CERN-TH/97-372
hep-ph/9712418

A simple method for multi-leg loop calculations 2: a general algorithm

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Abstract

The method introduced in a previous paper to simplify the tensorial reduction in multi-leg loop calculations is extended to generic one-loop integrals, with arbitrary internal masses and external momenta.

CERN-TH/97-372
December 1997

1 Introduction

In a previous paper [1], a technique was presented to simplify the tensorial reduction of m-point one-loop diagrams of the type

$$\mathcal{M}(p_1,\cdots,p_r;k_1,\cdots,k_{m-1}) = \sum_a \int d^n q \frac{\mathrm{Tr}^{(a)}\left[\not{q}\cdots\not{q}\cdots\right]}{D_1\cdots D_m}, \qquad (1)$$

where $p_{1\cdots r}$ are the external momenta of the diagram, $k_{1\cdots m-1}$ the momenta in the loop denominators, defined as

$$D_i = (q+s_{i-1})^2 - m_i^2, \qquad s_i = \sum_{j=0}^{i} k_j \qquad (k_0=0), \qquad (2)$$

and $\mathrm{Tr}^{(a)}$ traces over γ matrices, which may contain an arbitrary number of $\not{q}$'s.

It was shown that, by assuming at least two massless momenta in the set $k_{1\cdots m-1}$, the traces in eq. (1) can be rewritten in terms of the denominators appearing in the diagram, therefore simplifying the calculation.

Starting from m-point rank-l tensor integrals, the algorithm gave at most rank-1 m-point functions, plus n-point rank-p tensor integrals with $n < m$ and $p < l$.

In this paper, I show how to extend this technique when the momenta $k_{1\cdots m-1}$ are generic. On the one hand, this allows to apply the method to more general problems. On the other hand, the reduction procedure can therefore be iterated in such a way that, usually, only rank-1 integrals and scalar functions remain at the end.

In the next section, I introduce the algorithm and in section 3, I apply it to a specific example.

2 The general algorithm

The basic idea is simple. Given two vectors ℓ_1 and ℓ_2, one can 'extract' the q dependence from the traces with the help of the identity

$$\not{q} = \frac{1}{2\,(\ell_1\cdot\ell_2)}\left[2\,(q\cdot\ell_2)\,\not{\ell}_1 + 2\,(q\cdot\ell_1)\,\not{\ell}_2 - \not{\ell}_1\not{q}\not{\ell}_2 - \not{\ell}_2\not{q}\not{\ell}_1\right]. \qquad (3)$$

By further assuming $\ell_1^2 = \ell_2^2 = 0$, and making use of the completeness relations for massless spinors, the following result is obtained

$$\begin{aligned} \mathrm{Tr}[\not{q}\,\Gamma] &= \frac{1}{2\,(\ell_1\cdot\ell_2)}\,[\;2\,(q\cdot\ell_2)\,\mathrm{Tr}[\not{\ell}_1\Gamma] \\ &- \{q\}^{+-}_{1\;2}\,\{\Gamma\}^{+-}_{2\;1} - \{q\}^{-+}_{1\;2}\,\{\Gamma\}^{-+}_{2\;1} + (\ell_1 \leftrightarrow \ell_2)\;\Big]\,, \end{aligned} \tag{4}$$

where Γ represents a generic string of γ matrices and

$$\{\ell_1\,\ell_2\,\cdots\,\ell_n\}^{+-}_{i\;j} \;\equiv\; \{12\,\cdots\,n\}^{+-}_{i\;j} \;\equiv\; \bar{v}_+(\ell_i)\,\not{\ell}_1\,\not{\ell}_2\cdots\,\not{\ell}_n\,u_-(\ell_j)\,. \tag{5}$$

By iteratively applying the above procedure, together with the equations [1]

$$\begin{aligned} \{q\}^{-+}_{1\;2}\,\{q\}^{-+}_{2\;1} &= 4\,(q\cdot\ell_1)\,(q\cdot\ell_2) - 2\,q^2\,(\ell_1\cdot\ell_2) \\ \{q\}^{-+}_{1\;2}\,\{q\}^{-+}_{1\;2} &= \frac{2}{\{b\}^{+-}_{1\;2}}\Big[[q^2(\ell_1\cdot\ell_2) - 2(q\cdot\ell_1)(q\cdot\ell_2)]\,\{b\}^{-+}_{1\;2} \\ &+\; 2\,[(q\cdot\ell_1)(b\cdot\ell_2) - (q\cdot b)(\ell_1\cdot\ell_2) + (q\cdot\ell_2)(\ell_1\cdot b)]\,\{q\}^{-+}_{1\;2}\Big]\,, \end{aligned} \tag{6}$$

only one $\{q\}^{-+}_{1\;2}$ (or its complex conjugate $\{q\}^{-+}_{2\;1}$) survives in each term, and powers of q^2, $(q\cdot\ell_1)$, $(q\cdot\ell_2)$ and $(q\cdot b)$ factorize out.

The next step is to reconstruct the denominators from the above scalar products. By choosing, for example, $b = k_3$ one trivially gets

$$\begin{aligned} q^2 &= D_1 + m_1^2\,, \\ 2(q\cdot b) &= D_4 - D_3 + m_4^2 - m_3^2 - (k_1+k_2+k_3)^2 + (k_1+k_2)^2\,, \end{aligned} \tag{7}$$

but $(q\cdot\ell_1)$ and $(q\cdot\ell_2)$ still remain.

In ref. [1] the simple case was studied in which the diagram in eq. (1) is such that at least two k's (say k_1 and k_2) are massless. A solution to the problem is then to take $\ell_1 = k_1$ and $\ell_2 = k_2$:

$$\begin{aligned} 2(q\cdot\ell_1) &= D_2 - D_1 + m_2^2 - m_1^2\,, \\ 2(q\cdot\ell_2) &= D_3 - D_2 + m_3^2 - m_2^2 - (k_1+k_2)^2\,. \end{aligned} \tag{8}$$

If, in the set $k_{1\cdots m-1}$, only one momentum (say $k_1 \equiv \ell_1$) is massless, a solution can still be found by decomposing any other massive momentum (say k_2) in terms of massless vectors:

$$k_2 \;=\; \ell_2 + \alpha\,\ell_1\,. \tag{9}$$

The requirement that also ℓ_2 is massless, implies

$$\alpha = \frac{k_2^2}{2(k_1 \cdot k_2)}\,, \tag{10}$$

and therefore

$$\begin{aligned} 2(q \cdot \ell_1) &= D_2 - D_1 + m_2^2 - m_1^2\,, \\ 2(q \cdot \ell_2) &= D_3 - (1+\alpha)(D_2 + m_2^2) + \alpha(D_1 + m_1^2) + m_3^2 - (k_1 + k_2)^2\,. \end{aligned} \tag{11}$$

When there are no massless k's, a basis of massless vectors can yet be constructed:

$$k_1 = \ell_1 + \alpha_1 \ell_2\,, \quad k_2 = \ell_2 + \alpha_2 \ell_1\,. \tag{12}$$

In fact, requiring $\ell_1^2 = \ell_2^2 = 0$ gives

$$\begin{aligned} \alpha_1 &= \frac{(k_1 \cdot k_2) \pm \sqrt{\Delta}}{k_2^2}\,, \quad & \alpha_2 &= \frac{(k_1 \cdot k_2) \pm \sqrt{\Delta}}{k_1^2}\,, \\ \ell_1 &= \beta(k_1 - \alpha_1 k_2)\,, & \ell_2 &= \beta(k_2 - \alpha_2 k_1)\,, \\ \Delta &= (k_1 \cdot k_2)^2 - k_1^2 k_2^2\,, & \beta &= \frac{1}{1 - \alpha_1 \alpha_2}\,, \end{aligned} \tag{13}$$

from which one computes

$$\begin{aligned} \frac{2(q \cdot \ell_1)}{\beta} &= (1+\alpha_1)(D_2 - k_1^2 + m_2^2) - (D_1 + m_1^2) \\ &- \alpha_1[D_3 + m_3^2 - (k_1 + k_2)^2]\,, \\ \frac{2(q \cdot \ell_2)}{\beta} &= D_3 + \alpha_2(D_1 + m_1^2) - (1+\alpha_2)(D_2 - k_1^2 + m_2^2) \\ &+ m_3^2 - (k_1 + k_2)^2\,. \end{aligned} \tag{14}$$

When the loop integrals have to be evaluated in n dimensions, the substitution $q \to \underline{q} \equiv q + \tilde{q}$ is needed [1, 2], where q lives in 4 dimensions and $\tilde{q}$ is the $(n-4)$-dimensional part of the integration momentum, such that $(q \cdot \tilde{q}) = 0$. The only change in the previous formulas is that

$$q^2 = D_1 - \tilde{q}^{\,2} + m_1^2\,, \tag{15}$$

and the additional integrals, involving powers of $\tilde{q}^2$, can be easily handled as shown in ref. [1, 3].

Therefore, the described procedure completely solves the problem, for arbitrary k's appearing in the denominators of n-dimensional one-loop diagrams.

If, in the original trace, the number n_q of $\not{q}$'s is less than the number m of loop denominators, the algorithm can be iterated until rank-1 functions remain, at most. If $n_q \geq m$, owing to the lack of momenta k's to perform the denominator reconstruction, residual rank-p two-point integrals remain instead, with $p \leq (2 + n_q - m)$. However, two-point tensors are much easier to handle than generic m-point tensors, so that the diagram is anyhow simplified.

A last remark is in order. When some k's become collinear, one is faced with the usual problem of singularities generated by the tensor reduction (for an exhaustive study of this topic, see ref. [4]). In fact, denominators appear in eqs. (4) and (6), which may vanish, and the quantity Δ in eq. (13) is nothing but a Gram determinant. Even if the occurrence of such singularities cannot be completely avoided, a better control on them is in general possible [1], with respect to traditional techniques [5]. In addition, the analytic expressions can be kept rather compact, avoiding, at the same time, the appearance of large-rank tensors.

3 An example

To illustrate the method, I compute the reduction for the following integral with $n_q = 2$:

$$I = \int d^n q \frac{1}{D_1 \cdots D_m} \mathrm{Tr}[\, \underline{q}\, \Gamma\, \underline{q}\, \Lambda\,]\,, \tag{16}$$

where, to fix the ideas, Γ and Λ are strings containing an odd number of four-dimensional γ matrices. For convenience of notation, I omit to write the slashes in the traces.

Since the integration is performed in n dimensions, the denominators are given by eq. (2) with the substitution $q \to \underline{q} = q + \tilde{q}$.

When $m \geq 3$, the algorithm reduces I to a sum of scalar and rank-1 integrals. In fact, by splitting $\underline{q}$ in the numerator, one gets

$$\mathrm{Tr}[\,\underline{q}\,\Gamma\,\underline{q}\,\Lambda\,] \;=\; \mathrm{Tr}[q\Gamma q\Lambda] - \tilde{q}^2 \mathrm{Tr}[\Gamma\Lambda]\,, \tag{17}$$

and, by applying the formulas in the previous section,

$$\begin{aligned}
\mathrm{Tr}[q\Gamma q\Lambda] &= \frac{1}{2(\ell_1\cdot\ell_2)}\Big[2(q\cdot\ell_1)E(\ell_2)+2(q\cdot\ell_2)E(\ell_1)-q^2A-2(q\cdot k_3)\,G\Big]\,,\\
A &= 2\,\mathrm{Re}\Big[\{\Lambda\}^{+-}_{1\,1}\{\Gamma\}^{+-}_{2\,2}+\{\Lambda\}^{-+}_{2\,2}\{\Gamma\}^{-+}_{1\,1}-C\,\{k_3\}^{+-}_{2\,1}\Big]\,,\\
G &= 2\,\mathrm{Re}\Big[C\,\{q\}^{+-}_{2\,1}\Big]\,,\\
C &= \frac{1}{\{k_3\}^{+-}_{1\,2}}\Big[\{\Lambda\}^{-+}_{2\,1}\{\Gamma\}^{-+}_{2\,1}+\{\Lambda\}^{+-}_{1\,2}\{\Gamma\}^{+-}_{1\,2}\Big]\,,\\
E(\ell) &= \mathrm{Tr}[\ell\Gamma q\Lambda]-\frac{1}{2(\ell_1\cdot\ell_2)}\{\mathrm{Tr}[\ell_2 q\ell_1\Gamma\ell\Lambda]+\mathrm{Tr}[\ell_1 q\ell_2\Gamma\ell\Lambda]\\
&- 2(k_3\cdot\ell)\,G-(q\cdot\ell)\,A\}\,.
\end{aligned} \tag{18}$$

The above equations give the final answer:

$$\begin{aligned}
I &= \frac{1}{2(\ell_1\cdot\ell_2)}\int d^nq\frac{1}{D_1\,\cdots\,D_m}\Big\{(D_1+m_1^2)\,[E(\beta\alpha_2\ell_1-\beta\ell_2)-A]\\
&+ (D_2+m_2^2-k_1^2)\,E(\beta\ell_2+\beta\alpha_1\ell_2-\beta\ell_1-\beta\alpha_2\ell_1)\\
&+ \Big(D_3+m_3^2-(k_1+k_2)^2\Big)\,[E(\beta\ell_1-\beta\alpha_1\ell_2)+G]\\
&- \Big(D_4+m_4^2-(k_1+k_2+k_3)^2\Big)\,G+\tilde{q}^2\,(A-2(\ell_1\cdot\ell_2)\,\mathrm{Tr}[\Gamma\Lambda])\Big\}\,,\\
\beta &= \frac{1}{1-\alpha_1\alpha_2}\,.
\end{aligned} \tag{19}$$

When $k_{1,2}^2\neq 0$, $\ell_{1,2}$ and $\alpha_{1,2}$ are as in eq. (13).

If $k_1^2=0$ and $k_2^2\neq 0$, eq. (19) still holds with $\alpha_1=0$, $\ell_1=k_1$ and $\ell_2=k_2-\alpha_2 k_1$, where $\alpha_2=\alpha$ is given in eq. (10).

If $k_{1,2}^2=0$, then $\alpha_{1,2}=0$ and $\ell_{1,2}=k_{1,2}$.

When $m=3$, some terms vanish. This implies $C=G=0$ and

$$E(\ell) \;=\; \mathrm{Tr}[\ell\Gamma q\Lambda]+A\,\frac{(q\cdot\ell)}{2(\ell_1\cdot\ell_2)}\,. \tag{20}$$

4 Summary

In this paper, I extended the technique introduced in ref. [1] to reduce the tensorial complexity of the diagrams appearing in multi-leg loop calculations.

The method is now applicable to generic one-loop integrals, with arbitrary internal masses and external momenta.

The algorithm can usually be iterated in such a way that only scalar and rank-1 functions appear at the end of the reduction. At worst, higher-rank two-point tensors survive, independently from the initial number of denominators.

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