First unitarity-independent determination of the CKM matrix elements $\mathbf{V_{td}}$, $\mathbf{V_{ts}}$, and $\mathbf{V_{tb}}$ and the implications for unitarity

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The magnitudes of the CKM matrix elements V_{td}, V_{ts}, and V_{tb} are determined for the first time without any assumptions of unitarity. The implications for the unitarity of the CKM matrix as a whole are discussed.

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I. INTRODUCTION

The relationship between weak and mass eigenstates of quarks, assuming there are three generations, is described by the Cabibbo-Kobayashi-Maskawa (CKM) matrix [1]

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} . \tag{1}$$

Unless the CKM matrix is *assumed* to be unitary the bottom row elements, V_{td}, V_{ts}, and V_{tb}, are undetermined. There are, however, a number of measured quantities which are sensitive to combinations of these elements.

We use our recent determination of $|V_{tb}|$ from electroweak corrections to Z decays [2] to "unlock" the bottom row of the CKM matrix. From a combined fit to data from the LEP, SLC, CESR, Tevatron, and other experiments we determine all the CKM elements, independent of unitarity assumptions, and proceed to test unitarity of the full CKM matrix for the first time.

II. CONSTRAINTS ON THE CKM ELEMENTS

We use experimental results and the corresponding theoretical predictions to construct a chisquare which is then minimised to obtain all the CKM matrix elements. The following sections describe individual chisquare constraints which are ultimately summed to form the global chisquare.

A. Constraint on $|V_{tb}|$

In a previous paper [2] we described a new method for the determination of $|V_{tb}|$ from electroweak loop corrections, in particular to the process $Z \to b\bar{b}$. We define a chisquare $\chi^2_{tb}(m_Z, m_t, m_H, \alpha_s, \alpha, V_{tb})$ for the agreement of the theory with nineteen experimental measurements. Full details of the input parameter values, errors, and correlations are given in Ref. [2].

B. Constraint on $|V^*_{tb}V_{td}|$

The product $|V^*_{tb}V_{td}|$ is constrained by measurements of $B^0_d - \bar{B}^0_d$ mixing. Therefore, using these measurements and our constraint on $|V_{tb}|$ enables us to extract $|V_{td}|$. The mixing is expressed in terms of the mass difference Δm_d of the CP eigenstates [3]

$$\Delta m_d = \frac{G_F^2}{6\pi^2} |V^*_{tb}V_{td}|^2 m_W^2 m_{B_d} \left(B_{B_d} f^2_{B_d}\right) \eta_B S_0\left(x_t\right), \tag{2}$$

where $\sqrt{B_{B_d}} f_{B_d} = (201 \pm 42)\,\mathrm{MeV}$ [4] comes from a combined analysis of lattice calculations, which are compatible with predictions of QCD sum rules, and the QCD factor is $\eta_B = 0.55 \pm 0.01$ [5]. The function $S_0(x_t = \overline{m}_t^2/m_W^2)$ is given by [3]

$$S_0(x_t) = \frac{x_t}{4}\left[1 + \frac{3-9x_t}{(x_t-1)^2} + \frac{6x_t^2 \ln x_t}{(x_t-1)^3}\right], \tag{3}$$

where $\overline{m}_t$ is the running top mass [5] as required for consistency with η_B. It is related to the pole mass m_t as measured by the Tevatron, according to

$$\overline{m}_t = m_t\left(1 - \frac{4}{3}\frac{\alpha_s(m_t)}{\pi}\right) \approx m_t - 8\,\mathrm{GeV}. \tag{4}$$

Given the measurement of $\Delta m_d = 0.472 \pm 0.018\,\mathrm{ps}^{-1}$ [6] we define the chisquare constraint on $|V^*_{tb}V_{td}|$

$$\chi^2_{td*tb} = \frac{\left(\Delta m_d(m_t, m_W) - 0.472\right)^2}{\left(0.418\Delta m_d\left(m_t, m_W\right)\right)^2 + (0.018)^2} \tag{5}$$

where the theoretical prediction in the numerator depends on m_t and m_W. We vary m_t explicitly in the fit; m_W is varied implicitly since it is a function of the other electroweak parameters which vary explicitly [2]. The two terms in the denominator correspond respectively to the theoretical and experimental uncertainties, the former being dominated by the 20% relative error on $\sqrt{B_{B_d}} f_{B_d}$.

C. Constraint on $|V_{ts}^* V_{tb}|/|V_{cb}|$

The rate for the process $b \to s\gamma$, which involves a loop with a virtual top quark, is proportional to $|V_{ts}^* V_{tb}|^2$. Our constraint on $|V_{tb}|$ therefore enables us to determine $|V_{ts}|$. The theoretical prediction in the Standard Model has been calculated to be [7]

$$\mathrm{BR}(b \to s\gamma)_{\mathrm{th.}} = (3.28 \pm 0.33) \times 10^{-4} \times \left[\frac{|V_{ts}^* V_{tb}|/|V_{cb}|}{0.976}\right]^2. \tag{6}$$

The element V_{cb} enters through the normalisation to the measurement of $\mathrm{BR}(B \to X_c e \bar{\nu}_e) = (10.4 \pm 0.4)\%$ [8] which reduces the otherwise large error due to the b-quark mass uncertainty. CLEO has measured $\mathrm{BR}(b \to s\gamma) = (2.32 \pm 0.57 \pm 0.35) \times 10^{-4}$ [9], and ALEPH has presented a preliminary measurement of $\mathrm{BR}(b \to s\gamma) = (3.29 \pm 0.71 \pm 0.68) \times 10^{-4}$ [10]. We average these two results, assuming a common uncertainty of 0.3×10^{-4} for the theoretical modelling, to obtain $\mathrm{BR}(b \to s\gamma)_{\mathrm{ex.}} = (2.60 \pm 0.59) \times 10^{-4}$. The chisquare constraint on $|V_{ts}^* V_{tb}|/|V_{cb}|$ is then

$$\chi^2_{ts*tb/cb} = \frac{(\mathrm{BR}(b \to s\gamma)_{\mathrm{th.}} - \mathrm{BR}(b \to s\gamma)_{\mathrm{ex.}})^2}{\sigma^2_{\mathrm{th.}} + \sigma^2_{\mathrm{ex.}}}, \tag{7}$$

where $\sigma_{\mathrm{th.}}$ and $\sigma_{\mathrm{ex.}}$ are the theoretical and experimental errors on $\mathrm{BR}(b \to s\gamma)$ respectively. We neglect the explicit dependence on m_t and α_{s} but we do include their 3% contribution to the theoretical uncertainty of 10%.

D. Constraints on other CKM elements

The most precise constraints on the element $|V_{ud}|$ are from nuclear beta decay. We use the recent measurement of $|V_{ud}|$ from the Chalk River Laboratories [11] to define the chisquare constraining $|V_{ud}|$

$$\chi^2_{ud} = (|V_{ud}| - 0.9740)^2/(0.0005)^2. \tag{8}$$

We use combined results from measurements of K_{e3} and hyperon decays [8] to construct the $|V_{us}|$ chisquare

$$\chi^2_{us} = (|V_{us}| - 0.2205)^2/(0.0018)^2. \tag{9}$$

We use the results of experiments measuring the production of charm from (anti-)neutrino interactions with d valence quarks [8] to construct the $|V_{cd}|$ chisquare

$$\chi^2_{cd} = (|V_{cd}| - 0.224)^2/(0.016)^2. \tag{10}$$

Measurements of the D_{e3} decay $D \to \bar{K} e^+ \bar{\nu}$ [8] are used to construct the $|V_{cs}|$ chisquare

$$\chi^2_{cs} = (|V_{cs}| - 1.01)^2/(0.18)^2, \tag{11}$$

where the error is dominated by uncertainties in the hadronic form factors.

The existence of $b \to u$ transitions, which depend on $|V_{ub}|$, is well established from measurements of the end-point of the charged lepton energy spectrum in $b \to X \ell^- \bar{\nu}$ decays. Such analyses yield the ratio $|V_{ub}|/|V_{cb}|$ [8] from which we derive the chisquare

$$\chi^2_{ub/cb} = (|V_{ub}|/|V_{cb}| - 0.08)^2/(0.02)^2. \tag{12}$$

Recently CLEO reported the first measurements of $\mathrm{BR}\left(B^0 \to \pi^- \ell^+ \nu\right) = (1.8 \pm 0.4 \pm 0.3 \pm 0.2) \times 10^{-4}$ and $\mathrm{BR}\left(B^0 \to \rho^- \ell^+ \nu\right) = \left(2.5 \pm 0.4^{+0.5}_{-0.7} \pm 0.5\right) \times 10^{-4}$ where the errors correspond respectively to the statistical, systematic, and model-dependent uncertainties [12]. These results yield the chisquare

$$\chi^2_{ub} = (|V_{ub}| - 0.0033)^2/(0.0008)^2. \tag{13}$$

The element $|V_{cb}|$ is determined both from the average b-lifetime and from $B \to D^* \ell \nu$ decays in the limit of zero recoil. The resulting chisquare for $|V_{cb}|$ is [5]

$$\chi^2_{cb} = (|V_{cb}| - 0.040)^2/(0.003)^2. \tag{14}$$

III. UNITARITY-FREE FIT

We fit for all the CKM matrix elements without assuming unitarity of the CKM matrix by minimising the chisquare, defined as

$$\begin{aligned}\chi^2 = {} & \chi^2_{ud}(V_{ud}) + \chi^2_{us}(V_{us}) + \chi^2_{cd}(V_{cd}) + \chi^2_{cs}(V_{cs}) \\ & + \chi^2_{ub/cb}(V_{ub}, V_{cb}) + \chi^2_{ub}(V_{ub}) + \chi^2_{cb}(V_{cb}) \\ & + \chi^2_{td*tb}(V_{tb}, V_{td}, m_t, m_W) \\ & + \chi^2_{ts*tb/cb}(V_{ts}, V_{tb}, V_{cb}) \\ & + \chi^2_{tb}(m_Z, m_t, m_H, \alpha_{\mathrm{s}}, \alpha, V_{tb}),\end{aligned} \tag{15}$$

where for completeness we have included all the dependences of the individual chisquare definitions. The 14 parameters which vary in the fit are m_Z, m_t, $\log_{10}(m_H)$, $\alpha_{\mathrm{s}}(m_Z)$, α, and the magnitudes of the nine CKM elements. The results of the fit are shown in the second column of table I, where the errors include the effects of experimental and theoretical uncertainties and all correlations. The chisquare per degree of freedom for the unitarity-free fit is $15.1/(28-14) = 15.1/14$. The probability to obtain a value greater than this is 37% indicating that the data are in good agreement with the underlying model.

The bottom row CKM elements, $|V_{td}|$, $|V_{ts}|$, and $|V_{tb}|$ are determined for the first time, independent of any unitarity assumptions.

IV. UNITARITY-CONSTRAINED FIT

We also fit the data imposing CKM unitarity as a constraint. The four parameters which define a three dimensional unitary matrix can be uniquely determined from the moduli of the elements. We choose the parametrisation of the CKM matrix advocated by the Particle Data Group [8] which enforces exact unitarity

$$V_{ud} = c_{12}c_{13} \tag{16}$$
$$V_{us} = s_{12}c_{13} \tag{17}$$
$$V_{ub} = s_{13}e^{-i\delta_{13}} \tag{18}$$
$$V_{cd} = -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} \tag{19}$$
$$V_{cs} = c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} \tag{20}$$
$$V_{cb} = s_{23}c_{13} \tag{21}$$
$$V_{td} = s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} \tag{22}$$
$$V_{ts} = -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} \tag{23}$$
$$V_{tb} = c_{23}c_{13} \tag{24}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ for three angles, θ_{12}, θ_{13}, and θ_{23}, which lie in the first quadrant. The phase parameter δ_{13} lies in the range $0 < \delta_{13} < 2\pi$. The 9 parameters which vary in the unitarity-constrained fit are m_Z, m_t, $\log_{10}(m_H)$, $\alpha_s(m_Z)$, α, s_{12}, s_{13}, s_{23}, and $\cos\delta_{13}$. The results of the fit are shown in table I together with the corresponding values of the CKM elements.

The changes in m_t, m_H, and α_s, compared to the unitarity-free fit, are due to their correlation with $|V_{tb}|$, as discussed in Ref. [2]. Since only terms of the form $\cos\delta_{13}$ appear in this analysis, there is a two-fold ambiguity between the regions $0 < \delta_{13} < \pi$, as favoured by CP non-conservation in the kaon system, and $\pi < \delta_{13} < 2\pi$. Unfortunately, the fitted value of $\cos\delta_{13} = -0.26^{+0.82}_{-0.74}$ has rather large errors; in fact, the lower error corresponds to the physical limit of $\cos\delta_{13} = -1$.

The chisquare per degree of freedom for the unitarity-constrained fit is $25.5/(28-9) = 25.5/19$. The probability to obtain a value larger than this is 14% which indicates reasonable consistency of the data and the model, albeit with less probability than the unitarity-free fit.

V. TESTS OF CKM UNITARITY

Unitarity implies $VV^\dagger = V^\dagger V = U$, where U is the unit matrix, and hence the normality conditions

$$\rho_i \equiv |V_{i1}|^2 + |V_{i2}|^2 + |V_{i3}|^2 = 1; \quad \text{and} \tag{25}$$
$$\kappa_j \equiv |V_{1j}|^2 + |V_{2j}|^2 + |V_{3j}|^2 = 1; \tag{26}$$

for $i = 1, 2, 3$ rows and $j = 1, 2, 3$ columns. From only the magnitudes of the nine CKM elements it is not possible to test orthogonality of pairs of different rows or pairs of different columns. Allowing for correlations, the results of the six normality tests are

$$\rho_1 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.997 \pm 0.001; \tag{27}$$
$$\rho_2 \equiv |V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 1.105 \pm 0.367; \tag{28}$$
$$\rho_3 \equiv |V_{td}|^2 + |V_{ts}|^2 + |V_{tb}|^2 = 0.623 \pm 0.315; \tag{29}$$
$$\kappa_1 \equiv |V_{ud}|^2 + |V_{cd}|^2 + |V_{td}|^2 = 0.999 \pm 0.007; \tag{30}$$
$$\kappa_2 \equiv |V_{us}|^2 + |V_{cs}|^2 + |V_{ts}|^2 = 1.104 \pm 0.367; \tag{31}$$
$$\kappa_3 \equiv |V_{ub}|^2 + |V_{cb}|^2 + |V_{tb}|^2 = 0.629 \pm 0.316. \tag{32}$$

The respective similarities between the three row constraints and the three column constraints reflects the relatively small values of the off-diagonal elements. Only ρ_1 is significantly different from unity, being 2.1 standard deviations low. Given that there are six, albeit somewhat correlated, conditions we do not attach great significance to this although its future evolution is of interest.

The trend is for the ρ_i and κ_j to be somewhat lower than the expectations of a unitary matrix, as would be the case if there were more than three generations. To test this possibility with a single number we define the quantity

$$\Delta_{\rm CKM} = \sqrt{(\rho_1\rho_2\rho_3)(\kappa_1\kappa_2\kappa_3)} \tag{33}$$

which should be unity for a unitary matrix. We find $\Delta_{\rm CKM} = 0.69 \pm 0.43$ where all correlations have been taken into account. This is lower than unity but only by 0.7 standard deviations, indicating that there is no compelling evidence for a deviation from unitarity.

These results constitute the first tests of unitarity of the complete CKM matrix. In particular, ρ_3, κ_1, κ_2, κ_3, and $\Delta_{\rm CKM}$ are measured for the first time.

VI. OUTLOOK

The error on $|V_{td}|$ is limited by the large theoretical uncertainty on $(B_{B_d}f^2_{B_d})$, which is improving only slowly with time. In future $|V_{td}|$ will also be determined from the process $K^+ \to \pi^+\nu\bar{\nu}$, which is sensitive to the product $|V^*_{td}V_{ts}|$. Recently, the E787 experiment observed one candidate event with an estimated background of (0.08 ± 0.03) events, from which they determined $\mathrm{BR}(K^+ \to \pi^+\nu\ \bar{\nu}) = (4.2^{+9.7}_{-3.5}) \times 10^{-10}$ [13]. From this they derive, assuming unitarity, the constraint $0.006 < |V_{td}| < 0.060$, which is in agreement with our more precise unitarity-independent result. Ultimately, the measurement of $\mathrm{BR}(K^+ \to \pi^+\nu\ \bar{\nu})$ will yield precise and theoretically well understood constraints on $|V^*_{td}V_{ts}|$.

The uncertainty on $|V_{ts}|$ from the process $b \to s\gamma$ is statistics limited, so some improvement can be expected from CLEO and the B factory experiments. The parameter Δm_s describing $B_s - \bar{B}_s$ mixing is also sensitive to $|V_{ts}|$. The latest experimental limit is $\Delta m_s > 10.2\,\mathrm{ps}^{-1}$ at the 95% C.L. [6]. If Δm_s were to be measured, its interpretation in terms of $|V_{ts}|$ would suffer from the large errors on $(B_{B_s} f_{B_s}^2)$. The ratio $\Delta m_d/\Delta m_s$ is, however, considerably less sensitive to theoretical uncertainties and its measurement would yield an important constraint on the ratio $|V_{td}/V_{ts}|^2$. Similarly, a future measurement of the ratio $\mathrm{BR}(B \to (\rho/\omega)\gamma)/\mathrm{BR}(B \to K^*\gamma)$ would constrain $|V_{td}/V_{ts}|^2$.

Given that LEP has finished running on the Z, the uncertainty on $|V_{tb}|$ from our method of fitting electroweak data is unlikely to go below approximately 20% [2]. The single top quark production rate at hadron colliders is, however, also sensitive to $|V_{tb}|$. The estimated uncertainty at the end of the Tevatron Run II is $\delta|V_{tb}|/|V_{tb}| = 12\% - 19\%$, where the range reflects the uncertainty of the gluon structure functions [14].

VII. SUMMARY

The elements $|V_{td}|$, $|V_{ts}|$, and $|V_{tb}|$, and hence the full CKM matrix, are determined for the first time independent of any unitarity assumptions:

$$|V_{td}| = 0.0113^{+0.0060}_{-0.0029}; \tag{34}$$

$$|V_{ts}| = 0.045^{+0.022}_{-0.010}; \tag{35}$$

$$|V_{tb}| = 0.77^{+0.18}_{-0.24}. \tag{36}$$

We test unitarity and determine that all six CKM normality conditions are consistent with unitarity. Four of these tests are performed for the first time.

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[1] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963), M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

[2] J. Swain, and L. Taylor, **hep-ph/9712420** (unpublished).

[3] J. Rosner, in *B Decays (revised 2nd edition)*, edited by S. Stone, World Scientific, Singapore, (1994).

[4] J.M. Flynn and C.T. Sachrajda, **hep-lat/9710057** (1997). To appear in "Heavy Flavours II", World Scientific, Singapore, Eds. A.J. Buras and M. Lindner.

[5] A.J. Buras and R. Fleischer, **hep-ph/9704376** (1997). To appear in "Heavy Flavours II", World Scientific, Singapore, Eds. A.J. Buras and M. Lindner.

[6] M. Jimack, Invited talk at the *International Europhysics Conference on High Energy Physics*, Jerusalem, 1997, (unpublished).

[7] K. Chetyrkin, M. Misiak and M. Münz, Phys. Lett. **B400**, 206 (1997).

[8] R. M. Barnett *et al.*, Phys. Rev. **D54**, 1 (1996).

[9] M.S. Alam *et al.*, Phys. Rev. Lett. **74**, 2885 (1995).

[10] P. Kluit, Invited talk at the *International Europhysics Conference on High Energy Physics*, Jerusalem, 1997, (unpublished).

[11] E. Hagberg, J.C. Hardy, V.T. Koslowsky, G. Savard, and I.S. Towner, Invited talk at the *NNDF 96*, Osaka, 2-5 Sep, (1996), **nucl-ex/9609002** (unpublished).

[12] J.P. Alexander *et al.*, CLEO CONF **96-16** (unpublished); we do not use the updated result [CLEO CONF **97-30** (unpublished)] of $\mathrm{BR}\left(B^0 \to \rho^- \ell^+ \nu\right) = (2.0 \pm 0.3 \pm 0.3) \times 10^{-4}$ since the uncertainty from the model dependence has yet to be evaluated.

[13] S. Adler, *et al.*, Phys. Rev. Lett. **79**, 2204 (1997).

[14] A.P. Heinson, A.S. Belyaev and E.E. Boos, Phys. Rev. **D56**, 3114 (1997).

TABLE I. Results of the unitarity-free and unitarity-constrained fits described in the text. Values marked by a dagger (†) are not explicitly free in the fit but are derived from the other parameters in the same column.

	Free fit	Constrained fit
m_Z (GeV)	91.187 ± 0.002	91.187 ± 0.002
m_t (GeV)	174.2 ± 5.7	172.4 ± 5.3
$\log_{10}(m_H/\mathrm{GeV})$	$2.15^{+0.30}_{-0.38}$	$2.03^{+0.30}_{-0.37}$
$\alpha_s(m_Z)$	0.1171 ± 0.0025	0.1188 ± 0.0021
$\alpha^{-1}(m_Z)$	128.913 ± 0.092	128.904 ± 0.092
$\|V_{ud}\|$	0.9740 ± 0.0005	0.9748 ± 0.0003 †
$\|V_{us}\|$	0.2205 ± 0.0018	0.2230 ± 0.0014 †
$\|V_{ub}\|$	0.00325 ± 0.00058	0.0032 ± 0.0006 †
$\|V_{cd}\|$	0.224 ± 0.016	0.2228 ± 0.0014 †
$\|V_{cs}\|$	1.01 ± 0.18	0.9741 ± 0.0003 †
$\|V_{cb}\|$	0.0401 ± 0.0029	0.0397 ± 0.0029 †
$\|V_{td}\|$	$0.0113^{+0.0060}_{-0.0029}$	0.0093 ± 0.0018 †
$\|V_{ts}\|$	$0.045^{+0.022}_{-0.010}$	0.0387 ± 0.0029 †
$\|V_{tb}\|$	$0.77^{+0.18}_{-0.24}$	0.9992 ± 0.0001 †
s_{12}	–	0.2230 ± 0.0014
s_{13}	–	0.0032 ± 0.0006
s_{23}	–	0.0396 ± 0.0030
$\cos\delta_{13}$	–	$-0.26^{+0.82}_{-0.74}$