we display all the h-decompositions of $\underline{d_F}$ representations in the tables of the appendix.

3.1 8-dimensional cases

Apart from the standard solution (1.1) of a 3-form, one can have su_4 or g_2 invariance for the 4-form T of the Yang-Mills case ($p = 1$). These are the cases of interest in [4], [5], [6]. For $p = 1$, $D = 8$, T is a $D/2$-form and can thus be naturally decomposed into self and anti-self-dual parts. While $\underline{d_F} = \underline{28}$, one seeks $N_{eq} = 7$ equations, where the 4-form T, being itself a solution for (1.1), transforms as a $\underline{70} \sim \underline{35}^+ + \underline{35}^-$ of so_8.

We first consider the $g_2 \subset so_7$ theory. Under so_7, $\underline{35}^+$ and $\underline{35}^-$ remain irreducible, whereas $\underline{d_F}$ obviously decomposes into

$$\underline{28} = \underline{7} + \underline{21}. \tag{3.1}$$

Checking the decomposition of so_7's $\underline{35}$ under its sub-algebras selects g_2, under which

$$\begin{aligned} &\underline{35} = \underline{1} + \underline{7} + \underline{27} \\ &\underline{21} = \underline{7} + \underline{14} \\ &\underline{7} = \underline{7} : \text{ irreducible.} \end{aligned} \tag{3.2}$$

The spectrum for T's eigenvalues allows to impose exactly 7 equations (*i.e.* to project F onto a $\underline{7} + \underline{14}$). Indeed, one can check that T is the only invariant tensor upon which T^2 can decompose, so that it admits two eigenvalues (see sub-section 2.2), namely $\lambda_1 = -1$, $\lambda_2 = 3$ (see [9]). Moreover, T_{8ijk} (i, j, k are 7-dimensional indices) is antisymmetric and thus traceless with respect to the SO_7 metric. Therefore, the only possibility for the degeneracy of the λ's is the appropriate one: $21\lambda_1 + 7\lambda_2 = 0$, and imposing (1.2) with $\lambda = \lambda_1$ leads to the 7 equations on F. This case was first studied by Corrigan *et al.* [9] and the corresponding $TQFT$ was constructed in [4]. The set of 7 self-duality equations obtained are

$$\begin{aligned} F_{8i} &= c_{ijk} F_{jk}, \\ c_{ijk} &= T_{8ijk} \end{aligned} \tag{3.3}$$

where the c_{ijk} $(i,j,k = 1 \cdots 7)$ are the structure constants for octonions. More explicitly, in terms of 8-dimensional components of the curvature, they read

$$\begin{aligned} &F_{12} + F_{34} + F_{56} + F_{78} = 0, \\ &F_{13} + F_{42} + F_{57} + F_{86} = 0, \\ &F_{14} + F_{23} + F_{76} + F_{85} = 0, \\ &F_{15} + F_{62} + F_{73} + F_{48} = 0, \\ &F_{16} + F_{25} + F_{38} + F_{47} = 0, \\ &F_{17} + F_{82} + F_{35} + F_{64} = 0, \\ &F_{18} + F_{27} + F_{63} + F_{54} = 0. \end{aligned} \tag{3.4}$$

Let us now turn to the case $h \equiv su_4$ (it is not maximal and thus not recorded in the tables from [10]). It enables to complexify the 8-dimensional indices (with $z_a \equiv x_a + i x_{a+4}$, $a = 1 \cdots 4$):

$$F_{\mu\nu} \rightarrow (F_{z_a z_b},\ F_{z_a \bar{z}_b},\ F_{z_a \bar{z}_a}), \tag{3.5}$$

$$\underline{28} = \underline{6} + \underline{6} + \underline{15} + \underline{1}. \tag{3.6}$$

To end up with 7 equations, the projection of F on a $(\underline{1} + \underline{6})$ has to be cancelled. This is possible because there are three independant su_4 invariant 4-forms (see appendix B in [9]), and one considers T a linear combinaison of them. The coefficients can be adjusted at will to end up with the self-duality equations:

$$\text{(1 real eqs)} \qquad \sum_{a=1}^{4} F_{z_a \bar{z}_a} = 0 \tag{3.7}$$

$$\text{(6 real eqs)} \qquad F_{z_a z_b} + \frac{1}{2} \epsilon_{abcd} F_{\bar{z}_c \bar{z}_d} = 0. \tag{3.8}$$

Moreover, one can observe that equation (3.7) is the imaginary part of the complex equation

$$\sum_{a=1}^{4}\left(\partial_{z^a}A_{\overline{z}^a}+\frac{1}{2}[A_{z^a},A_{\overline{z}^a}]\right)=0. \tag{3.9}$$

The real part of (3.9) gives the Landau gauge condition $\partial_\mu A_\mu = 0$. It is remarkable that the former equations (3.7)–(3.8), when re-expressed in terms of SO_8 components $F_{\mu\nu}$, give the set of equations (3.4) obtained through g_2.

Notice that if one considers the maximal $su_4 \times u_1$ rather than su_4, only one T is possible, and one cannot adjust its eigenvalues. Equation (1.2) then leads to the sets of 13, 16 or 27 equations of [9] and criteria *ii)* cannot be satisfied. This illustrates the situation explained in subsection 2.3, where more flexibility comes from relaxing the hypothesis that h be maximal.

3.2 10-dimensional cases

Here, apart from the case of a 2-form (of the aforementioned type $so_d \times so_{D-d} \equiv so_4 \times so_6$),we have as a non-standard solution a 3-form gauge field with T invariant under su_5 (here again, the maximal sub-group is rather $su_5 \times u_1 \subset so_{10}$). The 4-form curvature is a $\underline{210}$, the 8-form T is a $\underline{45}$ and the self-duality equation should count for 84 conditions (so_{10} has two $\underline{210}$, only one of which is the 4-form we are interested in). This case can be investigated through complexification of the 10-dimensional indices (with $z_a \equiv x_a + ix_{a+5}$):

$$\begin{aligned} F_{\mu\nu\rho\sigma} &\rightarrow (F_{z_az_bz_cz_d},\ F_{z_az_bz_c\overline{z}_d},\ F_{z_az_bz_c\overline{z}_c},\ F_{z_az_b\overline{z}_c\overline{z}_d},\ F_{z_az_b\overline{z}_c\overline{z}_b},\ F_{z_az_b\overline{z}_a\overline{z}_b}), && (3.10)\\ \underline{210}^A &= (\underline{5}^+ + \underline{5}^-) + (\underline{40}^+ + \underline{40}^-) + (\underline{10}^+ + \underline{10}^-) + \underline{75} + \underline{24} + \underline{1}. && (3.11)\end{aligned}$$

An 84-dimensional sum of representations can be exhibited in various ways.

3.3 11-dimensional cases

Two interesting possibilities arise, for a Yang-Mills field as well as a 3-form, both involving the so-called principal su_2 of so_{11} as the invariance sub-algebra (there are many su_2 sub-algebras in so_D, but the only one that is maximal is the principal su_2 for odd D). Indeed, our first two requirements are fulfilled: the 1-form's curvature is a $\underline{55}$ and the 3-form's curvature a $\underline{330}$, while

$$\underline{55} = \underline{19} + \underline{15} + \underline{11} + (\underline{7} + \underline{3})^\star \qquad \text{for } p = 1, \tag{3.12}$$

$$\underline{330} = \underline{23} + \underline{21} + \underline{19} + 3\times\underline{17} + 3\times\underline{13} + 2\times\underline{11} + 3\times\underline{9} + \underline{7} + \underline{1}$$

$$+ (\underline{29} + \underline{25} + \underline{21} + 2\times\underline{15} + 3\times\underline{5})^\star \qquad \text{for } p = 3. \tag{3.13}$$

For the 1-form and 3-form, one needs respectively $N_{eq} = 10$ and 120 equations. There are several posibilities for the corresponding sum of representations, one of which we point out by $(\cdots)^\star$ in (3.12)–(3.13).

Notice that one can expect more than two eigenvalues for T in both cases (other su_2-invariant tensors appear in the decomposition of T^2). However, as to the $p = 3$ case, in view of the number of representations involved in the decomposition for $\underline{d_F}$, we do not know whether the spectrum of the eigenvalues can allow to project exactly on the complementary representation of N_{eq} in (3.13).

3.4 12-dimensional cases

In addition to the generic case (1.1) of a 5-form, $D = 12$ allows $p = 2$ and 3. For $p = 2$, the ($D = 8$, $p = 1$) pattern is reproduced: since it is a $D/2$-form, T can be decomposed into self and anti-self-dual parts ($\underline{462}^+ + \underline{462}^-$), which remain irreducible under so_{11}. Moreover, the 3-form curvature lies in the $\underline{220}$ of so_{12}, which, under so_{11}, automatically gives rise to the crucial $N_{eq} = 55$ dimensional representation:

$$\underline{220} \quad = \quad \underline{55} + \underline{165}. \tag{3.14}$$

Inspecting $\underline{d_T}$'s decompositions under so_{11}'s maximal sub-algebras selects $su_4{\times}sp_4 \sim so_6{\times}so_5$. The corresponding representations for the curvature F behave under $su_4{\times}sp_4$ as:

$$\underline{55} \quad = \quad (\underline{15},\underline{1}) + (\underline{6},\underline{5}) + (\underline{1},\underline{10}) \tag{3.15}$$

$$\underline{165} \quad = \quad (\underline{10}^+,\underline{1}) + (\underline{10}^-,\underline{1}) + (\underline{15},\underline{5}) + (\underline{6},\underline{10}) + (\underline{1},\underline{10}). \tag{3.16}$$

Unlike the ($p = 1$, $D = 8$) case, T is not the only invariant tensors to appear in T^2 so there can be more than two values for λ. We do not know whether their spectrum enables to get the wanted 55 equations from (1.2).

The remaining case $p = 3$ admits two possible invariance sub-algebras, namely su_6 (non-maximal) and $su_2{\times}sp_6$. The decomposition for $\underline{d_F} = \underline{d_T} = \underline{495}$ under su_6 is obtained by complexification of the indices and an $N_{eq} = 165$-dimensional representation can be extracted. As for the case of $p = 2$, other invariant tensors appear in the right hand side of (2.4).

3.5 13- and 14- dimensional cases

As for the 11-dimensional 1- and 3-forms, for $p = 3$ in 13 dimensions, it is the principal su_2 that arises. The representation $\underline{1287}$ for the 8-form T has two singlet components under su_2. This suggests that one has some freedom to arrange the eigenvalues, but we do not know whether F's projection onto the $N_{eq} = 220$-dimensional reducible representation can be cancelled.

The ($D = 14$, $p = 5$) su_7 possibility is another case of complexification of the indices and is easily worked out: T has only one singlet component and an $N_{eq} = 1287$-dimensional representation can be found in the decomposition for $\underline{d_F} = \underline{3003}$.

3.6 15-dimensional cases

$D = 15$ offers quite a few possibilities, for all possible form degree except the 2-form gauge field. The 6-form is of the aforementioned type $h \equiv so_{(D-1)}$.

For the case of a Yang-Mills field, F lies in the representation $\underline{105}$ of so_{15} and T in the $\underline{1365}$. The later gives two singlets in its decomposition under the maximal $su_2 \subset so_{15}$.

In the case of a 3-form gauge field, three maximal sub-algebras are suitable, namely, su_2, su_4 and $su_2 \times sp_4$. Here, $\underline{d_T} = \underline{6435}$ gives four singlets under su_2 against only one under the last two. The decompositions of $\underline{d_F} = \underline{1365}$ can be found in the appendix, where the $N_{eq} = 364$-dimensional representation is singled out. It involves an increasing number of low-dimensional representations of h, thus the fulfilling of *iv)* seems more and more difficult to us. However, for the case of su_2, T being a linear combination of four invariant tensors, there is more freedom for a relevant choice of the eigenvalues.

3.7 16-dimensional cases

In 16 dimensions, for $p = 3$, one has the case of $4(p + 1) = D$. It gives several possible sub-algebras $h \subset so_{15} \subset so_{16}$. Unlike the eight and twelve dimensional analogues, here T also contains a singlet in its decomposition under $sp_4 \times sp_4$, maximal within so_{16}. Finally, one has a 5-form with either su_8 (complexification of the indices), $sp_4 \times sp_4$ or $su_2 \times sp_8$. For all these cases, T has only one singlet component.

4 Conclusion

Recent developments have indicated that $TQFT$'s in various dimensions, *i.e.* theories with twisted supersymmetries, may play an important role. The case of the eight dimensional Yang-Mills $TQFT$ has indicated the relevance of a theory defined on a manifold with special holonomy group $H \subset SO_D$. The key to such $TQFT$'s is the existence of H-invariant "self-duality equations" (1.2) for the curvature of the gauge field: via $BRST$ quantization, they can serve as gauge fixing functions, provided they satisfy certain restrictions.

In this paper, we have pointed out restrictive conditions to be fulfilled by (1.2) to possibly admit instanton-like solutions and determine D-dimensional $TQFT$'s. We have not restricted ourselves to Yang-Mills fields, but have considered p-form gauge fields, in view of their importance in the definition of field theories for branes. Following the first steps exhibited in our analysis, we have established a table, listing for each p and $D \leq 16$, the possible subgroups $H \subset SO_D$ for which an SO_D-covariant and H-invariant self-duality equation can exist. We did not go any further in building the $TQFT$'s, but we believe that for some high dimensional cases, interesting and possibly unfamiliar theories could arise.

Acknowledgments:

We are grateful to R. Stora for interesting discussions related to this work. We also thank O. Babelon, E. Ragoucy, P. Roche, M. Talon for fruitful comments, and J. Patera for communication of unpublished results.

Appendix

Here, we give the h-decomposition of $\underline{d_F}$ for all the relevant cases for $D \geq 10$; one possibility for the N_{eq}-dimensional reducible representation is singled out with $(\cdots)^\star$.

$D=10$	$p=2,\ d_F=120,\ d_T=210,\ N_{eq}=36$
$su_2\times su_2\times su_4$	$\underline{120}=[(\underline{1},\underline{3},\underline{6})+(\underline{3},\underline{1},\underline{6})]^\star+(\underline{2},\underline{2},\underline{15})+(\underline{1},\underline{1},\underline{10}^+)+(\underline{1},\underline{1},\underline{10}^-)+(\underline{2},\underline{2},\underline{1})$
	$p=3,\ d_F=210,\ d_T=45,\ N_{eq}=84$
su_5	$\underline{210}=(\underline{40}^\pm+\underline{10}^++\underline{10}^-+\underline{24})^\star+\underline{75}+\underline{40}^\mp+\underline{5}^++\underline{5}^-+\underline{1}$
$D=11$	$p=1,\ d_F=55,\ d_T=330,\ N_{eq}=10$
su_2	$\underline{55}=(\underline{7}+\underline{3})^\star+\underline{19}+\underline{15}+\underline{11}$
	$p=3,\ d_F=330,\ d_T=165,\ N_{eq}=120$
su_2	$\underline{330}=(\underline{29}+\underline{25}+\underline{21}+2\times\underline{15}+3\times\underline{5})^\star+\underline{23}+\underline{21}+\underline{19}+3\times\underline{17}+3\times\underline{13}+2\times\underline{11}+3\times\underline{9}+\underline{7}+\underline{1}$
$D=12$	$p=2,\ d_F=220,\ d_T=462+462,\ N_{eq}=55$
so_{15}	$\underline{220}=\underline{55}^\star+\underline{165}$
$\supset su_4\times sp_4$	$\underline{55}=(\underline{15},\underline{1})+(\underline{6},\underline{5})+(\underline{1},\underline{10})$, $\quad \underline{165}=(\underline{10}^+,\underline{1})+(\underline{10}^-,\underline{1})+(\underline{15},\underline{5})+(\underline{6},\underline{10})+(\underline{1},\underline{10})$
	$p=3,\ d_F=495,\ d_T=495,\ N_{eq}=165$
su_6	$\underline{495}=(\underline{105}'^\pm+2\times\underline{15}^++2\times\underline{15}^-)^\star+\underline{189}+\underline{150}'^\mp+\underline{35}+\underline{1}$
$su_2\times sp_6$	$\underline{495}=[(\underline{1},\underline{90})+(\underline{5},\underline{14}')+(\underline{5},\underline{1})]^\star+(\underline{3},\underline{70})+(\underline{3},\underline{21})+(\underline{3},\underline{14}')+(\underline{1},\underline{14}')+(\underline{1},\underline{1})$
$D=13$	$p=3,\ d_F=715,\ d_T=1287,\ N_{eq}=220$
su_2	$\underline{715}^A=(\underline{37}+\underline{33}+\underline{31}+2\times\underline{29}+\underline{27}+\underline{25}+\underline{9})^\star+2\times\underline{25}+2\times\underline{23}+4\times\underline{21}+3\times\underline{19}+4\times\underline{17}$ $+3\times\underline{15}+5\times\underline{13}+2\times\underline{11}+3\times\underline{9}+2\times\underline{7}+3\times\underline{5}+2\times\underline{1}$
$D=14$	$p=5,\ d_F=3003,\ d_T=91,\ N_{eq}=1287$
su_7	$\underline{3003}=(\underline{490}^+_{(1)}+\underline{392}+\underline{224}^++\underline{112}^++\underline{48}+\underline{21}^+)^\star+\underline{784}+\underline{490}^-_{(1)}+\underline{224}^-+\underline{112}^-+(\underline{35}^++\underline{35}^-)+\underline{21}^-+\underline{7}^++\underline{7}^-+\underline{1}$
$D=15$	$p=1,\ d_F=105,\ d_T=1365,\ N_{eq}=14$
su_2	$\underline{105}=(\underline{11}+\underline{3})^\star+\underline{27}+\underline{23}+\underline{19}+\underline{15}+\underline{7}$
	$p=3,\ d_F=1365,\ d_T=6435,\ N_{eq}=364$
su_2	$\underline{1365}=(\underline{45}+\underline{41}+\underline{39}+2\times\underline{37}+\underline{35}+3\times\underline{33}+\underline{31})^\star+\underline{31}+4\times\underline{29}+3\times\underline{27}+5\times\underline{25}$ $+4\times\underline{23}+6\times\underline{21}+4\times\underline{19}+6\times\underline{17}+4\times\underline{15}+6\times\underline{13}+3\times\underline{11}+5\times\underline{9}+2\times\underline{7}+4\times\underline{5}+2\times\underline{1}$
su_4	$\underline{1365}=(\underline{105}+\underline{175}+\underline{84})^\star+(\underline{256}^++\underline{256}^-)+\underline{175}+\underline{84}+(\underline{35}^++\underline{35}^-)+(\underline{45}^++\underline{45}^-)+2\times\underline{20}+2\times\underline{15}$
$su_2\times sp_4$	$\underline{1365}=[(\underline{5},\underline{35}')+(\underline{3},\underline{35})+(\underline{5},\underline{14})+(\underline{1},\underline{14})]^\star+(\underline{3},\underline{81})+(\underline{7},\underline{35})+(\underline{5},\underline{35})+(\underline{1},\underline{35}')$ $+(\underline{7},\underline{10})+(\underline{9},\underline{5})+(\underline{3},\underline{14})+(\underline{5},\underline{10})+2\times(\underline{3},\underline{10})+(\underline{5},\underline{5})+(\underline{1},\underline{5})+(\underline{5},\underline{1})+(\underline{1},\underline{1})$

	$p=4,\ d_F=3003,\ d_T=3003,\ N_{eq}=1001$
su_4	$\underline{3003}=(\underline{300}+\underline{280}^{\pm}+\underline{105}+\underline{175}+\underline{45}^{+}+\underline{45}^{-}+\underline{20}+2\times\underline{15}+\underline{1})^{\star}$ $+\underline{280}^{\mp}+2\times\underline{256}^{+}+2\times\underline{256}^{-}+2\times\underline{175}+2\times\underline{84}+2\times\underline{45}^{+}+2\times\underline{45}^{-}$
	$p=5,\ d_F=5005,\ d_T=455,\ N_{eq}=2002$
su_2	$\underline{5005}=(\underline{55}+\underline{51}+\underline{49}+2\times\underline{47}+2\times\underline{45}+4\times\underline{43}+3\times\underline{41}+6\times\underline{39}+5\times\underline{37}+8\times\underline{35}+8\times\underline{33}+12\times\underline{31}+\underline{23}+\underline{7}+\underline{3})^{\star}+\underline{37}+10\times\underline{29}$ $+14\times\underline{27}+13\times\underline{25}+15\times\underline{23}+14\times\underline{21}+18\times\underline{19}+13\times\underline{17}+17\times\underline{15}+13\times\underline{13}+14\times\underline{11}+9\times\underline{9}+11\times\underline{7}+4\times\underline{5}+5\times\underline{3}$
su_4	$\underline{5005}=(2\times\underline{300}+\underline{256}^{+}+\underline{256}^{-}+5\times\underline{175}+\underline{15})^{\star}+\underline{729}+2\times(\underline{280}^{+}+\underline{280}^{-})$ $+(\underline{256}^{+}+\underline{256}^{-})+(\underline{35}^{+}+\underline{35}^{-})+3\times\underline{84}+3\times(\underline{45}^{+}+\underline{45}^{-})+\underline{20}+2\times\underline{15}$
$D=16$	$p=3,\ d_F=1820,\ d_T=6435+6435,\ N_{eq}=455$
$sp_4\times sp_4$	$\underline{1820}=[(\underline{35},\underline{10})+(\underline{10},\underline{10})+(\underline{5},\underline{1})]^{\star}+(\underline{14},\underline{14})+(\underline{10},\underline{35})+(\underline{35},\underline{5})+(\underline{5},\underline{35})+(\underline{14},\underline{5})$ $+(\underline{5},\underline{14})+(\underline{35}',\underline{1})+(\underline{1},\underline{35}')+(\underline{10},\underline{10})+(\underline{10},\underline{5})+(\underline{5},\underline{10})+(\underline{14},\underline{1})+(\underline{5},\underline{5})+(\underline{1},\underline{14})+(\underline{1},\underline{5})+(\underline{1},\underline{1})$
so_{15}	$\underline{1820}=\underline{1365}+\underline{455}$
$\supset su_2$	$\underline{455}=\underline{37}+\underline{33}+\underline{31}+\underline{29}+\underline{27}+2\times\underline{25}+\underline{23}+2\times\underline{21}+2\times\underline{19}+2\times\underline{17}+2\times\underline{15}+3\times\underline{13}+\underline{11}+2\times\underline{9}+\underline{7}+\underline{5}+\underline{1}$ $\underline{1365}=\underline{45}+\underline{41}+\underline{39}+2\times\underline{37}+\underline{35}+3\times\underline{33}+2\times\underline{31}+4\times\underline{29}+3\times\underline{27}+5\times\underline{25}+4\times\underline{23}+6\times\underline{21}+4\times\underline{19}$ $+6\times\underline{17}+4\times\underline{15}+6\times\underline{13}+3\times\underline{11}+5\times\underline{9}+2\times\underline{7}+4\times\underline{5}+2\times\underline{1}$
$\supset su_4$	$\underline{455}=\underline{175}+\underline{84}+\underline{35}^{+}+\underline{35}^{-}+\underline{45}^{+}+\underline{45}^{-}+\underline{20}+\underline{15}+\underline{1}$ $\underline{1365}=\underline{256}^{+}+\underline{256}^{-}+\underline{105}+2\times\underline{175}+2\times\underline{84}+\underline{35}^{+}+\underline{35}^{-}+\underline{45}^{+}+\underline{45}^{-}+2\times\underline{20}+2\times\underline{15}$
$\supset su_2\times sp_4$	$\underline{455}=(\underline{5},\underline{35})+(\underline{1},\underline{30})+(\underline{3},\underline{35})+(\underline{7},\underline{10})+(\underline{3},\underline{10})+(\underline{5},\underline{5})+(\underline{3},\underline{5})+(\underline{1},\underline{5})$ $\underline{1365}=(\underline{3},\underline{81})+(\underline{5},\underline{35}')+(\underline{5},\underline{35})+(\underline{7},\underline{35})+(\underline{1},\underline{35}')+(\underline{3},\underline{35})+(\underline{5},\underline{14})+(\underline{7},\underline{10})$ $+(\underline{9},\underline{5})+(\underline{3},\underline{14})+(\underline{5},\underline{10})+(\underline{1},\underline{14})+2\times(\underline{3},\underline{10})+(\underline{5},\underline{5})+(\underline{1},\underline{5})+(\underline{5},\underline{1})+(\underline{1},\underline{1})$
$\supset so_7\times so_8$	$\underline{455}=(\underline{7},\underline{28})+(\underline{21},\underline{8}^{V})+(\underline{1},\underline{56}^{C})+(\underline{35},\underline{1})$ $\underline{1365}=(\underline{21},\underline{28})+(\underline{7},\underline{56}^{C})+(\underline{35},\underline{8}^{V})+(\underline{1},\underline{35}^{S})+(\underline{1},\underline{35}^{C})+(\underline{35},\underline{1})$
	$p=5,\ d_F=8008,\ d_T=1820,\ N_{eq}=3003$
su_8	$\underline{8008}=(\underline{2352}+\underline{420}^{\pm}+\underline{70}+\underline{70}+\underline{63}+\underline{28}^{\pm})^{\star}+(\underline{1512}^{+}+\underline{1512}^{-})$ $+\underline{720}+(\underline{378}^{+}+\underline{378}^{-})+\underline{420}^{\mp}+\underline{28}^{\mp}+(\underline{28}^{+}+\underline{28}^{-})+\underline{1}$
$sp_4\times sp_4$	$\underline{8008}=[(\underline{35},\underline{35})+(\underline{81},\underline{5})+(\underline{10},\underline{35}')+(\underline{5},\underline{81})+(\underline{14},\underline{14})+(\underline{81},\underline{1})+(\underline{1},\underline{81})+(\underline{14},\underline{10})+(\underline{10},\underline{10})+2\times(\underline{10},\underline{1})]^{\star}$ $+(\underline{35}',\underline{10})+(\underline{30},\underline{10})+(\underline{10},\underline{30})+(\underline{35},\underline{14})+(\underline{14},\underline{35})+2\times(\underline{35},\underline{10})+2\times(\underline{10},\underline{35})+2\times(\underline{35},\underline{5})+2\times(\underline{5},\underline{35})$ $+(\underline{14},\underline{10})+2\times(\underline{10},\underline{14})+(\underline{5},\underline{14})+(\underline{14},\underline{5})+(\underline{35},\underline{1})+(\underline{1},\underline{35})+3\times(\underline{10},\underline{5})+3\times(\underline{5},\underline{10})+(\underline{5},\underline{5})+2\times(\underline{1},\underline{10})$
$su_2\times sp_8$	$\underline{8008}=[(\underline{5},\underline{315})+(\underline{3},\underline{315})+(\underline{5},\underline{42})+(\underline{7},\underline{27})+(\underline{3},\underline{27})+(\underline{3},\underline{1})]^{\star}+(\underline{1},\underline{825})+(\underline{3},\underline{792}^{(1)})$ $(\underline{3},\underline{308})+(\underline{1},\underline{315})+(\underline{5},\underline{36})+(\underline{3},\underline{42})+(\underline{5},\underline{27})+(\underline{1},\underline{36})+(\underline{3},\underline{27})+(\underline{7},\underline{1})$

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