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BRANES: FROM FREE FIELDS TO GENERAL BACKGROUNDS

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Abstract

Motivated by recent developments in string theory, we study the structure of boundary conditions in arbitrary conformal field theories. A boundary condition is specified by two types of data: first, a consistent collection of reflection coefficients for bulk fields on the disk; and second, a choice of an automorphism ω of the fusion rules that preserves conformal weights. Non-trivial automorphisms ω correspond to D-brane configurations for arbitrary conformal field theories. The choice of the fusion rule automorphism ω amounts to fixing the dimension and certain global topological features of the D-brane world volume and the background gauge field on it.

We present evidence that for fixed choice of ω the boundary conditions are classified as the irreducible representations of some commutative associative algebra, a generalization of the fusion rule algebra. Each of these irreducible representations corresponds to a choice of the moduli for the world volume of the D-brane and the moduli of the flat connection on it.

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1 Introduction

In this paper we investigate the structure of conformal field theories on (real two-dimensional) surfaces which have boundaries and/or are unorientable. These results are of relevance to various applications of such conformal field theories. For example, boundary effects are of interest in the description of two-dimensional critical systems in statistical mechanics, the quantum Hall effect, various impurity problems, or the Ising model with a defect line (see e.g. [1,2] and the literature cited there).

Our main motivation comes, however, from string theory. Our results pave the way to the study of open strings and D-brane configurations not only for backgrounds based on free field theories, but for arbitrary conformal field theories. Recall [3] that in string theory it has been known for a long time that there are theories of open strings, in which the ends of the open strings give rise to boundaries of the string world sheet. It is also known that consistency of open strings requires the consideration of unoriented world sheets as well. In the orbifold-inspired language that has been proposed in [4,5], the unorientable surfaces implement just the projection on symmetric respectively anti-symmetric states.

It has been known for quite some time, too, that the low energy effective actions of various string theories possess solitonic solutions. What has become apparent more recently is that there is also a string perturbation theory for the corresponding sectors and that this perturbation can be described using world sheets with boundaries ([6], see also [7]). In the case of free strings, a Lagrangian description in terms of the string coordinates (Fubini-Veneziano fields) X^i is available, in which the presence of these sectors corresponds to imposing boundary conditions on the X^i that are more general than the usual Neumann boundary conditions; see e.g. [8] and the literature cited there. While these considerations have provided non-trivial insight into the structure of string theory, they are also subject to certain limitations. In particular, most of them have been restricted to BPS sectors, or can so far be formulated at a fully nonperturbative level (on the world sheet) only for *free* conformal field theories. (Of course, for non-free theories that possess a geometric interpretation, one can still employ the sigma model approach, but then for many purposes one must resort to sigma model perturbation theory.)

In this paper we present a few steps towards a deeper understanding of interacting conformal field theories on surfaces that are relevant to open string theories. For brevity we will refer to conformal field theory on such world sheets which arise only in open string theories as *open conformal field theory*, while for conformal field theory on the world sheets that already appear for closed strings we use the term *closed conformal field theory*. We will show in this paper that theories on both types of world sheets can be described in the same formalism. The main objective of our paper is to outline a conceptual framework for open conformal field theories. Among other things, we will point out various detailed problems that deserve further study. Our results put these problems in the appropriate conceptual framework and can therefore help to initiate a programme for further research. In particular, we develop concepts and techniques that allow (at least in principle) to make exact, i.e. in particular non-perturbative, statements beyond the BPS sectors of string theories based on interacting conformal field theories.

A more specific goal of our work is to gain insight into the description and the structure

of boundary conditions. In the geometric description of free strings via the Fubini–Veneziano fields X^i (which are not proper conformal fields), D-branes are submanifolds of the target space, along with some additional structure. More precisely, such a D-brane is characterized by its dimension, its world volume and a vector bundle on it. We will give analogues of all these data for arbitrary conformal field theory backgrounds. Briefly, our basic observation in this direction is that for an arbitrary conformal field theory there exists a structure that, when specialized to the conformal field theory of free bosons, reproduces the various possible topologies of the D-brane. This structure is provided by certain automorphisms ω of the fusion rules, in much the same way as fusion rule automorphisms enter the classification of modular invariant torus partition functions in the case of closed conformal field theory.

Our next result concerns the finer structure of these boundary conditions. We exhibit certain algebras which provide a tool to classify the various possible boundary conditions. We present evidence that such a ‘classifying algebra’ is a commutative associative algebra that generalizes the fusion rule algebra, that for fixed choice of ω the boundary conditions are in one-to-one correspondence to its irreducible representations. Each of these irreducible representations corresponds to a choice of the moduli for the world volume of the D-brane and the moduli of the flat connection on it.

The plan of this article is as follows. Our basic aim is to arrive at a better understanding of open conformal field theory. Open conformal field theory has been studied for quite some time both in its own right [9–15] and in relation to strings [16–31, 7]. To some extent we can build on these existing discussions, but for a more comprehensive picture also some further ingredients are needed. In fact, it turns out that our first task is to generalize the very concept of a conformal field theory to the case of real two-dimensional surfaces which are allowed to possess boundaries and / or to be unorientable. In particular, a better understanding of open conformal field theory in part requires a rather detailed discussion of several features of closed conformal field theory. Accordingly, in section 2 we first review the structure of conformal field theory on a closed and orientable surface from a somewhat unorthodox point of view that concentrates on the relationship between ‘chiral’ and ‘full’ conformal field theory and as a consequence has the advantage that it generalizes in a straightforward manner to the open case. Those readers who expect that they can dispense of the information given there altogether may proceed directly to section 3, and later consult section 2 when necessary.

Afterwards, in section 3, we turn to the discussion of open conformal field theory. In subsection 3.1 we introduce the oriented cover of the world sheet of an open conformal field theory. Bulk fields then correspond to a suitable product of two chiral vertex operators on the oriented cover, as described in subsection 3.2. In subsection 3.3 we study one-point chiral blocks for the situation where the world sheet is the crosscap. Subsection 3.4 deals with the relation between the two chiral labels Λ and $\tilde{\Lambda}$ of a bulk field $\phi_{\Lambda, \tilde{\Lambda}}$; we argue that $\tilde{\Lambda} = \omega(\Lambda)$ with ω an automorphism of the fusion rules that preserves conformal weights, and discuss the implementation of this automorphism on chiral blocks and at the level of the operator formalism. This allows in particular to establish a formalism for branes in arbitrary conformal field theories; the specific case of free bosons, for which one recovers the known ordinary D-branes, is treated in subsections 3.4.4 and 3.4.6. The concepts

of boundary conditions and boundary fields are introduced in subsection 3.5, which allows us in particular to study, in subsection 3.6, chiral blocks on the disk. We are then in a position to proceed to the stage of full conformal field theory, which is done in subsection 3.7; in particular the concept of reflection coefficients is introduced there. In subsection 3.8 we address the issue of a classification of all possible boundary conditions (for fixed choice of the automorphism ω); this leads us to the concept of a ‘classifying algebra’, which we illustrate by various examples.

In section 4 we briefly add some remarks concerning the possible application of our results to string theory. We end in section 5 by outlining further lines of research, both for conformal field theory and for string theory.

Finally we mention that we have written this article with an eye towards applications in string theory. Accordingly, at several instances we streamline the arguments by leaving aside certain aspects that are presumably irrelevant in string theory. For example, we do not display explicitly the Weyl anomaly because it cancels out in critical string theory. We also suppress most mathematical issues that concern the topology on the space of physical states and domain questions for unbounded operators which act on that space. Since we are able to analyze open conformal field theory in a manner completely analogous to closed conformal field theory, these matters can be addressed by the same methods that were developed in the latter context, see for example [32, 33], and there is no reason to expect any further complications in the open case.

2 Closed conformal field theory revisited

2.1 Chiral versus full conformal field theory

2.1.1 Orientability versus orientedness

In quantum physics, one faces quite often the following situation. One is given a collection of data of which one expects that they should be sufficient to characterize some quantum theory completely (this expectation can e.g. be based on the observation that these data would already suffice to specify a corresponding classical theory), but closer inspection reveals that in order to be able to give a complete definition of the quantum theory in fact some additional auxiliary structure is needed. In such a situation, one has to make sure that in the end this auxiliary structure can again be eliminated without affecting the observable predictions of the theory.¹ A nice example of this phenomenon is provided by ‘topological’ field theories. One would like to define such a theory on any differentiable manifold M by only using the differentiable structure of M . But in order to formulate the theory as a quantum field theory (e.g. via a path integral), one actually needs to endow M with the auxiliary structure of a metric.

As it turns out, this pattern is also realized in the situation of our interest. Namely, our goal is to set up a two-dimensional conformal field theory, or more precisely, a conformal field theory on some (real) two-dimensional differentiable manifold C . But in order to achieve this goal, it is in fact necessary to consider manifolds that possess the structure of

¹ When this is not possible, then one speaks of an anomaly.

a complex curve, i.e. a *complex* manifold of dimension one. The origin of this requirement is that we have to impose invariance under local conformal transformations. Technically, this means that one must endow the (real) two-dimensional manifold with a *conformal structure* – that is, with an equivalence class of metrics that are related by local rescalings. In two dimensions, a conformal structure is, in turn, equivalent to a *complex structure*. It is important to realize that the choice of a complex structure requires in particular the choice of an orientation. Now even when the surface C is *orientable*, it does *not* come as an *oriented* surface; in other words, none of the two possible orientations is preferred over the other. It follows that once we have achieved the construction of a conformal field theory on an oriented surface, we finally have to eliminate any dependence on the chosen orientation.

2.1.2 The two stages of conformal field theory

From these observations we conclude in particular that in the study of conformal field theory there are two distinct conceptual levels that should better be carefully distinguished:

- Conformal field theory on a complex curve \hat{C} . We will refer to this stage as the *chiral conformal field theory*.²
- Conformal field theory on an unoriented real two-dimensional manifold C . We call this stage the *full conformal field theory*.

Furthermore, we proceed from the former to the latter by eliminating the choice of an orientation. Let us stress that our approach to this issue differs from the usual description that one finds in the literature. Conventionally, one imagines that one could study the theory directly on a real two-dimensional manifold. Now in many cases, in which one employs a path integral formulation of the theory, one observes that the space of solutions to the corresponding classical field equations factorizes. This is then taken as a motivation to ‘split’ the theory into two ‘chiral halves’, each of which is afterwards studied as an independent conformal field theory on a complex curve. This recipe is sometimes expressed by saying that one can ‘treat z and \bar{z} as independent complex variables’. In contrast, for us the starting point is the chiral theory on the curve \hat{C} , which can and should be studied in its own right, and the conformal field theory on the real two-dimensional manifold is obtained only at a later stage by eliminating the orientation dependence. In our opinion this provides in particular a more natural rationale for the emergence of ‘left-’ and ‘right-movers’. Moreover, as we will see it has the additional benefit that it works for the open case as well. (Also note in this context that all our considerations will be at a fully non-perturbative level for the world sheet theory; in particular we do not have to assume that the conformal field theories under consideration possess any Lagrangian formulation so that there need not be any analogue of field equations.)

To proceed, we will first have to exhibit the central pieces of structure that are present for chiral respectively full conformal field theory, and also those that arise in string theory. Somewhat unexpectedly, a careful discussion of these topics provides us with new insight already for the case of closed conformal field theory, namely concerning the relation be-

² It is worth mentioning that it is usually this kind of structure that mathematicians refer to when they talk about ‘conformal field theory’, see e.g. [34, 35, 38–40].

tween the chiral and the full level. For instance, the role of fusion rule automorphisms for the classification of modular invariants can be easily understood via this relationship. Moreover, it will turn out that once we have established a suitable – not entirely conventional – framework for closed conformal field theory, the extension to the open case, both for conformal field theory proper and for string theory, is much more straightforward than in more conventional approaches. In this framework the basic ingredient is the system of chiral blocks. We analyze these quantities from a point of view that does not presuppose the existence of an operator formalism, neither for the full nor just for the chiral conformal field theory. For technical reasons, some aspects of this approach (in particular the description of chiral blocks as so-called co-invariants) can so far be made fully explicit only for WZW theories. But it is generally expected (see e.g. [41]) that the relevant structures that are available in the WZW case are merely specific realizations of general structures that are indeed present in any arbitrary conformal field theory.

2.1.3 Perturbative string theory

On top of these two conceptual stages of conformal field theory, for the applications we have in mind there is a third, the one of (perturbative) string theory. At this additional stage the guiding principle is to get rid of *all* properties of the world sheet C while still keeping information about the quantum field theory that was defined on the world sheet. To implement this principle one first eliminates the Virasoro algebra (or one of its (super-)extensions) by taking the relevant (semi-infinite) cohomology. Next the choice of a conformal structure is eliminated by performing an integral over the moduli space of complex structures. And finally one eliminates the choice of topology of the world sheet by summing over all possible topologies, where the sum is weighted by a factor of $(\gamma_S)^{-\chi}$, with γ_S the string coupling constant and χ the Euler number of the world sheet. (Thus in particular, when one uses the correlation functions of the conformal field theory for the computation of string scattering amplitudes, the latter will always come combined with the relevant moduli integrals.)

2.2 Chiral conformal field theory

We start by discussing *chiral* conformal field theory. That is, we work on some manifold \hat{C} that has the structure of a complex curve. Technically, \hat{C} is an algebraic curve over the complex numbers \mathbb{C} that is complete and reduced and whose singularities are at worst ordinary double points. (The inclusion of the singular curves serves to compactify all moduli spaces $\mathcal{M}_{g,n}$ that will enter our investigations.)

2.2.1 Chiral sectors

The main structural data of a chiral conformal field theory are provided by the *system of chiral blocks*. It is worth stressing that a priori there may well exist such systems which cannot be constructed in an operator formalism, i.e. systems for which the chiral blocks cannot be interpreted directly as matrix elements of products of (chiral) vertex operators. But since the connection to more heuristic considerations is usually made via an operator

formalism, in the present and next subsection we first list those ingredients which occur in an operator interpretation of chiral blocks (compare e.g. [33, 42–46]).

- The observables form some infinite-dimensional associative algebra \mathfrak{W} , called the *chiral symmetry algebra*. \mathfrak{W} is \mathbb{Z} -graded, it is a $*$ -algebra (i.e. is endowed with an involutive anti-automorphism), and it contains (the enveloping algebra of) the Virasoro algebra as a subalgebra. The grading is provided by the zero mode L_0 of the Virasoro algebra. Often instead of this associative algebra one can equivalently consider the Lie algebra \mathfrak{L} whose Lie bracket is given by the commutator with respect to the associative product. But in general there is no guarantee that this already captures all features of the associative algebra \mathfrak{W} .

- The space of states of the chiral theory is the direct sum

$$\mathcal{H} = \bigoplus_{\Lambda \in \Xi} \mathcal{H}_\Lambda, \quad (2.1)$$

where each of the (chiral) *sectors* \mathcal{H}_Λ is some infinite-dimensional vector space, whose precise structure depends on the framework one chooses. In a representation theoretic approach, it is natural to assume that the space \mathcal{H}_Λ has a gradation over the integers that is compatible with the \mathbb{Z} -grading of the chiral algebra. Moreover, all subspaces of definite grade should be finite-dimensional, and their dimensions should grow less than exponentially. The latter condition ensures that the Virasoro-specialized character

$$\chi_\Lambda(\tau) := \text{tr}_{\mathcal{H}_\Lambda} e^{2\pi i \tau (L_0 - c/24)} \quad (2.2)$$

converges for any τ in the complex upper half-plane.

- We will further assume that the sectors carry a scalar product such that the action of the chiral algebra \mathfrak{W} is unitary, i.e. we restrict ourselves to unitary conformal field theories. In a more field theoretical spirit, one would further require that the sectors are endowed with the additional structure of a Hilbert space. Also, when the Lie algebra \mathfrak{L} associated to the chiral algebra possesses a triangular decomposition, the sectors \mathcal{H}_Λ will usually be irreducible highest weight modules over \mathfrak{L} .

- We will refer to the elements Λ of the index set Ξ that appears in (2.1) as ‘weights’ of the sectors, or also as ‘sector labels’. When the set Ξ of sector labels is finite, then the conformal field theory is called *rational*.

- There is a notion of *fusion product* [33, 47] which associates to any pair $\mathcal{H}_\lambda, \mathcal{H}_\mu$ of sectors a direct sum $\bigoplus_{\nu \in \Xi} \mathcal{N}_{\lambda\mu}^\nu \mathcal{H}_\nu$ of sectors. The fusion product preserves in particular the eigenvalues of central charges, and hence does not coincide with the ordinary tensor product of \mathfrak{L} -modules.

- The non-negative integers $\mathcal{N}_{\lambda\mu}^\nu$ are known as the *fusion rule coefficients* of the theory. Furthermore, there is a distinguished sector \mathcal{H}_Ω , the *vacuum sector*, which concerning fusion plays the role of a unit element, i.e. $\mathcal{N}_{\Omega\lambda}^\mu = \delta_\lambda^\mu$. Moreover, $\mathcal{N}_{\lambda\mu}^\Omega = \delta_{\lambda,\mu^+}$, where

$$\mu \mapsto \mu^+ \quad (2.3)$$

is a permutation that preserves the fusion rules as well as the conformal weights; the sector \mathcal{H}_{μ^+} is called *conjugate* (or also charge conjugate) to \mathcal{H}_μ .

2.2.2 Chiral vertex operators

In the usual language, which is borrowed from ordinary quantum field theory, one associates to each sector \mathcal{H}_Λ a primary ‘field’ and its descendants. However, such an ‘operator formalism’ must be introduced with great care. For chiral conformal field theories, the relevant concept is the one of *chiral vertex operators* [34, 48, 42], which constitute intertwiners for the fusion product [33, 49]. Technically, a chiral vertex operator for the sector \mathcal{H}_Λ is a linear map³

$$\begin{aligned} \mathcal{V}_\Lambda : \mathcal{H}_\Lambda &\rightarrow z^{\Delta_\nu - \Delta_\Lambda - \Delta_\mu} \text{Hom}(\mathcal{H}_\mu, \mathcal{H}_\nu)[[z, z^{-1}]] \\ |\psi_\lambda\rangle &\mapsto \mathcal{V}_\Lambda(|\psi_\lambda\rangle) =: \binom{\nu \mu}{\Lambda}(\psi_\lambda; z) \end{aligned} \quad (2.4)$$

for a fixed choice of sectors \mathcal{H}_μ (the ‘source’ sector) and \mathcal{H}_ν (the ‘range’ sector) that possesses certain intertwining properties for the chiral algebra \mathfrak{W} , or more precisely, for the action of \mathfrak{W} on the fusion product of \mathcal{H}_λ and \mathcal{H}_μ and on \mathcal{H}_ν , respectively. For a given triple Λ, μ, ν there can in general exist several independent maps of this type; the dimension of the space of such maps is just given by the fusion rule $\mathcal{N}_{\lambda\mu}^\nu$. Throughout the paper we suppress the corresponding multiplicity labels.

■ Usually we will have to deal only with *primary* chiral vertex operators, by which one means the image of the highest weight vector $|\psi_\Lambda\rangle$ of \mathcal{H}_Λ ; we denote this linear map by

$$\binom{\nu \mu}{\Lambda}(z) \equiv \binom{\nu \mu}{\Lambda}(\psi_\Lambda; z) := \mathcal{V}_\Lambda(|\psi_\Lambda\rangle). \quad (2.5)$$

Moreover, we will follow the common practice to abbreviate this quantity by the symbol

$$\varphi_\Lambda(z) \equiv \binom{* *}{\Lambda}(z), \quad (2.6)$$

i.e. suppress the source and range labels, whenever this makes the formulas more intelligible.

In the case of free bosons X^i , the fusion rules are abelian, i.e. $\mathcal{N}_{\lambda\mu}^\nu = \delta_{\nu, \lambda + \mu}$, and the primary chiral vertex operators are nothing but the usual abelian vertex operators [3], which can be written as normal ordered exponentials

$$\varphi_\lambda(z) = :e^{i\lambda \cdot X(z)}: \quad (2.7)$$

of the free (chiral) boson fields.

■ At genus zero, products of m chiral vertex operators $\phi_{\Lambda_i}(z_i)$ can be defined when the variables z_i lie in the subset $\{(z_i) \mid |z_m| > \cdots > |z_2| > |z_1|, \arg(z_i) > 0\}$ of \mathbb{C}^m , and via the intertwining properties the domain of definition can be extended to the image of this set under projective transformations, which is the universal covering of $\mathbb{C}^m \setminus \{z_i = z_j \text{ for some } i \neq j\}$ [33]. In rational theories, one is actually dealing with a finite covering. This shows in particular that in a rational theory the variable z in (2.4) must be interpreted as $z = w^\ell$

³ Incidentally, this is one instance where we should worry about the completion of modules. For dealing with operators that make sense for a sufficiently large set of z -values, in this formula at least the range sector \mathcal{H}_ν must be understood as being suitably completed.

with some $\ell \in \mathbb{Z}$ and w a (quasi-)global coordinate on this finite covering. Very roughly, one can think of z_i as the value that the quasi-global coordinate function z on \mathbb{P}^1 takes at the insertion point p_i of the chiral vertex operator.⁴ An analogous interpretation is definitely no longer possible at higher genus, where there is no analogue of the quasi-global coordinate z .

■ It is commonly expected that the fusion product and the chiral vertex operators can be constructed in a mathematically rigorous manner via the representation theory of vertex operator algebras [50–53], where the indeterminate z in (2.4) is regarded as a formal variable. It is, however, not clear to us whether that settles the problems that potentially arise from the fact that the sectors \mathcal{H}_λ are not complete (in the norm topology) and that the chiral vertex operators are unbounded operators. This remark applies likewise to the supposed operator formalism for open conformal field theories, where the situation is complicated by the fact that one also has to take care of the various possible boundary conditions.

2.2.3 Chiral blocks

Even though the chiral vertex operators are quite directly accessible to field theoretic intuition, the more important structure in chiral conformal field theory is actually provided by the *chiral blocks*. (This can e.g. be inferred from the fact that even in the operator formalism the chiral blocks are the prime quantities of interest as soon as it comes to concrete calculations.) In the operator formalism, chiral blocks are easily constructed for arbitrary conformal field theories for curves C of genus 0 or 1. At genus 0, where the concept of radial ordering makes sense, one can in particular multiply chiral vertex operators in a well defined manner; the chiral blocks are then given by the expectation values

$$V_0(\vec{\Lambda}; \vec{\mu}) \equiv \langle \varphi_{\Lambda_1}(z_1) \varphi_{\Lambda_2}(z_2) \cdots \varphi_{\Lambda_{n-1}}(z_{n-1}) \varphi_{\Lambda_n}(z_n) \rangle^{(\vec{\mu})}, \quad (2.8)$$

of such products of primary chiral vertex operators; here $\langle \cdots \rangle \equiv \langle \Omega | \cdots | \Omega \rangle$ denotes the vacuum expectation value, while the additional label $\vec{\mu}$ indicates the chosen collection of ‘intermediate states’ including possible multiplicities.⁵ (In a pictorial description of the blocks by graphs with trivalent vertices, the intermediate states correspond to the labels of the internal lines, while possible multiplicities correspond to labels of the vertices; for a review of this pictorial representation see e.g. section 1.2 of [54]). Similarly, for a curve of genus 1 with modular parameter τ the chiral blocks can be obtained as suitably weighted traces of products of chiral vertex operators,

$$V_{1;\nu}(\vec{\Lambda}; \vec{\mu}) \equiv \text{tr}_{\mathcal{H}_\nu} \left(e^{2\pi i \tau (L_0 - c/24)} \varphi_{\Lambda_1}(z_1) \cdots \varphi_{\Lambda_n}(z_n) \right)^{(\vec{\mu})}, \quad (2.9)$$

where $\vec{\mu}$ has an analogous meaning as at genus zero and ν corresponds to yet another internal line that closes the corresponding graph to a ‘1-loop diagram’; a special example

⁴ Note that we must distinguish carefully between the point p_i and the coordinate value $z_i = z(p_i)$. This is nicely illustrated by the appearance of powers z_i^n below, which make sense for coordinates, while of course there does not exist any concept of multiplying the points p_i .

⁵ In the special case that the fusion rule coefficients satisfy $\sum_{\nu \in \Xi} \mathcal{N}_{\lambda\mu}^\nu = 1$ for all $\lambda, \mu \in \Xi$ (e.g. for free bosons, where one has $\mathcal{N}_{\lambda\mu}^\nu = \delta_{\nu, \lambda+\mu}$), only a single such choice is possible, but generically there are $\sum_{\mu_1, \mu_2, \dots \in \Xi} \mathcal{N}_{\Lambda_1 \Lambda_2}^{\mu_1} \mathcal{N}_{\mu_1 \Lambda_3}^{\mu_2} \cdots \mathcal{N}_{\mu_{n-3} \Lambda_{n-1} \Lambda_n}$ many distinct possibilities.

is provided by the characters (2.2), which can be interpreted as zero-point blocks on the torus, $\chi_\lambda \equiv V_1(\emptyset; \lambda)$.

In principle, via the implementation of suitable factorization rules it should be possible to establish an operator formalism that allows for a definition of chiral blocks for higher genus surfaces similarly as in (2.8) and (2.9). Unfortunately, so far such an operator formalism has not been worked out for general interacting conformal field theories. On the other hand, such a formulation is already available to some extent for free field theories, or more specifically, mainly for free bosons and for b - c -systems. These constructions are either based on a path integral formulation [55–58] or employ the representation theory of Krichever–Novikov type algebras [59]. Some specific non-free theories have been analyzed as well, namely orbifolds of free boson theories [60] and free bosons with a background charge [20]. In fact, the results obtained in the latter case (which are based on the earlier work [61, 62]) might provide guiding principles for the extension to more complicated conformal field theories. Some other results that are relevant to the case of general conformal field theories can be found in [63]; they are based on the sewing prescriptions of [64, 65].

For our purposes, the proper way to think about the chiral blocks is as the space of solutions to certain algebraic and differential equations which are known as the *Ward identities* of the theory. The Ward identities for chiral blocks form a collection of algebraic and differential equations that express the symmetries of the theory. In the operator formalism they arise by combining the special properties of the vacuum state (e.g. that it is annihilated by the Virasoro modes L_m with $m \geq -1$) with the intertwining properties of the chiral vertex operators. It is worth stressing that here we speak of *all* Ward identities, not only those particularly simple ones (such as the projective Ward identities, which correspond to the modes L_0 and $L_{\pm 1}$ of the Virasoro algebra) that are obtained by considering those modes of the chiral algebra which annihilate both the ‘in’ and the ‘out’ vacuum state. (The chiral blocks also satisfy a collection of identities that are commonly called null vector equations. In the present description all the information contained in those null vector equations is already implemented by the Ward identities together with fact that the sectors are irreducible modules.) However, except for the latter special cases, it proves to be a difficult task to write out the Ward identities explicitly. This suggests to look for an alternative formulation of the Ward identities that does not make use of the concept of chiral vertex operators and that, ideally, allows to combine all Ward identities into a single prescription. As it turns out, one can make this idea concrete in the special case of WZW theories. The relevant structure turns out to be given by what is known as the *block algebra*, which provides a ‘global’ version of the chiral algebra that is only defined with reference to some local coordinate. As will be explained in some detail in subsection 2.3, the (co-)invariants with respect to the block algebra indeed implement all the Ward identities of the WZW theory.

2.2.4 Bundles of chiral blocks

By now, we have succeeded in associating to each point $(g; p_1, p_2, \dots, p_n) \equiv (g; \vec{p})$ in the moduli space $\mathcal{M}_{g,n}$ of complex curves of genus g with n distinct marked smooth points

p_1, p_2, \dots, p_n some complex vector space $V \equiv V_{\hat{C}_g}(\vec{p}, \vec{\lambda})$, the space of chiral blocks. It is expected that these vector spaces $V_{\hat{C}_g}(\vec{p}, \vec{\lambda})$ for all $(g; \vec{p}) \in \mathcal{M}_{g,n}$ fit together to the total space of a vector bundle⁶ over $\mathcal{M}_{g,n}$, the bundle $\mathcal{V} \equiv \mathcal{V}(\vec{\lambda})$ of chiral blocks; in the WZW case this can be proven rigorously ([35, 36], for a review see [54]). More precisely, for any finite sequence $\vec{\lambda} \equiv (\lambda_1, \lambda_2, \dots, \lambda_n)$ of n sector labels $\lambda_1, \lambda_2, \dots, \lambda_n \in \Xi$ of a rational chiral conformal field theory, the solutions to the Ward identities provide us with a vector bundle \mathcal{V} of finite rank

$$\text{rank}_{\mathbb{C}} V_{\hat{C}_g}(\vec{p}, \vec{\lambda}) =: \mathcal{N}_g(\vec{\lambda}) \quad (2.10)$$

over each of the moduli spaces $\mathcal{M}_{g,n}$.

This system of vector bundles has a number of highly non-trivial properties. First, we assume that each of the vector bundles $\mathcal{V}(\vec{\lambda})$ comes equipped with a projectively flat unitary connection, the *Knizhnik–Zamolodchikov connection*. The existence of a projectively flat connection has been shown rigorously [35, 36] for any genus in the case of WZW theories. (A generalization for genus zero is discussed in [37].) The existence of this connection motivates the habit (that we will also follow) to use the term chiral block not only for an element of the fiber over a specific point of $\mathcal{M}_{g,n}$, but also in a closely related, but still conceptually different meaning, namely as some definite horizontal section of the bundle \mathcal{V} .

Moreover, we will assume that the system of blocks obeys so-called *factorization rules* [66, 35]. They relate the chiral blocks on a singular curve and the blocks on its *normalization* for which the singularity is resolved. Suppose e.g. that the curve \hat{C} is singular and that the singularity is an ordinary double point at $q \in \hat{C}$. In the normalization \tilde{C} of \hat{C} two points q_1, q_2 of \tilde{C} lie over the singular point q . Then the factorization rule reads

$$V_{\hat{C}}(\vec{p} \cup \{q\}, (\vec{\lambda}, \Omega)) = \bigoplus_{\mu \in \Xi} V_{\tilde{C}}(\vec{p} \cup \{q_1, q_2\}, \vec{\lambda} \cup \{\mu, \mu^+\}). \quad (2.11)$$

It can happen that the normalization has two different connected components. This is e.g. the case when one pinches a curve \hat{C} between two sets \vec{p} and \vec{p}' of points in such a manner that it becomes singular. The normalization \tilde{C} of \hat{C} has in this case two connected components \tilde{C}_1 and \tilde{C}_2 , with the lift of \vec{p} in \tilde{C}_1 and the lift of \vec{p}' in \tilde{C}_2 . As we will see in subsection 2.4.3, the blocks for a curve which is not connected are given by the tensor products of blocks for the different connected components. As a consequence, in this special situation the factorization rule (2.11) reads

$$V_{\hat{C}}(\vec{p} \cup \vec{p}', \vec{\lambda} \cup \vec{\lambda}') = \bigoplus_{\mu \in \Xi} V_{\tilde{C}_1}(\vec{p} \cup \{q_1\}, \vec{\lambda} \cup \{\mu\}) \otimes V_{\tilde{C}_2}(\vec{p}' \cup \{q_2\}, \vec{\lambda}' \cup \{\mu^+\}). \quad (2.12)$$

The Knizhnik–Zamolodchikov connection and factorization rules have deep consequences because they provide relations between chiral conformal field theories on *different* curves \hat{C} . As a consequence, they allow us to speak in a meaningful way about the ‘same’ chiral conformal field theory on different curves. (In contrast, to speak about the ‘same quantum

⁶ Actually, since the moduli spaces $\mathcal{M}_{g,n}$ have singularities, one is dealing with locally free sheaves instead of vector bundles.

field theory' on two arbitrary space-times M_1 and M_2 typically does not make any sense at all.)

The factorization rules (2.11) and (2.12) also have evident consequences on the fusion rules, i.e. the ranks of the vector bundles of chiral blocks. More precisely, the integers $\mathcal{N}_g(\vec{\lambda})$ (2.10) inherit evident properties from the requirements (2.11) and (2.12); in particular, we have

$$\mathcal{N}_g(\vec{\lambda}) = \sum_{\mu \in \Xi} \mathcal{N}_{g-1}(\vec{\lambda} \cup \{\mu, \mu^+\}). \quad (2.13)$$

There exists closed formulas, the so-called Verlinde [67] formulas, for the dimensions $\mathcal{N}_g(\vec{\lambda})$, which are automatically compatible with the factorization rules. To explain their origin, we note that the fundamental group of the moduli space of non-singular tori is the modular group $\mathrm{PSL}(2, \mathbb{Z})$; it can be written as the complex upper half plane, parametrized by τ , modulo the action of $\mathrm{PSL}(2, \mathbb{Z})$, acting as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}. \quad (2.14)$$

The existence of a unitary Knizhnik–Zamolodchikov connection implies that the space of zero-point blocks on the torus, i.e. the characters, carries a projective unitary representation of the modular group (compare also [68]). The unitary symmetric matrix S that describes the transformation of the characters under the transformation $\tau \mapsto -1/\tau$ is of particular importance. Another consequence of the existence of the Knizhnik–Zamolodchikov connection is that – as is already implicit in the chosen notation – the dimension of the space of chiral blocks does not depend on the precise choice of the moduli, but only on the genus g of the curve and the type $\vec{\lambda}$ of insertions.⁷ The Verlinde conjecture states that for every rational conformal field theory the rank of the vector bundle of chiral blocks on a curve of genus g with sectors $\lambda_1, \dots, \lambda_n$ as insertions is given by the expression

$$\mathcal{N}_g(\vec{\lambda}) = \sum_{\mu \in \Xi} |S_{\mu, \Omega}|^{2-2g} \prod_{l=1}^n \frac{S_{\lambda_l, \mu}}{S_{\Omega, \mu}}. \quad (2.15)$$

2.3 WZW blocks as co-invariants

2.3.1 WZW theories

We have noted above that the description of blocks in terms of solutions to the Ward identities is not completely trivial. However, there is one subclass of conformal field theories, namely the WZW theories, for which the chiral blocks can be obtained in that framework in very concrete terms, such that e.g. the Verlinde conjecture (2.15) can be proven rigorously [40, 38, 54]. (Since this class actually serves as the starting point for obtaining many other conformal field theories, such as coset models, it is to be expected that much of the structure that we will describe for the WZW case can be generalized.)

For these models the underlying Lie algebra \mathfrak{L} is an affine Kac–Moody algebra \mathfrak{g} , which is generated over \mathbb{C} by modes J_n^a with $n \in \mathbb{Z}$ and by a central element K , with Lie

⁷ Actually, this follows already from the fact that the sheaf of chiral blocks is locally free.

brackets

$$[J_n^a, J_m^b] = \sum_c f_c^{ab} J_{n+m}^c + K \kappa^{ab} n \delta_{n+m,0}. \quad (2.16)$$

Here f_c^{ab} and κ^{ab} are the structure constants and Killing form, respectively, of the horizontal subalgebra of \mathfrak{g} , i.e. of the finite-dimensional simple Lie algebra $\bar{\mathfrak{g}}$ that is spanned by the zero modes J_0^a . For our purposes, it is convenient to allow also for the case that all structure constants vanish, $f_c^{ab} \equiv 0$. The corresponding models describe the conformal field theory of free bosons; the modes J_n^a are then the modes of the abelian currents $J^a(z) \equiv i\partial X^a(z)$ and span the Heisenberg algebra $\hat{u}(1)$ in place of an affine Kac–Moody algebra \mathfrak{g} . (As a side remark, we mention that the system of chiral blocks for WZW theories is most intimately related to the space of states of the three-dimensional Chern–Simons theory that is based on the simple compact connected and simply connected Lie group whose Lie algebra is the real form of $\bar{\mathfrak{g}}$ [69, 63].) For a WZW theory the index set Ξ consists of all weights Λ of \mathfrak{g} that are integrable at some fixed value k (the level) of the canonical central element $K \in \mathfrak{g}$; to each Λ there is associated a unitarizable irreducible highest weight module \mathcal{H}_Λ , which constitutes the space of states in the corresponding sector. Given the level k , a highest weight Λ is already determined by its horizontal part, i.e. by the weight $\bar{\Lambda}$ with respect to the subalgebra $\bar{\mathfrak{g}}$. The highest weight for the vacuum sector is given by k times the zeroth fundamental weight of \mathfrak{g} , $\Omega = k\Lambda_{(0)}$, and the conjugate Λ^+ of Λ is the unique weight that has the same level k as Λ and whose horizontal part is the conjugate (as $\bar{\mathfrak{g}}$ -weight) of $\bar{\Lambda}$.

2.3.2 Global symmetries: the block algebra

Let us describe in some detail how one can characterize the chiral blocks of a WZW theory. To keep the discussion as elementary as possible, we take C to be the complex projective space $\mathbb{P}^1 \equiv \mathbb{P}\mathbb{C}$, which we represent as the complex z -plane plus a point at infinity. Given n (distinct) points p_i on \mathbb{P}^1 , to which for brevity we still refer as the insertion points even in the absence of an operator formalism,⁸ and the corresponding n sectors labelled by integrable weights Λ_i of \mathfrak{g} , we consider the tensor product

$$\mathcal{H}_{\bar{\Lambda}} := \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2} \otimes \cdots \otimes \mathcal{H}_{\Lambda_n} \quad (2.17)$$

of the \mathfrak{g} -modules \mathcal{H}_{Λ_i} . From this big vector space we obtain the blocks by imposing the Ward identities. These identities should constitute the global realization of the symmetries of the theory and involve states ‘inserted’ at different points. As a consequence, we need a new algebra that encodes these symmetries. In the case of WZW theories such an algebra is readily available: one takes the algebra of all $\bar{\mathfrak{g}}$ -valued holomorphic functions on \hat{C} , with singularities not worse than poles of finite order at the insertion points p_i . Note that here we use the concept of holomorphic function, which is well-defined only once we have chosen a complex structure on the manifold \hat{C} .

The space of $\bar{\mathfrak{g}}$ -valued functions actually forms a Lie algebra; the Lie bracket is just the commutator with respect to the associative product that is given by pointwise multiplication. We call this algebra the *block algebra* that is associated to the chosen sequence

⁸ In mathematics, also the term ‘parabolic points’ is common.

of integrable weights, and denote it by

$$\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1) \equiv \bar{\mathfrak{g}} \otimes_{\mathbb{C}} \mathcal{F} \quad (2.18)$$

with $\mathcal{F} \equiv \mathcal{F}(\mathbb{P}^1 \setminus \{p_1, p_2, \dots, p_n\})$ ($\mathcal{F}(U)$ stands for the space of holomorphic functions on an open subset $U \subset \hat{C}$).

To arrive at the Ward identities, we have to define an action of the block algebra on $\mathcal{H}_{\bar{\Lambda}}$, i.e. endow $\mathcal{H}_{\bar{\Lambda}}$ with the structure of a $\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1)$ -module. The idea is to perform an expansion in local coordinates and to identify the local coordinates with the indeterminate of the loop space construction of the affine Lie algebra \mathfrak{g} . Accordingly, we introduce local coordinates ξ_i such that $\xi_i(p_i) = 0$, e.g.

$$\xi_i := z - z_i \quad (2.19)$$

when the points p_i correspond to values z_i of the quasi-global coordinate z on \mathbb{P}^1 . In terms of these coordinates we have local expansions

$$f(\xi_i) = \sum_{n \geq n_0} f_n^{(i)} \xi_i^n \quad (2.20)$$

of the functions $f \in \mathcal{F}$ (here the infinite sum starts at some finite value n_0 which may be negative). For every insertion point p_i this local expansion induces a Lie algebra homomorphism (actually, even an injection) j_i from $\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1)$ to the loop algebra $\mathfrak{g}_{\text{loop}}$. Namely, the loop algebra is given by $\bar{\mathfrak{g}} \otimes \mathbb{C}[[t, t^{-1}]]$ with some indeterminate t (thus $\mathfrak{g}_{\text{loop}}$ is essentially the affine algebra \mathfrak{g} without central extension), and j_i acts as

$$\bar{x} \otimes f \mapsto j_i(\bar{x} \otimes f) := \sum_{n \geq n_0} f_n^{(i)} \bar{x} \otimes t^n. \quad (2.21)$$

Actually, by making use of the residue theorem, one checks that this construction embeds the block algebra $\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1)$ as a Lie subalgebra into the n -fold direct sum \mathfrak{g}^n of the affine Lie algebra; moreover, one can check that this way the \mathfrak{g}^n -module $\mathcal{H}_{\bar{\Lambda}}$ indeed acquires the structure of a module over $\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1)$.

2.3.3 WZW blocks

In this framework, the space of chiral blocks can essentially be characterized as the space of singlets in the tensor product $\mathcal{H}_{\bar{\Lambda}}$. Closer inspection shows that the relevant representation theoretic concept is in fact the one of *co-invariants* $[\mathcal{H}_{\bar{\Lambda}}]_{\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1)}$ of $\mathcal{H}_{\bar{\Lambda}}$ with respect to the block algebra [35, 39, 40]. The idea is to divide out the submodule that consists of all vectors in $\mathcal{H}_{\bar{\Lambda}}$ that can be obtained by acting with an element of $\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1)$ on some other vector of $\mathcal{H}_{\bar{\Lambda}}$; thus

$$[\mathcal{H}_{\bar{\Lambda}}]_{\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1)} = \mathcal{H}_{\bar{\Lambda}} / \mathbf{U}^+(\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1))\mathcal{H}_{\bar{\Lambda}}, \quad (2.22)$$

where $\mathbf{U}^+(\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1)) \equiv \bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1)\mathbf{U}(\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1))$ is the so-called augmentation ideal of the universal enveloping algebra $\mathbf{U}(\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1))$ of the block algebra.

For the benefit of those readers who are not familiar with this description of chiral blocks (that is frequently used in the mathematical literature), let us provide two pieces of evidence that these are indeed the same objects that are encountered as chiral blocks in the more familiar operator formalism. First, the co-invariants are *generalized singlets*. Namely, imagine that the tensor product $\mathcal{H}_{\bar{\lambda}}$ were fully reducible as a module over the block algebra $\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1)$, i.e. that it can be decomposed into a direct sum of irreducible $\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1)$ -modules. Now in full generality, for irreducible modules \mathcal{H}_{λ} over any Lie algebra \mathfrak{h} we have $U^+(\mathfrak{h})\mathcal{H}_{\lambda} = \mathcal{H}_{\lambda}$ unless $\lambda = 0$, in which case \mathcal{H}_0 (the singlet) is one-dimensional while $U^+(\mathfrak{h})\mathcal{H}_0 = 0$, and hence $[\mathcal{H}_{\lambda}]_{\mathfrak{h}} = 0$ except for the singlet. Thus in the situation at hand, quotienting out the subspace $U^+(\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1))\mathcal{H}_{\bar{\lambda}}$ of $\mathcal{H}_{\bar{\lambda}}$ would precisely leave us with the singlets in the tensor product space $\mathcal{H}_{\bar{\lambda}}$. In other words, we would pick precisely those vectors in $\mathcal{H}_{\bar{\lambda}}$ that are invariant under all operators in the block algebra $\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1)$; accordingly, this quotienting procedure indeed should correspond to implementing the invariance of the chiral blocks under the symmetries of the theory.

As a second hint, let us specialize to the particular case where $n = 2$ with $z_2 = 0$ and where $\bar{x} = J^a \in \bar{\mathfrak{g}}$ is the element of the horizontal subalgebra that corresponds to $J_0^a \in \mathfrak{g}$. Then $x := z^{m \otimes} J^a \in \bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1)$ acts on the elements of $\mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2}$ as

$$\begin{aligned} (z^{m \otimes} J^a) \otimes \mathbf{1} + \mathbf{1} \otimes (z^{m \otimes} J^a) &= ((\xi_1 + z_1)^{m \otimes} J^a) \otimes \mathbf{1} + \mathbf{1} \otimes ((\xi_2)^{m \otimes} J^a) \\ &= \sum_{j=0}^{\infty} \binom{m}{j} z_1^{m-j} J_p^a \otimes \mathbf{1} + \mathbf{1} \otimes J_m^a. \end{aligned} \quad (2.23)$$

The so obtained expression is nothing but the ‘modified coproduct’ $\Delta_{z_1}(J_m^a)$ that implements [33] the fusion product of the sectors at the level of the chiral algebra. Thus the action of the block algebra as defined above generalizes the modified coproduct Δ_z to the situation with $n > 2$ insertions.

As a matter of fact it is *not* true that the tensor product $\mathcal{H}_{\bar{\lambda}}$ is fully reducible as a $\bar{\mathfrak{g}}(\mathbb{P}_{(n)}^1)$ -module; to obtain genuine singlets, one would have to start with the algebraic dual $(\mathcal{H}_{\bar{\lambda}})^*$ of $\mathcal{H}_{\bar{\lambda}}$, which is a much larger space than $\mathcal{H}_{\bar{\lambda}}$. Also, concerning the fiber bundle description of blocks one must be aware of the fact that while the algebraic dual gives rise to the trivial bundle $(\mathcal{H}_{\bar{\lambda}})^* \otimes \mathcal{M}_{g,n}$, the singlets must be determined fiberwise so that the resulting subbundle of chiral blocks is generically a non-trivial bundle. Actually, this bundle is naturally defined over an extended moduli space $\mathcal{M}_{g,n}^{\text{ext}}$ that on top of the ordinary moduli (insertion points and moduli of the curve) also includes the choice of local coordinates at the insertion points. We therefore consider the subbundle

$$(\mathcal{H}_{\bar{\lambda}})_0^* \rightarrow \mathcal{M}_{g,n}^{\text{ext}} \quad (2.24)$$

of singlets in the trivial bundle $(\mathcal{H}_{\bar{\lambda}})^* \times \mathcal{M}_{g,n}^{\text{ext}} \rightarrow \mathcal{M}_{g,n}^{\text{ext}}$. On the total space of the bundle (2.24) we have a free and fiber-preserving action of U^n , where U is the group of reparametrizations of the local coordinate. When we evaluate the functions in (2.24) on the highest weight vectors and take the quotient with respect to the action of U^n , we just end up with bundles that are isomorphic to the bundles $V_{\hat{C}_g}(\vec{p}, \vec{\lambda})$ considered in subsection 2.2.4. The bundle (2.24), however, contains more information, in particular about the descendants. Notice that, since the descendants transform non-trivially under changes of

local coordinates, it is natural to keep track of local coordinates and to work over $\mathcal{M}_{g,n}^{\text{ext}}$. Actually, the singlets in the algebraic dual $(\mathcal{H}_{\bar{\lambda}})^*$ are closely related to the so-called *multi-reggeon vertex* in the case of free conformal field theories (for references as well as the inclusion of background charges, see [20]). In this formalism one does not specify the sectors for the insertions; rather, one takes the full chiral space $\mathcal{H} = \bigoplus_{\Lambda \in \Xi} \mathcal{H}_{\Lambda}$ (2.1) of states, which is the direct sum of all (chiral) sectors. The n -reggeon vertex is then given by the space of co-invariants of the block algebra that acts on the n -fold tensor product of \mathcal{H} with itself. The chiral blocks are then obtained by evaluating these singlets (which are elements of $(\mathcal{H}_{\bar{\lambda}})^*$, i.e. linear forms on $\mathcal{H}_{\bar{\lambda}}$) on the highest weight states of specific sectors, which precisely projects out the contribution of the relevant sector. (Notice that this formalism is frequently not formulated on the level of *chiral* conformal field theory.)

2.3.4 Block algebras for general conformal field theories

We close this subsection with the following side remark. For a general conformal field theory, we postulate the existence of an analogue of the block algebra, i.e. of an algebra that encodes the global realizations of all symmetries of the theory. Such a global algebra $\mathcal{A}_{g,n}(m)$ should exist for every point m in the moduli space $\mathcal{M}_{g,n}$. Roughly speaking, the relation between this global algebra and the chiral algebra should generalize the relation between the algebra of global functions (possibly subject to certain conditions at marked points) and local germs of functions. As a consequence, it should be possible to embed $\mathcal{A}_{g,n}(m)$ in the direct sum of n copies of the chiral algebra. The chiral blocks are then described in the same way as for WZW theories, namely as co-invariants in the tensor product of n sectors under the action of $\mathcal{A}_{g,n}(m)$. Concretely, it should be possible to obtain such a formulation of chiral blocks in terms of co-invariants for general conformal field theories by translating the analogous coproduct formula that generalizes the expression (2.23) [33, 47] to the geometric description in terms of the functions \mathcal{F} , respectively of vector and tensor fields. In the case of the Virasoro algebra, a first step towards such a formulation is implicit in [70]. We also note that the type of structure we have just sketched also appears in several other contexts; e.g. it allows to exhibit [71] a close analogy between chiral blocks of WZW theories and the theory of automorphic forms.

2.4 Full conformal field theory

2.4.1 The oriented cover \hat{C}

To study the relation between chiral and full conformal field theory, we now assume that some consistent collection of vector bundles (respectively sheaves) of chiral blocks is given. We first observe that while chiral conformal field theory is a mathematically deep structure, it evidently does not provide the correlators of a physical field theory on the non-oriented manifold C . Specifically:

- The chiral blocks are (generically) multi-valued functions of the insertion points (they are sections in a non-trivial bundle) whereas correlation functions should be single-valued as functions of the insertion points ('locality'). Moreover, the correlation functions should also be essentially (i.e., up to the Weyl anomaly) functions of the moduli of the curve.

- The space of chiral blocks generically has dimension larger than one, whereas the physical correlators should be unique.
- Moreover, we have artificially fixed an orientation, whereas both orientations should be completely equivalent.

While the locality requirement for the dependence on the insertion points is obvious, the corresponding statement for the moduli dependence deserves a further comment. Indeed, as it turns out the correlation functions are *not* genuine functions of the moduli of the curve. Let us illustrate this with the example of zero-point correlators, i.e. partition functions Z . One way in which a moduli dependence shows up is the dependence of Z on the representative of the world sheet metric γ within a conformal equivalence class; namely,

$$Z[e^f \gamma] = Z[\gamma] \cdot e^{c S_L(f, \gamma)}, \quad (2.25)$$

where c is the Virasoro central charge and the Liouville action S_L is characterized by the ‘trace-anomaly’ $\frac{\partial}{\partial f} S_L(f, \gamma) = \frac{1}{48\pi} \sqrt{\gamma} R_\gamma$. As a consequence, the partition function $Z(m)$ is a section in a (real) line bundle over $\mathcal{M}_{g,0}$. In the case of genus $g=1$, it is related to the bilinear expression

$$Z(\tau) = \sum_{\lambda, \mu \in \Xi} Z_{\lambda, \mu} \chi_\lambda(\tau) (\chi_\mu(\tau))^* \quad (2.26)$$

of characters by $Z(m) = Z(\tau) |\sigma|^2$, where σ is a nowhere vanishing section of a projective line bundle over $\mathcal{M}_{1,0}$. For more details, we refer to [66]. Note that the Weyl anomaly in (2.25) is proportional to the value of the Virasoro central element; in critical string theory this vanishes when also the ghost sector is included; accordingly we will neglect this subtlety.

As it turns out, the three issues listed above are intimately linked and are resolved by one and the same construction. The starting point is the idea that in order to obtain results that do not depend on the orientation, at a first stage we should keep track of both possible orientations; this leads us to introduce the notion of the *oriented cover* \hat{C} of the manifold C . Let us describe the construction of \hat{C} directly for the general case; that is, we neither assume that C is orientable and we allow for boundaries. Thus we denote by C a real two-dimensional manifold, which is possibly unorientable and can have boundaries. For the tangent space at each point p in the interior of C we have two different orientations; we construct another manifold from C by taking over every such point p two points, one for each orientation. This way we obtain a two-sheeted cover \hat{C} of C . We stress that \hat{C} is not only *orientable*, but is even naturally *oriented*. The underlying manifold C is orientable if and only if the two sheets of \hat{C} are disconnected. When C has a boundary, then \hat{C} is a *branched cover* of C . C can be obtained from \hat{C} as the quotient by the involution I that exchanges the two sheets. The boundaries of C are just the fixed points of this involution I . The involution reverses the orientation, hence it is an anti-conformal map.⁹

⁹ Note that the lift from C to \hat{C} is defined on the level of complex structures. Concerning the metric structure, the following statement can be made. For a metric γ on C to represent, upon lifting to \hat{C} , the complex structure on \hat{C} , all boundary components of C must be geodesics in the metric γ . For example, two half-spheres can be matched smoothly, whereas this is not possible for two opposite ‘caps’ on the sphere.

As an illustration, take \hat{C} to be the Riemann sphere S^2 , represented by the complex plane plus one point at infinity and endowed with a (quasi-)global coordinate z . The involution

$$I_d: z \mapsto 1/z^* \quad (2.27)$$

then corresponds to C being the disk, while for

$$I_c: z \mapsto -1/z^* \quad (2.28)$$

the manifold C is the real projective space $\mathbb{P}\mathbb{R}^2$, also known as the crosscap.

2.4.2 The chiral theory on \hat{C}

We now restrict our attention again to the case where C is orientable and without boundaries. Then the cover is a non-connected curve that is the disjoint union

$$\hat{C} = C_1 \dot{\cup} C_2 \quad (2.29)$$

of two copies $C_{1,2}$ of the orientable curve C which are endowed with the two opposite orientations; the involution I interchanges the components C_1 and C_2 . Our approach to formulate the full conformal field theory on C in terms of chiral objects is to first construct the oriented cover \hat{C} , then establish a chiral conformal field theory on \hat{C} , and finally arrive at the full theory by getting rid of the orientation dependence. Concretely, this requires that for every $n \in \mathbb{Z}_{\geq 0}$ we have to lift an n -point situation on C to \hat{C} . This results in a $2n$ -point situation on \hat{C} , where the $2n$ insertion points p_i and \tilde{p}_i with $i = 1, 2, \dots, n$ are related by

$$\tilde{p}_i = I(p_i), \quad (2.30)$$

so that in particular they satisfy (say) $p_i \in C_1$ and $\tilde{p}_i \in C_2$ for all $i = 1, 2, \dots, n$.

To make contact to the more conventional formulation, we remark that this effective doubling of the number of insertion points is usually described by the statement that ‘in a conformal field theory on a closed orientable surface there are left- and right-movers.’ It is worth mentioning that the terms left-mover and right-mover have their origin in the study of the spaces of solutions to classical field equations; in contrast, here such structures arise without the need to require that there exists a Lagrangian description of the theory. Also note that in the case of the sphere S^2 , the relation (2.30) is precisely what one usually wants to express when one says that z and \tilde{z} , previously regarded as two independent complex variables (the latter being conventionally denoted by \bar{z}), are to be considered as each others’ complex conjugates,

$$\tilde{z} = z^*. \quad (2.31)$$

Note that superficially our description of closed conformal field theory merely constitutes a minor modification of more conventional expositions. But still this innocent change of perspective allows us to explain the existence of left- and right-movers (which elsewhere are often introduced in a somewhat heuristic fashion) in a concise way via the connection with the fixing of the orientation. To dispose of the dependence on the orientation we simply have to divide out the anti-conformal involution I . The main benefit of

our approach, however, will be that it allows us to treat open conformal field theory to a large extent along precisely the same lines.

The geometrical unoriented world sheet C can be identified with the quotient \hat{C}/I of the oriented cover by the anti-conformal involution I . Correspondingly we regard the full conformal field theory on C as being obtained by lifting this quotienting procedure to the level of chiral blocks or, when thinking in terms of an operator formalism, of ‘fields’. Thus a field of the full conformal field theory corresponds to two chiral fields on the oriented cover \hat{C} . For the chiral objects we can apply the theory developed previously. But we have to take into account the relation (2.30), which means in particular that in the full theory manipulations with insertion points, such as limiting processes, have to be taken in a correlated way.

2.4.3 Bi-blocks

The next observation is that while the prescription to obtain \hat{C} from C is unique at the geometrical level, typically there will be an ambiguity on how to lift this prescription to the field theoretical level. First of all, it need not necessarily be required that one has one and the same chiral algebra on the two sheets C_1 and C_2 ; taking different algebras leads to so-called *heterotic* theories. But even if the chiral algebras on both sheets are identical, there is a priori no reason to take the sector label $\tilde{\Lambda}$ on C_2 equal to the sector label Λ on C_1 , or in other words, we are still allowed to choose a non-trivial pairing between the sectors on the two sheets. Before we study in more detail the consistency requirements that we must impose on the pairings, we wish to present a few more comments on the chiral blocks on the oriented cover.

The individual chiral blocks as described in formula (2.8) are, generically, not single-valued. To arrive at a single-valued correlation function, we must look for a specific horizontal section of the bundle $\mathcal{V} = \mathcal{V}_{\vec{\Lambda}, \vec{\Lambda}}$ of chiral blocks for the prescribed sequence of ‘external’ sectors $\mathcal{H}_{\Lambda_i} \otimes \mathcal{H}_{\tilde{\Lambda}_i}$ for both connected components of the orientable cover. This means that the correlation function is to be obtained by forming a suitable linear combination of the blocks with fixed external sectors $\vec{\Lambda}$, $\vec{\Lambda}$ and arbitrary allowed ‘internal’ sectors $\vec{\mu}$, $\vec{\mu}$. In short, the correlators of the full theory on C are *linear* combinations of the chiral blocks on \hat{C} . Now since in the case of closed conformal field theory the oriented cover \hat{C} is not connected, the Ward identities on the two components of \hat{C} factorize. In the case of WZW theories this follows from the fact that

$$\bar{\mathfrak{g}}(C_1 \cup C_2) = \bar{\mathfrak{g}}(C_1) \oplus \bar{\mathfrak{g}}(C_2). \quad (2.32)$$

Namely, when $p_1, \dots, p_n \in C_1$ are the insertion points corresponding to sectors $\mathcal{H}_{\Lambda_1}, \mathcal{H}_{\Lambda_2}, \dots, \mathcal{H}_{\Lambda_n}$ and $\tilde{p}_1, \dots, \tilde{p}_n \in C_2$ are the insertion points for $\mathcal{H}_{\tilde{\Lambda}_1}, \mathcal{H}_{\tilde{\Lambda}_2}, \dots, \mathcal{H}_{\tilde{\Lambda}_n}$, respectively, then the relevant tensor product space is $\vec{\mathcal{H}} = \mathcal{H}_{\vec{\Lambda}} \otimes \mathcal{H}_{\vec{\Lambda}}$ with

$$\mathcal{H}_{\vec{\Lambda}} := \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2} \otimes \dots \otimes \mathcal{H}_{\Lambda_n} \quad \text{and} \quad \mathcal{H}_{\vec{\Lambda}} := \mathcal{H}_{\tilde{\Lambda}_1} \otimes \mathcal{H}_{\tilde{\Lambda}_2} \otimes \dots \otimes \mathcal{H}_{\tilde{\Lambda}_n}, \quad (2.33)$$

so that as a consequence of (2.32) also the co-invariants factorize,

$$[\mathcal{H}_{\vec{\Lambda}} \otimes \mathcal{H}_{\vec{\Lambda}}]_{\bar{\mathfrak{g}}(C_1 \cup C_2 \setminus \{p_1, \dots, \tilde{p}_n\})} = [\mathcal{H}_{\vec{\Lambda}}]_{\bar{\mathfrak{g}}(C_1 \setminus \{p_1, \dots, p_n\})} \otimes [\mathcal{H}_{\vec{\Lambda}}]_{\bar{\mathfrak{g}}(C_2 \setminus \{\tilde{p}_1, \dots, \tilde{p}_n\})}. \quad (2.34)$$

In short, the chiral WZW blocks on \hat{C} have a factorized form, so that it is appropriate to refer to them as *bi-blocks*. Such a factorized form of the blocks is expected for all other closed conformal field theories, too.

The correlators of the full theory are then linear combinations of these bi-blocks. In the case of a non-heterotic theory, this looks of course like *bilinear* combinations of blocks of one ‘chiral half’, which is the description used in the more conventional treatment.

Now recall that for chiral conformal field theories there exists an operator formalism (which is fully established at genus 0 and 1, while at higher genus it still has to be worked out for general conformal field theories, see the remarks in subsection 2.2.3). In particular, the chiral vertex operators (2.5) can be multiplied and possess chiral operator product expansions [33]. It is usually taken for granted that analogous expansions exist also for the full theory, but we would like to stress that this is a rather non-trivial assumption even once an operator formalism has been established at the chiral level. But let us nevertheless assume for the moment that indeed we are given an operator formalism not only for the chiral, but also for the full theory. Then the above result can be understood as follows. In order that the correlation functions can be single-valued, it is necessary that the physical fields of the full theory are linear combinations of bi-chiral objects of the type

$$\binom{\nu}{\Lambda}^{\mu}(z) \otimes \binom{\tilde{\nu}}{\tilde{\Lambda}}^{\tilde{\mu}}(\tilde{z}) \in \text{Hom}(\mathcal{H}_{\mu}, \mathcal{H}_{\nu})[[z, z^{-1}]] \otimes \text{Hom}(\mathcal{H}_{\tilde{\mu}}, \mathcal{H}_{\tilde{\nu}})[[\tilde{z}, \tilde{z}^{-1}]]; \quad (2.35)$$

Thus the role of the (primary) chiral vertex operators on C is taken over by analogous tensor product maps, and these are to be combined linearly. In particular, to every field on C one thereby associates a pair $(\lambda, \tilde{\lambda}) \in \Xi \times \Xi$ of sector labels rather than a single label $\lambda \in \Xi$.

Using the abbreviation (2.6) for primary chiral vertex operators, the factorization of bi-chiral blocks is then interpreted as

$$\begin{aligned} & \langle \varphi_{\Lambda_1}(z_1) \otimes \varphi_{\tilde{\Lambda}_1}(\tilde{z}_1) \varphi_{\Lambda_2}(z_2) \otimes \varphi_{\tilde{\Lambda}_2}(\tilde{z}_2) \varphi_{\Lambda_3}(z_3) \otimes \varphi_{\tilde{\Lambda}_3}(\tilde{z}_3) \varphi_{\Lambda_4}(z_4) \otimes \varphi_{\tilde{\Lambda}_4}(\tilde{z}_4) \rangle^{(\mu, \tilde{\mu})} \\ &= \langle \varphi_{\Lambda_1}(z_1) \varphi_{\Lambda_2}(z_2) \varphi_{\Lambda_3}(z_3) \varphi_{\Lambda_4}(z_4) \rangle^{(\mu)} \cdot \langle \varphi_{\tilde{\Lambda}_1}(\tilde{z}_1) \varphi_{\tilde{\Lambda}_2}(\tilde{z}_2) \varphi_{\tilde{\Lambda}_3}(\tilde{z}_3) \varphi_{\tilde{\Lambda}_4}(\tilde{z}_4) \rangle^{(\tilde{\mu})} \\ &=: \mathcal{F}_{\Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4}^{(\mu)}(z_1, z_2, z_3, z_4) \mathcal{F}_{\tilde{\Lambda}_1 \tilde{\Lambda}_2 \tilde{\Lambda}_3 \tilde{\Lambda}_4}^{(\tilde{\mu})}(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4) \end{aligned} \quad (2.36)$$

for $n=4$ (here the labels μ and $\tilde{\mu}$ refer to the intermediate state, compare the explanation of the corresponding notation in (2.8)), and analogously for general n . Furthermore, comparison with the putative operator product expansion¹⁰

$$\begin{aligned} & \Phi_{\Lambda_1, \tilde{\Lambda}_1}(z_1, \tilde{z}_1) \Phi_{\Lambda_2, \tilde{\Lambda}_2}(z_2, \tilde{z}_2) \\ & \sim \sum_{\Lambda_3, \tilde{\Lambda}_3} C_{\Lambda_1 \tilde{\Lambda}_1, \Lambda_2 \tilde{\Lambda}_2}^{\Lambda_3 \tilde{\Lambda}_3} (z_1 - z_2)^{\Delta_3 - \Delta_1 - \Delta_2} (\tilde{z}_1 - \tilde{z}_2)^{\tilde{\Delta}_3 - \tilde{\Delta}_1 - \tilde{\Delta}_2} \Phi_{\Lambda_3, \tilde{\Lambda}_3}(z_2, \tilde{z}_2) + \dots \end{aligned} \quad (2.37)$$

for the primary fields $\Phi_{\Lambda, \tilde{\Lambda}}$ of the full theory tells us that the coefficients that appear in

¹⁰ The ellipsis stands for the contributions of descendant fields. Also, any multiplicity labels that may be present are suppressed.

the expansion

$$\mathcal{G}_{\Lambda_1\Lambda_2\Lambda_3\Lambda_4;\tilde{\Lambda}_1\tilde{\Lambda}_2\tilde{\Lambda}_3\tilde{\Lambda}_4} = \sum_{\mu,\tilde{\mu}} C_{\Lambda_1\tilde{\Lambda}_1,\Lambda_2\tilde{\Lambda}_2,\Lambda_3\tilde{\Lambda}_3,\Lambda_4\tilde{\Lambda}_4}^{(\mu,\tilde{\mu})} \mathcal{F}_{\Lambda_1\Lambda_2\Lambda_3\Lambda_4}^{(\mu)} \mathcal{F}_{\tilde{\Lambda}_1\tilde{\Lambda}_2\tilde{\Lambda}_3\tilde{\Lambda}_4}^{(\tilde{\mu})} \quad (2.38)$$

of the correlation functions with respect to the (bi-)blocks (2.36) are nothing but suitable products of operator product coefficients.

2.4.4 Consistency conditions

Concerning the correlation functions of the full theory, even independently of any operator formalism it is clear that they will be suitable combinations of the chiral blocks. More precisely, certainly not any arbitrary combination of blocks will qualify as a sensible correlation function. Rather, various strong restrictions apply; they are of the following three types:

- *Locality*: While the chiral blocks are sections in a (generically) non-trivial bundle over $\mathcal{M}_{g,n}$, the correlators of the full theory must be ordinary functions of the insertion points which provide (part of) the coordinates on $\mathcal{M}_{g,n}$.
- Analogously, we impose the requirement that the same statement applies to the dependence on the other coordinates on $\mathcal{M}_{g,n}$, i.e. on the moduli of the complex curves C_i . (The moduli that correspond to the two disconnected components of \hat{C} are to be identified via an anti-conformal involution, too. For instance, in the case where C is the torus and hence $\hat{C} = C_\tau \cup C_{\tilde{\tau}}$, for the partition function $Z(\tau) = \sum_{\lambda,\mu} Z_{\lambda,\mu} \chi_\lambda(\tau) (\chi_\mu(\tau))^*$ the relevant identification is taken into account by the complex conjugation of χ_μ , which amounts to identifying $\tilde{\tau}$ with $-\tau^*$.)
- *Factorization*: The theory should be compatible with singular limits on the moduli spaces, in such a way that all coefficients that appear in the expansions of correlation functions in terms of chiral blocks (such as the $C_{\Lambda_1\tilde{\Lambda}_1,\Lambda_2\tilde{\Lambda}_2,\Lambda_3\tilde{\Lambda}_3,\Lambda_4\tilde{\Lambda}_4}^{(\mu,\tilde{\mu})}$ that appear in (2.38)) are expressible through the coefficients for the three point functions. By comparing different sequences of factorizations that lead to one and the same final result, this requirement leads to various consistency relations, which are known as factorization or also as sewing constraints.

The presence of such locality and factorization constraints is in fact a necessary prerequisite for the existence of operator product expansions like (2.37). In particular, the correct factorization of four-point correlators (2.38) amounts to the statement that the operator product (2.37) is associative.

Finally there is also another type of constraints:

- *Integrality*: The coefficients that appear in some specific linear combinations must be integral and non-negative. This applies to the zero-point (bi-)blocks for surfaces with Euler characteristic zero (including those with boundaries or crosscaps). The reason is that the corresponding correlators of the full theory should acquire the physical meaning of partition functions, i.e. of generating functions for multiplicities of states of the full theory.

It is believed that these constraints admit a unique solution for the correlation functions; this has been checked in various non-trivial examples. But note that the locality

constraints alone can in general possess several distinct solutions, corresponding e.g. to different possible torus partition functions; for concrete realizations in the case of $\mathfrak{su}(2)$ WZW theories and $c \leq 1$ minimal models see e.g. [72–74].

We should also mention that the integrality constraints are in fact not independent of the locality and factorization constraints. In the case of closed conformal field theory, where they apply to the torus partition function, the crucial structure is provided by the fusion algebra, and integrality can be derived as a consequence of the Verlinde formula. Similarly, in the open case the relevant concept turns out to be the one of a *classifying algebra*, which is an associative algebra that generalizes the fusion algebra. This structure, introduced in [75], will be studied in subsection 3.8.

Moreover, as intermediate ‘channels’ that appear in a factorization formula – e.g., as combinations $(\mu, \tilde{\mu})$ in a factorization of the four-point function (2.38) into three-point functions – only those combinations $(\mu, \tilde{\mu})$ of sector labels appear which are compatible with the chosen pairing ω , i.e. for which $\tilde{\mu} = \omega(\mu)$. This requirement has an immediate conceptual consequence: just like we already did at the level of chiral conformal field theory, we can now also talk about considering one and the same full conformal field theory on different surfaces. Then the factorization constraints imply in particular that for each full conformal field theory the pairing of labels has to be identical on all closed oriented surfaces. In particular, we can then think of the pairing as being prescribed by the form

$$Z_{\lambda, \mu} = \delta_{\mu, \omega(\lambda)} \equiv \delta_{\mu, \tilde{\lambda}}. \quad (2.39)$$

for the torus partition function (2.26).

From now on we restrict our attention to torus partition functions of the particular form (2.39). This is justified by the fact that as soon as we talk about fusion rules and factorization at the level of the *full* theory, we implicitly assume that no further extension of the chiral algebra is possible that would alter the structure of the (chiral) fusion rules.

The factorization conditions severely constrain the possible pairings ω . Namely, the various factorization limits must be compatible for the chiral theories on both sheets simultaneously. On the other hand, at the chiral level much of the information on the factorization is encoded in the fusion rules. As a consequence, the pairing ω has to be an automorphism of the fusion rules.¹¹ Furthermore, the locality constraint requires that in addition the automorphism of the fusion rules does not change the conformal weight modulo integers of the primary fields. This observation constitutes a natural origin for the appearance of different modular invariants and puts the general result [76] about the possible structure of modular invariants for theories with maximally extended chiral algebras in its natural context.

2.4.5 Fusion rule automorphisms

We have seen that the factorization constraint requires that the pairing ω must be an *automorphism of the fusion rules*. This means that ω satisfies

$$\mathcal{N}_{\omega(\lambda), \omega(\mu)}^{\omega(\nu)} = \mathcal{N}_{\lambda \mu}^{\nu} \quad \text{and} \quad \omega(\Omega) = \Omega. \quad (2.40)$$

¹¹ In the case of heterotic theories, analogously we need an isomorphism between the fusion rules of the two chiral theories.

Moreover, locality implies that ω must commute with the action of the modular transformation T that sends τ to $\tau+1$. These fusion rule automorphisms ω of course form a group, with the unit element provided by the identity map on Ξ . But apart from this property, the identity automorphism does not seem to play any distinguished role, in particular there is no reason to regard the corresponding *diagonal* pairing

$$\tilde{\Lambda} = \Lambda \tag{2.41}$$

as more fundamental than any non-trivial pairing. Indeed, it is typically not even clear whether two full conformal field theories that are obtained through different choices of the pairing correspond to physically distinct situations. The standard non-trivial example for a fusion rule automorphism ω of the required form is the *charge conjugation* automorphism ω_C which is given by the mapping (2.3), i.e.

$$\tilde{\Lambda} = \omega_C(\Lambda) \equiv \Lambda^+ \tag{2.42}$$

for all $\Lambda \in \Xi$. This automorphism is present for every conformal field theory (of course it coincides with the identity map when all sectors are self-conjugate). As far as we know, in all applications the charge conjugation cannot be distinguished from the identity automorphism by any physical property.

We also remark that in certain cases the fusion rule automorphism ω that is encountered in this context possesses a field theoretic realization. Namely, it can happen that ω is induced by the operation of forming the fusion product with some specific sector \mathcal{H}_J . The relevant sectors \mathcal{H}_J constitute so-called *simple currents* [77–80], which in turn are closely related [81] to automorphisms of the chiral algebra. Via this connection fusion rule automorphisms can correspond to automorphisms of \mathfrak{W} , but clearly in the situation at hand it is the automorphism of the fusion rules that matters, independently of whether it has a counterpart for the chiral algebra.¹² The identity automorphism trivially possesses such a field theoretic realization, the relevant simple current being just the vacuum sector. In contrast, the charge conjugation automorphism (when non-trivial) can never be interpreted in this manner.

Finally, as an illustration consider the theory of a single free boson. It is readily verified that in the uncompactified case, where the sectors are labelled by their charge $q \in \mathbb{R}$, the charge conjugation

$$q \mapsto -q \tag{2.43}$$

¹² In this context one should note that simple currents also play a role already in chiral conformal field theory, where they give rise to extensions of the chiral algebra \mathfrak{W} [77]. Moreover, *every* simple current corresponds to an outer automorphism of \mathfrak{W} which, however, usually does not leave the Virasoro algebra invariant. One might expect that such automorphisms show up in open conformal field theory as well; for some comments on this issue see [82].

Note that even when an automorphism ω of the fusion rules can be described in terms of some simple current J , their actions on a general sector need not coincide. For instance, the $\mathfrak{su}(2)$ WZW theory at level k has a simple current J which maps the sector with highest weight Λ (which takes values in $\Xi = \{0, 1, \dots, k\}$) to the sector $k - \Lambda$. In contrast, the automorphism of the fusion rules leaves sectors with $\Lambda \in 2\mathbb{Z}$ (i.e., integral isospin) invariant and only maps fields with odd Λ to their simple current transform $k - \Lambda$.

is in fact the only non-trivial fusion rule automorphism that preserves the conformal weights. This remains true for a compactified boson, for which the (chiral) charges lie on a one-dimensional lattice. When we have several, say d , free bosons, the situation is a bit more complicated. In the uncompactified case, where the fusion product just amounts to addition of vectors $\vec{q} \in \mathbb{R}^d$, now every invertible linear map of \mathbb{R}^d constitutes an automorphism of the fusion rules, i.e. instead of $\mathbb{R}^\times \equiv \text{GL}(1, \mathbb{R})$ they now form the group $\text{GL}(d, \mathbb{R})$. Imposing the additional requirement that the automorphism preserves the conformal weights $\Delta(\vec{q}) = \vec{q}^2/2$ of the sectors, this gets restricted to the orthogonal group $\text{O}(d)$ which generalizes $\mathbb{Z}_2 \equiv \text{O}(1)$. Finally we note that the group $\text{O}(d)$ contains in particular the element $-\mathbb{1}$. The corresponding automorphism is just the charge conjugation automorphism.

3 Open conformal field theory

3.1 The oriented cover

The presentation of our somewhat non-standard view of closed conformal field theory took quite some time. We do think that this effort is rewarding for the study of closed conformal field theory itself. But it pays off even more once we turn to the study of open conformal field theory. Namely, we will now see that once the appropriate formulation of closed conformal field theory has been achieved, the extension to open conformal field theory does not pose any major conceptual problems any more. Indeed, to formulate conformal field theory on a world sheet that has boundaries or is unoriented, we follow exactly the same steps as in the closed case, the main difference being that now the oriented cover \hat{C} of the unoriented surface C is connected. In particular, we have again a map

$$I: \hat{C} \rightarrow \hat{C} \tag{3.1}$$

which is an anti-conformal involution.

We have already presented this involution above for the examples of the crosscap and the disk. Similarly, for the annulus, the Klein bottle and the Möbius strip the oriented cover is a torus. More precisely, these three surfaces are all characterized by a modular parameter $t \in \mathbb{R}_{\geq 0}$, and the annulus can be obtained from a torus with modular parameter $\tau = it/2$ by quotienting out $I_a: z \mapsto 1 - z^*$, the Klein bottle from a torus with modular parameter $\tau = 2it$ via $I_k: z \mapsto 1 - z^* + \tau/2$, and the Möbius strip from a torus with modular parameter $\tau = (1 + it)/2$ via $I_m: z \mapsto 1 - z^*$. For a more detailed exposition of surfaces with negative Euler characteristic we refer to [21].

We also note that the oriented surfaces that arise this way as oriented covers of open or unorientable surfaces do not exhaust all possible complex curves of the appropriate genus; rather, their Teichmüller space can be embedded into the Teichmüller space of the oriented cover. The latter Teichmüller spaces are actually not only complex, but also symplectic manifolds; it is believed that the Teichmüller spaces of the open surfaces form lagrangian submanifolds in these spaces.

3.2 Bulk fields

Concerning the lift of the geometric prescription for going from C to \hat{C} to the field theoretic level, as compared to the closed case two additional features have to be taken into account. In the language of the operator formalism, these features are expressed as follows. First, to fields that are supported in the interior of C , which are called *bulk fields* [9], one has to associate again a pair of chiral vertex operators¹³ on the cover \hat{C} ; the new aspect is that now one is no longer dealing with a tensor product map as in (2.35), but rather one has to consider a product

$$\binom{\nu}{\Lambda}(z) \odot \binom{\tilde{\nu}}{\tilde{\Lambda}}(\tilde{z}) \in \text{Hom}(\mathcal{H}_\mu, \mathcal{H}_\nu)[[z, z^{-1}]] \odot \text{Hom}(\mathcal{H}_{\tilde{\mu}}, \mathcal{H}_{\tilde{\nu}})[[\tilde{z}, \tilde{z}^{-1}]] \quad (3.2)$$

that is formal in the sense that its precise meaning depends on the particular surface under consideration as well as on the possible presence of further fields and has to be made more concrete below. At first sight, this might look like a rather big difference to the operator formalism of the closed case, but in fact it is but another realization of the simple fact that the oriented cover is now a connected manifold. The lesson to be drawn from this observation is then again that we should better aim at a formulation of the chiral theory directly in terms of the blocks. In fact, we will again try to employ the concepts of block algebras and co-invariants. More precisely, the relevant modules will again be based on *ordinary* tensor products of sectors \mathcal{H}_Λ and $\mathcal{H}_{\tilde{\Lambda}}$ with some suitable pairing of Λ and $\tilde{\Lambda}$, but in distinction from the closed case the block algebra will no longer have the form of a direct sum of two subalgebras.

The second new feature arises when C has boundaries, which happens when the involution I (3.1) does not act freely any more. In this case there is an additional structure that was not present in the case of closed conformal field theory, namely the so-called *boundary fields* [9] which live on the boundary of C and accordingly correspond only to a single chiral vertex operator on \hat{C} . These objects will be discussed in more detail later; for the moment, we concentrate on the case without boundaries where this complication is absent.

In the sequel we will study chiral blocks on various surfaces. Among them, the one-point blocks are actually the most important ones, because more complicated situations can be reduced to them with the help of factorization arguments [13] that are completely analogous to those already employed in the case of closed conformal field theory. Indeed, invoking factorization, arbitrary n -point blocks on arbitrary surfaces can be expressed in terms of only three types of special building blocks: The three-point chiral blocks for the sphere (these are all that is needed in the case of closed conformal field theory) as well as, as new ingredients needed for the open case, the one-point blocks (‘tadpoles’) on the sphere and the one-point blocks on the crosscap.

In the next subsections we will study the chiral conformal field theory on \hat{C} in some detail. Afterwards, in subsection 3.7 we move on to the full theory on C . Let us stress that in particular the one-point blocks on the crosscap and on the disk that we present

¹³ The use of the oriented cover with two pre-images for points in the bulk is reminiscent of the method of mirror charges that is employed to deal with other problems with boundary conditions, e.g. in electrodynamics.

in subsections 3.3 and 3.6 are *not* the physical one-point functions, but rather the latter are to be obtained as suitable multiples of the former.

3.3 Blocks on the crosscap

The simplest surface without boundaries that has to be studied for open conformal field theory is the *crosscap*, i.e. the real projective space $\mathbb{P}\mathbb{R}^2$. (We speak about ‘the’ crosscap because there is no modular parameter for this surface.) The oriented cover of $\mathbb{P}\mathbb{R}^2$ is the complex projective space \mathbb{P}^1 . Thus an n -point situation on the crosscap is mapped to a $2n$ -point situation on \mathbb{P}^1 . Writing \mathbb{P}^1 as the complex plane (compactified by a point at infinity) with coordinate z , the relevant involution I is given by I_c (2.28), i.e.

$$I_c(z) = -\frac{1}{z^*}. \quad (3.3)$$

A fundamental domain for the action of I is given by the disk $|z| \leq 1$ with identification of diametrically opposite points of the circle $|z| = 1$.

Our aim is to compute the one-point blocks for a bulk field with sector labels Λ and $\tilde{\Lambda}$. This situation corresponds to a two-point situation on \mathbb{P}^1 , with sectors \mathcal{H}_Λ and $\mathcal{H}_{\tilde{\Lambda}}$. In the spirit of our approach to chiral blocks, we want to construct this chiral block as a co-invariant of the tensor product space $\mathcal{H}_\Lambda \otimes \mathcal{H}_{\tilde{\Lambda}}$ with respect to an appropriate action of some block algebra. But before doing so, let us first mention what we expect from a more heuristic viewpoint. Namely, from the knowledge about two-point blocks on \mathbb{P}^1 in the case of closed conformal field theory, one expects immediately that the one-point block on the crosscap is non-zero if and only if

$$\tilde{\Lambda} = \Lambda^+. \quad (3.4)$$

To investigate this issue in a more rigorous manner, let us first specialize to the case of WZW models. In this case we can definitely apply the language of co-invariants to the study of the modules $\mathcal{H}_\Lambda \otimes \mathcal{H}_{\tilde{\Lambda}}$. For simplicity let us assume in addition that the insertions are at $z_1 = 0$ and $z_2 = I_c(z_1) = \infty$. Moreover, let us for the moment restrict our attention to trivial pairing $\tilde{\Lambda} \equiv \omega(\Lambda) = \Lambda$ (much of our discussion will, however, translate with only minor changes to the case of general pairing ω). Then the block algebra consists of elements of the form $\bar{x} \otimes f$ with $\bar{x} \in \bar{\mathfrak{g}}$ and f a function with poles only at 0 and ∞ , and hence is spanned by the elements $x := \bar{x} \otimes z^n$ with $\bar{x} \in \bar{\mathfrak{g}}$ and $n \in \mathbb{Z}$. The local expansion of x at 0 is $\bar{x} \otimes \xi_1^n$ with ξ_1 the local coordinate at 0. To find the local expansion at ∞ , we first realize that the antiholomorphic local coordinate is $-1/z^*$, so the correct holomorphic coordinate is $\xi_2 = -1/z$ [16]; as a consequence, the local expansion of x is $\bar{x} \otimes (-1)^n \xi_2^{-n}$. Upon identifying the local coordinates $\xi_{1,2}$ with the indeterminate t of the loop construction (just as we did in (2.21)), we then conclude that we must consider co-invariants of $\mathcal{H}_\Lambda \otimes \mathcal{H}_{\tilde{\Lambda}}$ with respect to the action of the Lie algebra that is spanned by

$$J_n^a \otimes \mathbf{1} + (-1)^n \mathbf{1} \otimes J_{-n}^a \quad (3.5)$$

with $n \in \mathbb{Z}$ and $a = 1, 2, \dots, \dim \bar{\mathfrak{g}}$. (Note that this looks somewhat similar to the formula (2.23) that we encountered in the closed case.)

Moreover, again as in the closed case, the co-invariants are generalized singlets. In particular if one had complete reducibility, one would be able to identify every vector in the space of co-invariants with some vector $|C_\Lambda\rangle$ in the tensor product $\mathcal{H}_\Lambda \otimes \mathcal{H}_{\bar{\Lambda}}$. One would then write the defining condition for co-invariants as an equation for that vector $|C_\Lambda\rangle$; introducing the short-hands

$$J_n^a \otimes \mathbf{1} =: J_n^a \quad \text{and} \quad \mathbf{1} \otimes J_n^a =: \tilde{J}_n^a, \quad (3.6)$$

this would read

$$(J_n^a + (-1)^n \tilde{J}_{-n}^a) |C_\Lambda\rangle = 0 \quad (3.7)$$

for all $a = 1, 2, \dots, \dim \bar{\mathfrak{g}}$ and all $n \in \mathbb{Z}$. In the literature the solution to these equations has been called the *crosscap state*, but since it is not a genuine state but rather a chiral block, we prefer the more accurate term crosscap one-point block, or for brevity, *crosscap block*.

Expressed in a bit more mathematical terms, the relation between the local expansions at the insertion points 0 and ∞ is given by the action of an anti-automorphism σ_c of the affine Lie algebra \mathfrak{g} that acts as¹⁴

$$\sigma_c : J_m^a \mapsto (-1)^m J_{-m}^a. \quad (3.8)$$

Via the affine Sugawara relation $L_m \propto \kappa_{ab} \sum_{n \in \mathbb{Z}} :J_n^a J_{m-n}^b:$ this extends to an anti-automorphism of the Virasoro algebra that acts as

$$L_m \mapsto -(-1)^m L_{-m}. \quad (3.9)$$

(Also, for a general chiral *Lie* algebra \mathfrak{L} , the analogous formula will read $Y_m^i \mapsto (-1)^{m+\Delta_i-1} Y_{-m}^i$.) It follows that the crosscap state has the property that it is preserved by the Virasoro algebra, in the twisted sense that

$$(L_n - (-1)^n \tilde{L}_{-n}) |C_\Lambda\rangle = 0. \quad (3.10)$$

A *formal* solution to the equation (3.7) can be given [11] for every Λ that in the chosen pairing ω gets combined with its conjugate sector, i.e. for which

$$\omega(\Lambda) = \Lambda^+; \quad (3.11)$$

since our formulas were adapted to the case where the pairing is trivial, i.e. $\omega(\Lambda) \equiv \tilde{\Lambda} = \Lambda$, this just means that we need $\Lambda = \Lambda^+$ and hence reproduces the heuristic result (3.4). This formal solution is often called an ‘Ishibashi state’. But this expression is *not* a state in the usual sense; it is not only an *infinite sum* of basis elements of the tensor product vector space $\mathcal{H}_\Lambda \otimes \mathcal{H}_{\Lambda^+}$, but, moreover, each of the terms in the sum has length one. As a consequence, the formal expression is not even contained in the completion of this tensor product vector space with respect to its standard scalar product. (Also, an interpretation

¹⁴ Actually we should think of this as an anti-automorphism of the nilpotent subalgebra \mathfrak{g}_- of the affine Lie algebra \mathfrak{g} . This extends to an anti-automorphism of the whole affine Lie algebra \mathfrak{g} provided that one also changes the sign of the central element.

as an eigendistribution to some operator with sensible conformal properties, which could serve as a conceptual explanation of the non-normalizability, does not seem to be possible. In the free boson case, a candidate for such an operator does exist, namely the Fubini-Veneziano field X itself, but this is not a proper field of the conformal field theory.)

We do not write down the explicit form of the crosscap one-point block for the WZW case, which can be found in the literature [11, 13]. More interesting for the application to string theory is the case of an (uncompactified) free boson (recall that we may think of the corresponding conformal field theory as the WZW model based on $\hat{u}(1)$). Then the chiral sectors are labelled by their charge $q \in \mathbb{R}$. Assuming again that the relation between the two labels q and \tilde{q} that are attached to a bulk field is provided by the diagonal pairing (2.41), i.e. that $\tilde{q} = q$, there is only a single sector that is paired with its conjugate, namely the vacuum sector \mathcal{H}_0 . The formal solution for the one point-blocks takes in this case the form [18]

$$|C\rangle = \exp\left(-\sum_{n>0} \frac{(-1)^n}{n} \alpha_{-n} \tilde{\alpha}_{-n}\right) |0\rangle_{\otimes} |0\rangle, \quad (3.12)$$

where the vacuum $|0\rangle$ is the highest weight state of \mathcal{H}_0 . Apart from the fact that the two sets of oscillators appear in a ‘coupled’ manner, this just has the form of a coherent state ($\alpha_n \equiv J_n$ are the Fourier modes of $J = i\partial X$).

In the operator formalism, the presence of the non-trivial anti-automorphism (3.8) translates into the inclusion of an additional operator \mathcal{O}_c which, heuristically, ‘creates a crosscap’ [18] and is to be inserted at $|z| = 1$, so that for each bulk field there is one chiral part to the right and the other chiral part to the left of this operator. The meaning of the formal product \odot that we introduced in the formula (3.2) can then be made concrete. Namely, for one-point blocks one arrives at the recipe that they are to be interpreted as (using the shorthand (2.6) for chiral vertex operators) $\langle \varphi_{\Lambda}(z) \mathcal{O}_c \varphi_{\tilde{\Lambda}}(I_c(z)) \rangle$ (for $|z| < 1$) – regarded as ordinary chiral blocks on \mathbb{P}^1 . For higher-point blocks the prescription is

$$\langle \varphi_{\Lambda_1}(z_1) \cdots \varphi_{\Lambda_n}(z_n) \mathcal{O}_c \varphi_{\tilde{\Lambda}_n}(I_c(z_n)) \cdots \varphi_{\tilde{\Lambda}_1}(I_c(z_1)) \rangle. \quad (3.13)$$

The operator \mathcal{O}_c is known as the *crosscap operator* and can be sensibly described only at the level of the full rather than the chiral theory.¹⁵ Its explicit form was displayed in [27]; it involves in particular coefficients Γ_{μ} which are determined by a system of linear equations involving fusing matrices and operator products (in the one-point case the presence of Γ_{μ} only changes the over-all normalization of the block).

3.4 Automorphism types

3.4.1 The choice of pairing

At this point it is appropriate to point out that apart from the restriction to diagonal pairing in the previous subsection – which was in fact chosen merely for keeping the presentation as simple as possible – in our discussion of open conformal field theory so far we did not bother to say anything about the way that the two labels λ and $\tilde{\lambda}$ of a bulk

¹⁵ The crosscap operator should implement the anti-automorphism (3.8). Therefore it can presumably be regarded as a twisted (anti-)intertwiner in the sense that $\mathcal{O}_c J_m^a = \sigma_c(J_m^a) \mathcal{O}_c$.

field $\phi_{\lambda, \tilde{\lambda}}$ are related. Clearly, the precise form of this relationship plays a crucial role for the theory. For instance, when in the case of a WZW theory instead of the diagonal pairing the charge conjugation pairing ω_C (2.42) is chosen, then the condition (3.7) gets replaced by

$$(J_n^a - (-1)^n \tilde{J}_{-n}^a) |C_\Lambda\rangle = 0 \quad (3.14)$$

which again possesses a solution precisely when (3.11) holds, but now that condition is satisfied for *every* λ , i.e. each bulk field $\phi_{\lambda, \tilde{\lambda}}$ possesses a non-vanishing one-point block on the crosscap.

To study this issue more systematically, we compare to the analogous situation in closed conformal field theory. In that case the relation was described (see subsection 2.4.5) by a pairing ω , which for consistency with factorization and locality was required to constitute an automorphism of the fusion rules and to be compatible with L_0 modulo integers. Surfaces with boundaries (or unorientable surfaces) generically possess moduli spaces as well, which have singular limits and therefore give rise to factorization constraints. They should take an analogous form as in the closed case. Correspondingly, from now on we will assume that the pairing

$$\omega : \lambda \mapsto \tilde{\lambda} \quad (3.15)$$

is again a fusion rule automorphism. In short, in order to fully specify the theory on the oriented cover \hat{C} it is necessary to specify an automorphism ω of the fusion rules, which then leads in particular to a definite prescription for the block algebras. Accordingly, for definiteness we will refer to the crosscap block – and more generally to the open conformal field theory in which it is computed – to be of *automorphism type* ω . The factorization constraints will in particular link the pairing for bulk fields on various surfaces. In a first step, however, we will consider a single surface in its own right.

Just as in the case of closed conformal field theory, it is not clear whether every arbitrary fusion rule automorphism can be used. Similarly to the closed case we will address this issue on two levels. We first work at the level of chiral blocks, and later we discuss the possible realization in an operator calculus. We assume that not only in the crosscap one-point situation, but in full generality, for every allowed automorphism type ω there exists a locally free sheaf (i.e., roughly, a vector bundle) \mathcal{V}_ω of chiral blocks analogous to the sheaf $\mathcal{V} \equiv \mathcal{V}_{\text{id}}$ of blocks that arises for the diagonal pairing $\tilde{\lambda} = \lambda$.

3.4.2 Implementation on chiral blocks

In this section we consider again the system of vector bundles belonging to all chiral blocks for any genus and any number of insertion points. The fact that we deal with an automorphism of the fusion rules, which means in particular that

$$\mathcal{N}_{\omega(\lambda), \omega(\mu)}^{\omega(\nu)} = \mathcal{N}_{\lambda\mu}^\nu, \quad (3.16)$$

tells us that for all possible values of g, n and $\lambda_1, \lambda_2, \dots, \lambda_n$ the rank of the vector bundle $\mathcal{V}_{\lambda_1 \dots \lambda_n}$ over the moduli space $\mathcal{M}_{g, n}$ coincides with the rank of the vector bundle $\mathcal{V}_{\omega(\lambda_1) \dots \omega(\lambda_n)}$ over the same moduli space.

To make meaningful statements, we have to postulate a bit more structure. Namely, we require that the automorphism is *implementable* in the sense that for every value of the parameters there is an associated isomorphism $\Theta \equiv \Theta_{\vec{\lambda}}$ between these vector bundles $\mathcal{V}_{\lambda_1 \dots \lambda_n}$ and $\mathcal{V}_{\omega(\lambda_1) \dots \omega(\lambda_n)}$. It is not clear under what conditions an automorphism of the fusion rules is implementable.¹⁶ Moreover, there are automorphisms of the fusion rules which can be implemented in several inequivalent ways; we will encounter examples for such automorphisms later on when we discuss the corresponding aspects in an operator calculus.

In the sequel, we will always restrict our attention to the group of implementable automorphisms, which we denote by G , counting of course inequivalent implementations separately. This group contains several interesting subgroups. First, there is the subgroup G_T of G that consists of elements that commute with the modular matrix T , i.e. $\Delta_{\omega(\lambda)} = \Delta_{\lambda} \bmod \mathbb{Z}$. As we have seen, this subgroup is the one that is relevant in closed conformal field theory. Second, we have the subgroup G_0 of G_T that consists of all elements that preserve the conformal weight exactly, i.e. $\Delta_{\omega(\lambda)} = \Delta_{\lambda}$, not just modulo integers. If we assume that the boundary conditions¹⁷ are required to preserve the conformal symmetry exactly, this subgroup G_0 is the relevant group for open conformal field theory.

The group G_0 always contains the charge conjugation automorphism as a central element, but otherwise it depends, of course, on the theory under consideration. In some cases, like e.g. the free boson, it turns out to be a Lie group, which can have different connected components. In that case, the choice of a connected component will look like a ‘topological’ choice in a space-time interpretation. In case the dimension of the Lie group G_0 is non-zero, there are continuous moduli as well. From the conformal field theory point of view, both types of moduli are on the same footing, while their geometrical interpretation might look quite different.

3.4.3 Operator formalism

Let us now make again contact to the operator formalism and assume that the chiral blocks of the theory can be understood in terms of matrix elements of chiral vertex operators. We call an automorphism *implementable* at the operator level if there exists a family

$$\theta_{\omega}^{(\lambda)} : \mathcal{H}_{\lambda} \rightarrow \mathcal{H}_{\omega(\lambda)} \quad (3.17)$$

of maps that is consistent with the chiral block structure. In the case of WZW theories, where we can describe chiral blocks by means of co-invariants, compatibility with the

¹⁶ On the other hand, it can even happen that a fusion rule automorphism can be pulled back to the chiral algebra \mathfrak{W} . As we will see shortly, this is the case for all automorphisms that are implementable in an operator calculus. A special example is given by charge conjugation for the free boson, which corresponds to $J_n \mapsto -J_n$. In this context we remark that in the literature other proposals for a classification of branes have been made (for certain specific classes of conformal field theories) [83, 84] in which the guiding principle is an automorphism of the chiral algebra instead of a fusion rule automorphism. As we will see, an automorphism of the chiral algebra alone does not provide enough information to specify a boundary condition. Accordingly we do not expect that those ideas can be extended to generic conformal field theories.

¹⁷ This concept will be introduced in subsection 3.5 and be studied in great detail thereafter.

chiral block structure means that on all finite tensor products

$$\mathcal{H}_{\vec{\Lambda}} = \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2} \otimes \cdots \otimes \mathcal{H}_{\Lambda_n} \quad (3.18)$$

of irreducible highest weight \mathfrak{g} -modules the map

$$\theta_{\omega}^{(\vec{\Lambda})} := \bigotimes_{i=1}^n \theta_{\omega}^{(\Lambda_i)} \quad (3.19)$$

factorizes to the co-invariants, or more precisely, to *all* co-invariants for *any* genus. Clearly, every automorphism that is implementable at the operator level induces an implementable automorphism of the system of conformal blocks. In contrast, even if a theory admits an operator calculus, the converse statement is highly non-trivial. (As an aside, we remark that the analogous maps on modules that are induced by simple currents J_i , respectively by the associated automorphisms of the chiral algebra [81], can be implemented on the chiral blocks only if the product of the simple currents is the identity, $J_1 \star J_2 \star \cdots \star J_n = \mathbf{1}$ [85]. This illustrates once again that in the present context the basic structure is given by automorphisms of fusion rules, not by automorphisms of the chiral algebra.)

In the language of chiral vertex operators, the implementability of the map (3.17) should amount to the statement that the diagram

$$\begin{array}{ccc} \mathcal{H}_{\mu} & \xrightarrow{\binom{\nu \ \mu}{\lambda}(v; z)} & \mathcal{H}_{\nu} \\ \theta_{\omega}^{(\mu)} \downarrow & & \downarrow \theta_{\omega}^{(\nu)} \\ \mathcal{H}_{\omega(\mu)} & \xrightarrow{\binom{\omega(\nu) \ \omega(\mu)}{\omega(\lambda)}(\theta_{\omega}^{(\lambda)} v; z)} & \mathcal{H}_{\omega(\nu)} \end{array} \quad (3.20)$$

commutes for every $v \in \mathcal{H}_{\lambda}$, which in turn will imply that conformal blocks are mapped to conformal blocks.

Actually, it is not difficult to see that the knowledge of the chiral blocks for three insertions of the vacuum sector on \mathbb{P}^1 for all choices of insertion points and local coordinates around these points is equivalent to the complete knowledge of the chiral algebra of the theory. (Notice that as a vector space the vacuum sector \mathcal{H}_{Ω} is just the vector space underlying the chiral algebra itself.) As a consequence, the fact that an implementable automorphism θ_{ω} preserves this vacuum three-point block implies that

$$\theta_{\omega} : \mathcal{H}_{\Omega} \rightarrow \mathcal{H}_{\Omega} \quad (3.21)$$

even constitutes an automorphism of the chiral algebra.

We are now finally in a position to give an example for an automorphism of the fusion rules that can be implemented in two inequivalent ways. To this end we consider the identity automorphism of the fusion rules of a chiral conformal field theory for which each sector is self-conjugate, but in which some sectors have a negative Frobenius–Schur indicator. (Such sectors should be thought of as analogues of symplectic (quasi-real) representations in the theory of Lie algebras. For more information about the Frobenius–Schur

indicator see [86].) On the modules belonging to these primary fields, the implementing map Θ can be chosen to be either the identity or the natural involution that is provided by the symplectic form. Another class of examples is provided by conformal field theories with fixed points of simple current actions [14, 75]; on the fixed point labels the action of the automorphism of the fusion rules is trivial, but again Θ can be chosen to be non-trivial.

These examples also nicely illustrate that an automorphism of the chiral algebra is not a sufficient datum to specify an automorphism type. Namely, the associated automorphism of the chiral algebra is the identity map in both cases. However, the extension of the automorphism to sectors other than the vacuum contains non-trivial information that cannot be reconstructed from the automorphism of the chiral algebra alone. In particular, the continuation of the automorphism to non-trivial sectors is not by simply twisting the representation R_Λ of the chiral algebra on the sector \mathcal{H}_Λ by the automorphism, i.e. considering the action $R_{\omega(\Lambda)} \circ \theta_\omega^{(\Lambda)}$ on the same vector space that underlies \mathcal{H}_Λ . All these aspects are somewhat hidden in the case of the free boson, because in that case the only information about the algebra of chiral vertex operators that is not already fixed by the chiral algebra can be encoded in the zero mode of the Fubini–Veneziano field $X(z)$.

3.4.4 Free bosons

Let us now illustrate these points in the example of the free boson. In this case we have a primary field for every $q \in \mathbb{R}$. Since the fusion rules just realize charge conservation, G is the group of all non-zero real numbers α , where the group operation is multiplication; it acts as $q \mapsto \alpha q$ with $\alpha \neq 0$. Since the conformal weight is $\Delta_q = q^2/2$, the subgroup G_T contains just two elements corresponding to $\alpha = \pm 1$. Thus there are two automorphism types; they turn out to correspond to Dirichlet and Neumann boundary conditions for the free boson X . The Neumann condition amounts to $\partial X = \tilde{\partial} X$, which means that momentum is conserved; in contrast, in the Dirichlet case one deals with a brane that ‘carries momentum’. Sometimes therefore the D-brane blocks are regarded as eigenstates of the momentum operator and are therefore called ‘delocalized’ D-brane states [7]. Heuristically, one would look for a kind of Fourier transformation to find ‘D-brane states’ that have a sharp position. We will see below how this is afforded naturally in our formalism on the level of full conformal field theory and how to choose the appropriate linear combinations of D-brane blocks.

Next we consider the theory of d free bosons. The primary fields of the chiral conformal field theory are labelled by their charges, which as already discussed at the end of subsection 2.4.5, now constitute a vector q in \mathbb{R}^d . The fusion product is addition in \mathbb{R}^d , and the group of automorphisms of the fusion rules is $G = \text{GL}(d, \mathbb{R})$. As a consequence, we can consider crosscap blocks $|C\rangle$, respectively the analogous objects for the disk, the so-called boundary blocks $|B\rangle$ (see subsection 3.6 below), that are invariant under the action of the relevant block algebra. In the case of $|B\rangle$ this leads to the equation

$$(\partial X^i - \sum_{j=1}^d M_j^i \tilde{\partial} X^j)|B\rangle = 0 \quad \text{for all } i = 1, 2, \dots, d, \quad (3.22)$$

where $M \in \text{GL}(d, \mathbb{R})$. The subgroup of automorphisms that preserve L_0 is $G_0 = \text{O}(d)$. One can check that in geometric terms the choice of M corresponds to choosing the dimension of a D-brane and a field strength on the brane.

In the case of free bosons, chiral vertex operators can be given explicitly using the string coordinates X^i and their derivatives. Now of course operators living on disconnected sheets of the oriented cover commute. In the case of open surfaces, however, the oriented cover is connected, and hence all chiral vertex operators must be concatenated. This is sometimes expressed by saying that in the closed orientable case (i.e. for bi-blocks) there are two independent sets $\alpha_n \equiv J_n$ and $\tilde{\alpha}_n \equiv \tilde{J}_n$ of oscillators, while for all other surfaces there is just one set of oscillators.

The action of the charge conjugation pairing on the chiral vertex operators is given by $e^{iqX(z)} \mapsto e^{-iqX(z)}$ for a single free boson. For d free bosons, the general automorphism is

$$e^{iq \cdot X(z)} \mapsto e^{i(Mq) \cdot X(z)}, \quad (3.23)$$

where $M \in \text{O}(d)$. The eigenspaces of M to the eigenvalue $+1$ give the directions in which one has Neumann boundary conditions. Eigenvalues -1 correspond to Dirichlet boundary conditions, while the rest corresponds to a field strength on the world volume of the brane. As an example, the matrix $\text{diag}((+1)^{p+1}, (-1)^{d-p-1}) \in \text{O}(d)$ corresponds to a Dirichlet p -brane for which the field strength on the $p+1$ -dimensional world volume of the brane vanishes.

It is common to rewrite

$$(Mq) \cdot X(z) = q \cdot (M^T X(z)) \quad (3.24)$$

so that the set of all transformations (3.23) (for all relevant values of q) just reduces to a single map

$$X(z) \mapsto MX(z). \quad (3.25)$$

In the case of a general conformal field theory it will no longer be possible to encode the fusion rule automorphism in such a simple formula, because no analogue of the quantity $X(z)$ (which is not a genuine conformal field) is available any longer.

3.4.5 Compatibility

Up to this point, we have only discussed the pairing ω on a single given surface. We now address the additional aspects that must be taken into account when one requires consistent factorization of all blocks on arbitrary surfaces. Just like in the case of closed conformal field theory, only after imposing this constraint we can sensibly talk about the ‘same’ conformal field theory simultaneously on all surfaces.

In this context an important observation is that given a definite torus partition function, non-trivial solutions for the one-point blocks will exist only for a subset of the bulk fields, and it will depend on the choice of the automorphism type what this subset looks like. In particular, not any automorphism type will allow for solutions for sectors other than the vacuum sector. More precisely, once we have fixed both the torus partition function and the automorphism ω , we can study for which values of λ the tensor product

$\mathcal{H}_\lambda \otimes \mathcal{H}_{\tilde{\lambda}}$ possesses any co-invariants for the relevant block algebra. In the case of the crosscap with trivial automorphism type, for WZW theories this is just the Lie algebra spanned by the modes (3.5), and we can immediately conclude that there exists at least one λ for which there is a (formal) solution, namely $\lambda = \Omega$, simply because the combination (Ω, Ω) is contained in every sensible torus partition function and every automorphism of the fusion rules leaves the vacuum invariant, $\omega(\Omega) = \Omega$. But apart from this special case the existence of solutions is not guaranteed. (Similarly, for generic $M \in O(d)$ the equation (3.22) will not have solutions other than $q = \tilde{q} = 0$.) On the other hand, it turns out that when the torus partition function is given by charge conjugation and the automorphism ω is the charge conjugation automorphism as well, then a solution exists for every λ .

As a side remark, we note that the element $-\mathbb{1}$ of $O(d)$ allows to flip between (generalized) Dirichlet and (generalized) Neumann boundary conditions because it changes the relative sign of left movers and right movers. Now recall from subsection 2.4.4 that we can restrict our attention to theories for which the torus partition function is given by a fusion rule automorphism π . We will sometimes abuse terminology and employ the terms *Neumann* and *Dirichlet* automorphism type to refer to the situation where π preserves conformal weights and where the pairing ω coincides with π and with its ‘charge conjugate’ $\pi \circ \omega_C$, respectively.

Finally we note that the precise form of the co-invariants will of course depend on the chosen automorphism type, too. Consider for instance again the case of a single free boson. Denoting the highest weight state of \mathcal{H}_q by $|q\rangle$, for Neumann automorphism type, i.e. $\tilde{q} = q$, the crosscap one-point blocks read

$$|C_q\rangle = \exp\left(-\sum_{n>0} \frac{(-1)^n}{n} \alpha_{-n} \tilde{\alpha}_{-n}\right) |q\rangle \otimes |-q\rangle \quad (3.26)$$

(which of course includes (3.12) as a special case), while for Dirichlet automorphism type, i.e. $\tilde{q} = q^+ \equiv -q$, they are

$$|C_q\rangle_D = \exp\left(+\sum_{n>0} \frac{(-1)^n}{n} \alpha_{-n} \tilde{\alpha}_{-n}\right) |q\rangle \otimes |q\rangle. \quad (3.27)$$

Moreover, according to the previous remarks in the case of the diagonal torus partition function the Neumann block (3.26) exists only for $q = 0$ while the Dirichlet block (3.27) exists for arbitrary charge $q \in \mathbb{R}$, and in the case of the charge conjugation torus partition function (which is just the T-dual of the diagonal partition function that is usually chosen to describe the uncompactified free boson) the situation is reversed.

3.4.6 Compactified free bosons

To illustrate these considerations, we consider again a system of d free bosons, this time compactified on some d -dimensional torus T^d . On this torus, we choose a basis $\{e_i\}$ and denote by $\{e_i^*\}$ a dual basis. Such a compactification is characterized by the metric $g_{ij} := (e_i, e_j)$ and an antisymmetric tensor B_{ij} . The torus partition function relates the

left and right moving charges $q_{L,R}$ according to

$$q_{L,R} = \sum_{i=1}^d (m_i e_i^* \pm \frac{1}{2} n_i e_i) - \frac{1}{2} \sum_{i,j=1}^d B_{ij} e_i^* n_j, \quad (3.28)$$

where n_i and m_i are integers. The vectors $q_L \oplus q_R$ form a self-dual sublattice $\Gamma(g, B)$ of $\mathbb{R}^{n,n}$ (i.e., of \mathbb{R}^{2n} with signature (n, n)). In string theory terms, m_i is a momentum number and n_i a winding number; the background field B couples to the winding.

We now study a fixed automorphism type, described by some orthogonal matrix M . To find out which bulk fields will lead to boundary blocks, we consider the d -dimensional subspace

$$\mathcal{D}_M := \{(q, Mq)\} \quad (3.29)$$

of \mathbb{R}^{2d} . For a chosen background g, B , the set of those pairs of left and right charges for which there are non-vanishing one-point blocks for the automorphism type specified by M is given by the sublattice

$$\Gamma(g, B, M) := \Gamma(g, B) \cap \mathcal{D}_M \quad (3.30)$$

of $\Gamma(g, B)$. For fixed g, B the rank of $\Gamma(g, B, M)$ depends on M ; generically it is zero. For example, for Neumann automorphism type, i.e. $M = \mathbb{1}$, only for discrete values of the Kalb–Ramond background field B non-trivial one-point blocks exist at all.¹⁸

We conclude this subsection with a brief comparison to the geometric description of branes. At the level of chiral conformal field theory, only aspects of the topology of the brane (in particular its dimensionality) and the field strength on it enter. Other features like e.g. the value of the moduli of the world volume of the brane will only show up on the level of full conformal field theory, see subsection 3.8.3 below.

3.5 Boundary conditions and boundary fields

Having clarified the structure of chiral conformal field theory for the case of the crosscap, we now proceed to study what happens when we allow for the surface C to have boundaries. To start, we still restrict our attention to bulk fields. As pointed out in subsection 3.1, due to factorization the quantities of prime interest are the one-point correlation functions of the bulk fields.

Now when imposing the various consistency conditions detailed in subsection 2.4.4, notably the integrality constraint for the partition function on the annulus, there will be severe restrictions on the possible one-point functions. As it turns out, these restrictions typically allow not only for one, but for several solutions. (This is in sharp contradistinction to the closed case, where the locality and factorization constraints are believed to possess a unique solution.) To every such solution, i.e. to every consistent collection of one-point correlation functions for all bulk fields, one commonly associates a corresponding *boundary condition*.

¹⁸ This quantization of B plays is crucial in the description of type I duals of the CHL string [87].

Except for the case of free fields, where one just deals with boundary conditions in the usual geometric sense, it is a priori not clear how to translate the notion of boundary condition that is introduced this way directly to a prescription on how the fields behave when they come close to the boundary. In fact, this is an issue that concerns the full rather than the chiral conformal field theory, and accordingly will be studied in more detail in subsection 3.7 below. But already at this point we can say that in the operator formalism the appropriate characterization of a boundary condition is [9] as a collection of certain *reflection coefficients*, whose conceptual status is similar to the one of operator product coefficients.

Roughly, one can imagine that the boundary carries some kind of ‘charge’. The bulk fields ‘feel’ this charge when they approach the boundary. But even without invoking an operator formalism for the full nor even for the chiral theory, we can already be more specific at the level of the chiral blocks. Namely, just like in the case of the crosscap, we have the freedom to choose an automorphism type that fixes the pairing of the bulk labels, and certainly the blocks will depend on the chosen automorphism type. Thus the boundary condition includes in particular a definite prescription for the automorphism type that is to be chosen, i.e. is to some extent specified by a label ω . In addition, however, the boundary condition may contain further information, corresponding to some other label for which we generically use the symbol a . Accordingly, we should think of a boundary condition as a *pair* of labels ω and a .¹⁹

Since the presence of the label a is tied to the very existence of boundaries, one may refer to the corresponding freedom as the *boundary type*. On the other hand, in order to be close to conventional terminology, we continue to refer to the pair consisting of both ω and a as a boundary condition, and accordingly the term boundary type may be slightly misleading. Therefore we prefer to use a different term, namely *Chan–Paton type*; this is inspired by the role that these quantities will play in string theory, where to every pair (ω, a) one can associate an independent so-called Chan–Paton multiplicity (see subsection 4.2 below). Of course, when we restrict our attention to ordinary Neumann automorphism type (which is the situation usually considered in the conformal field theory literature, e.g. in [9, 13]), we may wish to suppress the label for the automorphism type and speak of the boundary condition and the corresponding Chan–Paton type a interchangeably.

Note that whereas we have a quite clear prescription that tells us what the various possible automorphism types are, so far we have been somewhat loose concerning the possible values of the Chan–Paton type a . Indeed we do not want to study this issue in any detail here; rather, this will be done in subsection 3.8, yielding the concept [75] of a *classifying algebra*.

Next we turn our attention to boundary fields, which we commonly denote by $\Psi(x)$. The distinguished feature of boundary fields is that they ‘live’ on the boundary of the surface C [9], i.e. x is a coordinate on ∂C . Upon lifting to the oriented cover \hat{C} , boundary fields should therefore correspond to a single chiral vertex operator, and accordingly they carry a single sector label Λ . (In string theory terms, where the bulk fields – more precisely, Virasoro-primary bulk fields of conformal dimension 1 – correspond to the vertex operators for closed strings, boundary fields correspond to the vertex operators for open

¹⁹ Similar ideas have been expressed in the recent paper [31].

strings.²⁰) In addition, however, since they are confined to the boundary, the boundary fields will possess an intrinsic dependence on the boundary condition that is attached to the connected component of $\partial\mathcal{C}$ on which they are inserted, and accordingly they will also carry a corresponding label $A \equiv (\omega, a)$. More precisely, in fact there must be *two* such labels. The reason is that the boundary fields also play the role [9] of effecting a change of boundary conditions, say from A to B . (Such changes of boundary conditions indeed occur not only in string theory, but also in various situations that are of interest in condensed matter physics, see e.g. [1] and the literature cited there.) Thus we should describe boundary fields as

$$\Psi(y) = \Psi_{\Lambda}^{AB}(y) \quad (3.31)$$

when we have boundary conditions A for $x < y$ and B for $x > y$, respectively.

Note that an immediate consequence of this description is that besides the boundary fields that change only the Chan–Paton type a which have been studied e.g. in [9, 13, 14], there must in fact also exist boundary operators that change the automorphism type. (Incidentally, such ‘topology changing’ operators are also implicit in [88].) Geometrically, this corresponds to the situation that a boundary of the world sheet jumps from one brane to the other, at a point where the world volumes of the branes intersect.

Finally we note that we have not been too specific about the label Λ that is attached to boundary fields. Below we will usually assume that Λ already appears as a sector label of the bulk theory, i.e. that $\Lambda \in \Xi$. But in principle more general choices can lead to consistent theories as well. Roughly speaking, if we use a chiral conformal field theory \mathcal{C} to construct chiral blocks on surfaces without boundaries (including unorientable surfaces), we can choose for a given automorphism type ω a conformal field theory $\tilde{\mathcal{C}}$ with a ‘bigger’ fusion algebra to construct chiral blocks in the presence of boundaries. The chiral theory $\tilde{\mathcal{C}}$ cannot be chosen independently from \mathcal{C} , though. Namely, consider in the space \mathcal{H} of chiral states of \mathcal{C} the subspace $\mathcal{H}^{(\omega)}$ of those states which are allowed as bulk fields for the automorphism type ω . We then require that one can embed $\mathcal{H}^{(\omega)}$ into $\tilde{\mathcal{H}}$ and that $\tilde{\mathcal{C}}$ has some implementable automorphism $\tilde{\omega}$ of the fusion rules that prolongs ω under the embedding. In short, one imposes an additional projection in the bulk as compared to the boundary.

As an example, we consider again the free boson compactified on a circle of radius R . For $\tilde{\mathcal{C}}$ we choose the free boson compactified on a circle of radius ℓR , where ℓ is a natural number. The Dirichlet blocks of $\tilde{\mathcal{C}}$ are given by $|B_q\rangle_{\text{D}}$, defined as in (3.38) below, with $q = n/\ell R$ (for more details, see subsection 3.8.3). The embedding of the D-allowed bulk fields of \mathcal{C} into the D-allowed bulk fields of $\tilde{\mathcal{C}}$ is obvious, and these are all obvious embeddings of D-allowed sectors of a free boson into D-allowed sectors of another free boson. We want to argue that this situation corresponds to a brane that wraps ℓ times around the circle. First, we observe that the boundary conditions in a standard theory based on $\tilde{\mathcal{C}}$ are given by $\tilde{a} \bmod 2\pi\ell R$. As a consequence, the range for the theory $\tilde{\mathcal{C}}$ wraps ℓ times around the range of the theory \mathcal{C} . However, the bulk cannot probe these details:

²⁰ Recall that in the free boson case the primary chiral vertex operators are nothing but the ordinary vertex operators $\bullet e^{i\lambda \cdot X(z)} \bullet$.

its momenta are only $q \in \mathbb{Z}/R$. As a consequence,

$$e^{iq(\tilde{a}+2\pi R)} = e^{iq\tilde{a}}, \quad (3.32)$$

and since we probe the localization with bulk fields (we use left and right movers), \tilde{a} and $\tilde{a}+2\pi R$ are localized at the same point. But still, these are different boundary conditions: we have different 3-point functions for three insertions on the boundary, and on the boundary fields with different conformal weights appear.

3.6 Blocks on the disk

We now briefly discuss chiral one-point blocks for bulk fields on the disk. Like for the crosscap, the oriented cover of the disk is \mathbb{P}^1 , which we write again as the complex plane plus a point at infinity, so that the relevant anti-conformal involution is given by I_d (2.27), i.e.

$$I_d(z) = \frac{1}{z^*}. \quad (3.33)$$

Just as for the crosscap there is no modular parameter for the disk.²¹

We first restrict our attention to Neumann automorphism type. Then the analysis of the one-point blocks parallels the analogous derivation for the crosscap in subsection 3.3. The only difference is that in place of the anti-automorphism σ_c we have to use a corresponding anti-automorphism σ_d . This differs from σ_c just in the omission of the factor $(-1)^m$ that is a sign of the non-orientability of $\mathbb{P}\mathbb{R}^2$. Thus e.g. in the case of WZW theories it reads

$$\sigma_d : J_m^a \mapsto J_{-m}^a \quad (3.34)$$

instead of (3.8), for the Virasoro algebra one has the corresponding analogue of (3.9), and similarly for other chiral algebras.

Just like in the case of the crosscap, the one-point block is constructed as a co-invariant of the tensor product $\mathcal{H}_\Lambda \otimes \mathcal{H}_{\tilde{\Lambda}}$ with respect to the action of the block algebra. And just like in that case it is common to ignore the fact that this tensor product isn't fully reducible as a module over the block algebra, and correspondingly use the notation $|\mathbb{B}_\Lambda\rangle$ for the one-point block and write the defining condition of the co-invariants as

$$(J_n^a + \tilde{J}_{-n}^a)|\mathbb{B}_\Lambda\rangle = 0 = (L_n - \tilde{L}_{-n})|\mathbb{B}_\Lambda\rangle \quad (3.35)$$

analogous to (3.7) and (3.10). Clearly, a formal solution to these conditions is obtained from the one for $|\mathbb{C}_\Lambda\rangle$ by simply removing the appropriate factors of $(-1)^n$. For instance in the case of a single free boson, instead of (3.12) one now has

$$|\mathbb{B}\rangle = \exp\left(-\sum_{n>0} \frac{1}{n} \alpha_{-n} \tilde{\alpha}_{-n}\right) |0\rangle \otimes |0\rangle, \quad (3.36)$$

²¹ Thus all disks with arbitrary values of the radius are conformally equivalent. For definiteness here the disk is always taken to have unit radius. To describe a disk of arbitrary radius R , one simply would have to replace the map (3.33) by $z \mapsto R^2/z$, or equivalently, rescale the local coordinates around both pre-images of an insertion point by R .

while the analogues of the more general crosscap blocks (3.26) and (3.27) read

$$|B_q\rangle = \exp\left(-\sum_{n>0} \frac{1}{n} \alpha_{-n} \tilde{\alpha}_{-n}\right) |q\rangle \otimes |-q\rangle \quad (3.37)$$

and

$$|D_q\rangle \equiv |B_q\rangle_D = \exp\left(+\sum_{n>0} \frac{1}{n} \alpha_{-n} \tilde{\alpha}_{-n}\right) |q\rangle \otimes |q\rangle, \quad (3.38)$$

respectively.

In the literature this formal solution is known as a boundary state (or also Ishibashi state); but we stress again that it has the status of a *chiral block* and accordingly prefer to call $|B\rangle$ a *boundary block*.

The boundary block (3.36) is the one-point block for the vacuum bulk field. Depending on the chosen pairing, the conditions (3.35) may or may not possess a (formal) solution also for other bulk fields. Moreover, one can perform an analogous analysis also for other automorphism types than the Neumann type, in which case there is a different (action of the) block algebra, leading to generalized boundary blocks $|B\rangle_\omega$. In the special case of Dirichlet automorphism type, and also more generally when they correspond to brane configurations in string theory, these are often denoted by $|D\rangle$ and referred to as *brane states*. These issues can again be treated completely parallel to the corresponding discussion in the crosscap case, and we will not repeat this here. What *is* different to the crosscap case is that for a given collection of *one-point blocks* (with fixed automorphism type) there typically exist several distinct consistent collections of *one-point correlators*, namely precisely one for each Chan–Paton type; this will be studied in more detail in the next subsection.

3.7 Correlators of the full theory

For the very same reasons as in the case of closed conformal field theory, in the open case the correlation functions of the full conformal field theory on C are again to be constructed as specific linear combinations of chiral blocks on \hat{C} . And again they are severely constrained by locality, factorization and integrality constraints. Since the oriented cover is now connected, the blocks can no longer be written in the form of bi-blocks, though.

In particular, the one-point correlators of bulk fields $\phi_{\Lambda, \bar{\Lambda}}$ on the disk and on the crosscap are linear combinations of the relevant boundary and crosscap blocks, respectively. Now in this particular case according to the results of the previous subsections there exists only (at most) a single chiral block, and hence these one-point correlators are simply proportional to the one-point blocks. In the case of the crosscap with Neumann automorphism type, the constant of proportionality is nothing but the coefficient Γ_Λ that appears (compare the remark after (3.13)) in the crosscap operator [14]. In contrast, in the case of the disk the proportionality constant depends on the boundary condition $A \equiv (\omega, a)$. More precisely, up to the Λ -independent factor ²²

$$N_\Omega^{AA} := \langle \Psi_\Omega^{AA} \rangle \quad (3.39)$$

²² For the presence of this factor and the explanation why this normalization constant is generically different from unity we refer to [14, 82].

it is given by the *reflection coefficients* $R_{\Lambda\tilde{\Lambda};\Omega}^A$ of the bulk field with respect to the vacuum boundary field; thus the one-point correlation function of $\phi_{\Lambda,\tilde{\Lambda}}$ on the disk with boundary condition A reads

$$\langle \phi_{\Lambda,\tilde{\Lambda}} \rangle_A = N_{\Omega}^{AA} R_{\Lambda\tilde{\Lambda};\Omega}^A |B_{\Lambda}\rangle_{\omega}. \quad (3.40)$$

The reflection coefficients that show up here are usually introduced (and receive their name) in the operator formalism, namely by the expansion [13, 14]

$$\phi_{\Lambda,\tilde{\Lambda}}(re^{i\sigma}) \sim \sum_{\mu \in \Xi} \sum_{a \in \mathcal{I}_{\omega}} (r^2 - 1)^{-2\Delta_{\Lambda} + \Delta_{\mu}} R_{\Lambda\tilde{\Lambda};\mu}^A \Psi_{\mu}^{AA}(e^{i\sigma}) \quad \text{for } r \rightarrow 1 \quad (3.41)$$

in terms of boundary fields, which the bulk field should possess when $|z| \sim 1$. Thus they encode how the bulk field behaves close to a boundary component, or in other words, to what extent it ‘excites’ the boundary fields when it approaches the boundary. Note that in (3.41) it is assumed that the automorphism type ω is fixed (in particular $\tilde{\Lambda} \equiv \omega(\Lambda)$ with ω the prescribed automorphism type); the second summation in (3.41) is then over the set \mathcal{I}_{ω} of all those Chan–Paton types a for which there is a boundary condition $A = (\omega, a)$. Moreover, to be precise, the expansion (3.41) was established in [13, 14] only for the case of boundary conditions of Neumann automorphism type; here we assume that it remains true for any other automorphism type, which, employing the existence of an operator formalism, follows [31] by the same arguments as in the Neumann case. Similar remarks apply to all other relations that follow in this subsection; an argument why this assumption should be valid will be presented in the next subsection.

The formula (3.40) nicely displays the physical meaning of the boundary conditions A . Namely, they constitute a degeneracy index A that labels the various consistent collections of one-point correlators on the disk. Note that the allowed linear combinations are severely restricted by the locality and factorization requirements, and in particular by the integrality and positivity constraints for the annulus, Möbius strip and Klein bottle; these have been discussed in great detail in [13, 14]. Now whereas it is expected that the linear combinations of bi-blocks that constitute the correlators of a closed conformal field theory are uniquely determined by the various constraints, as it turns out, in the presence of boundaries there typically indeed exist several different consistent solutions, i.e. several distinct boundary conditions, even for prescribed automorphism type ω . In general, it is a rather difficult task to obtain a classification of all possible boundary conditions by studying the various constraints. As we will propose in the next subsection, there exists, however, a general structure which neatly encodes the set of all boundary conditions, namely the so-called classifying algebra.

Another quantity in which the reflection coefficients $R_{\Lambda\tilde{\Lambda};\Omega}^A$ that appear in (3.40) enter is the zero-point correlation function on the annulus, which is briefly called the annulus amplitude. To investigate this quantity, we consider the combinations

$$|B^A\rangle := \sum_{\Lambda \in \Xi} \langle \phi_{\Lambda,\tilde{\Lambda}} \rangle_A = \sum_{\Lambda \in \Xi} N_{\Omega}^{AA} R_{\Lambda\tilde{\Lambda};\Omega}^A |B_{\Lambda}\rangle_{\omega}. \quad (3.42)$$

Here and in the sequel we shall adopt the convention that the boundary blocks are normalized in such a way that

$${}_{\omega} \langle B_{\Lambda} | q^{L_0 + \tilde{L}_0 - c/12} | B_{\Lambda'} \rangle_{\omega} = \delta_{\Lambda,\Lambda'} \left(\frac{S_{\Omega\Lambda}}{S_{\Omega\Omega}} \right)^{-1} \chi_{\Lambda}(\tau), \quad (3.43)$$

where $\chi_\Lambda(\tau)$ is the Virasoro-specialized character. Note that in the case of free conformal field theories, the prefactor $S_{\Omega\Lambda}/S_{\Omega\Omega}$, the so-called quantum dimension, is equal to one. In the literature, the particular linear combinations (3.42) of boundary blocks are again often referred to as boundary states.

These boundary states do not possess any immediate physical meaning. They should rather be compared to the n -reggeon vertex that was mentioned at the end of subsection 2.3.3, for which one also considers expressions that are direct sums over all sectors of the space of chiral states. Indeed, saturating one leg of the n -reggeon vertex with a boundary state $|\mathcal{B}^A\rangle$ amounts to introducing a boundary of type A on the world sheet. Also in this case the physical correlators are to be obtained by applying suitable projections to the boundary state. For example, the classical p -brane solutions of the type II superstring can be recovered this way [7]. Using factorization, we can express the annulus amplitude as the ‘product’ of two such boundary states and one ‘propagator’:

$$\begin{aligned} A^{AB}(t) &= \langle \mathcal{B}^A | e^{-\frac{2\pi}{t}(L_0 + \tilde{L}_0 - c/12)} | \mathcal{B}^B \rangle \\ &= \sum_{\mu \in \Xi} \left(\frac{S_{\Omega\mu}}{S_{\Omega\Omega}} \right)^{-1} (B_\mu^A N_\Omega^{AA})^* \chi_\mu\left(\frac{2i}{t}\right) (B_\mu^B N_\Omega^{BB}) = \sum_{\mu \in \Xi} A_\mu^{AB} \chi_\mu\left(\frac{it}{2}\right). \end{aligned} \quad (3.44)$$

Here the second expression and the last one are related by a modular transformation. In the string context, (3.44) is the partition function for the open string states (before orientifold projection), and the second expression corresponds to the closed string channel while the last one describes the open string channel. This short calculation also nicely illustrates that due to factorization arguments we can obtain a complete overview over the boundary conditions from considerations on the disk alone.

To conclude this subsection, let us mention a few other aspects of the operator formalism [13, 14]. First there is an operator product expansion for two boundary fields; it reads ²³

$$\Psi_\lambda^{AB}(x_1) \Psi_\mu^{BC}(x_2) \sim \sum_{\nu \in \Xi} (x_1 - x_2)^{\Delta_\nu^b - \Delta_\lambda^b - \Delta_\mu^b} C_{\lambda\mu\nu}^{ABC} \Psi_\nu^{AC}(x_2) \quad \text{for } x_1 \rightarrow x_2. \quad (3.45)$$

Combining this formula with the requirement that the only boundary fields with non-vanishing one-point function are the vacuum fields Ψ_Ω^{AA} and with the relation (3.39), it follows that the two-point blocks of boundary fields read

$$\langle \Psi_\lambda^{AB}(x_1) \Psi_\mu^{BA}(x_2) \rangle = (x_1 - x_2)^{-2\Delta_\lambda^b} N_\lambda^{AB} \delta_{\lambda, \mu^+}, \quad (3.46)$$

with

$$N_\lambda^{AB} = C_{\lambda\lambda^+\Omega}^{ABA} \cdot N_\Omega^{AA}. \quad (3.47)$$

Finally we consider the sewing constraint that comes from the correlation function $\langle \phi_{\lambda, \bar{\lambda}}(z_1) \phi_{\mu, \bar{\mu}}(z_2) \Psi_\nu^{AA}(x_3) \rangle$ for two bulk fields and one boundary field. This correlator can be factorized either by first using the operator product of two bulk fields and afterwards

²³ Here Δ_λ^b , which according to (3.46) governs the decay of a boundary field two-point function along the boundary, need not coincide with the ordinary (bulk) conformal dimension Δ_λ of the sector \mathcal{H}_λ [9].

the reflection coefficient for the resulting bulk field, or else using twice the reflection coefficients. According to [13, 14] this provides in particular the relation

$$C_{\lambda\tilde{\lambda},\mu\tilde{\mu}}^{\nu\tilde{\nu}} R_{\nu\tilde{\nu};\Omega}^A N_{\Omega}^{AA} = \sum_{\kappa \in \Xi} N_{\kappa}^{AA} \epsilon_{\lambda\mu}^{\nu} e^{i\pi(\Delta_{\mu} - \tilde{\Delta}_{\mu} + \Delta_{\kappa})} R_{\lambda\tilde{\lambda};\kappa}^A R_{\mu\tilde{\mu};\kappa}^A F_{\kappa\nu}^{\mu\mu^+}[\lambda\lambda^+], \quad (3.48)$$

which besides the coefficients introduced above contains the entries of fusing matrices F and certain sign factors ϵ (the latter are related to the Frobenius–Schur [86] indicator).

3.8 Classifying algebras

3.8.1 Boundary conditions as representations of an algebra

Our goal is now to make more definite statements about the possible boundary conditions. To this end we implement the information from the conventional operator formalism for the full conformal field theory [9, 13, 14] that we collected in the previous subsection. But while we need this as a heuristic input, we expect that our conclusions in fact do not rely on the existence of an operator formalism at all and can be replaced by statements about chiral blocks and the structure of their singularities when a curve degenerates.

Specifically, we take the formula (3.48) and contract it by the inverse fusing matrix $F_{\nu\Omega}^{-1}[\lambda\lambda^+]$. This amounts to [14]

$$R_{\lambda\tilde{\lambda};\Omega}^A R_{\mu\tilde{\mu};\Omega}^A = \sum_{\nu \in \Xi} {}^{(\omega)}\mathcal{N}_{\lambda\mu}^{\nu} R_{\nu\tilde{\nu};\Omega}^A \quad (3.49)$$

with some numbers ${}^{(\omega)}\mathcal{N}_{\lambda\mu}^{\nu}$ which are combinations of operator product coefficients, entries of fusing matrices and normalization constants.

The equation (3.49) proves to be an extremely useful result. In contrast to the crosscap constraint [27] which is linear, it provides us with non-linear relations and therefore also allows to fix the normalization of the reflection coefficients.²⁴ First recall that (3.48) has been obtained in the literature [14] under the (implicit) assumption of dealing with boundary conditions of Neumann automorphism type. But as already indicated above, we expect that it holds in fact for any other automorphism type as well. As will be discussed next, the relation (3.49), gives rise to the structure of an associative algebra that looks similar to the fusion algebra which governs the behaviour of ordinary (Neumann) blocks. We regard this as an indication that the vector bundles of blocks for arbitrary automorphism type carry structures analogous to those of the Neumann blocks, such as a Knizhnik–Zamolodchikov connection and fusing and braiding properties.²⁵ Given

²⁴ Actually, the fact that three crosscap insertions are topologically equivalent to one crosscap and a handle provides an additional constraint. Namely, the crosscap coefficients Γ_{Λ} must obey a relation in which they enter both cubically and linearly. This constraint should also allow to fix their absolute normalization.

²⁵ We also expect that just like in the case of Neumann automorphism type, the analogue of the Knizhnik–Zamolodchikov connection is projectively flat and unitary. Moreover, as already pointed out above, there should be factorization rules that relate different vector bundles over different moduli spaces. The blocks should then still furnish a representation of the modular group; it would be interesting to see whether one can describe the rank of these vector bundles of blocks through a generalization of the Verlinde formula.

these structures, the derivation of (3.49) will be completely parallel to the Neumann case. Further evidence comes from the study of examples, see below.

Anyhow, we take it for granted that the relation (3.49) holds for every automorphism type, and moreover, that it holds independently of the existence of an operator formalism. We interpret the formula (3.49) as follows. It tells us that once an automorphism type ω has been prescribed, then for each fixed boundary condition $A \equiv (\omega, a)$ the reflection coefficients $R_{\Lambda\bar{\Lambda};\Omega}^A$ furnish a one-dimensional irreducible representation of an associative algebra with structure constants ${}^{(\omega)}\mathcal{N}_{\lambda\mu}{}^\nu$.²⁶ We call this algebra the *classifying algebra* for the boundary conditions of type (ω, a) with fixed automorphism type ω and denote it by \mathfrak{C}_ω . The determination of all possible boundary conditions, i.e. of all possible values of a for given automorphism type ω , is thereby reduced to the study of the representation theory of the finite-dimensional algebra \mathfrak{C}_ω .

3.8.2 Properties of \mathfrak{C}_ω

A distinguished basis of the classifying algebra \mathfrak{C}_ω is labelled by the collection of bulk fields $\phi_{\Lambda,\bar{\Lambda}}$ that for the chosen automorphism type ω have a non-vanishing one-point function on the disk (that is, a non-vanishing boundary block $|\mathbb{B}_\Lambda\rangle_\omega$). We denote the corresponding basis elements by $t_\Lambda^{(\omega)}$, so that the relations of the algebra read

$$t_\lambda^{(\omega)} t_\mu^{(\omega)} = \sum_{\nu \in \Xi} {}^{(\omega)}\mathcal{N}_{\lambda\mu}{}^\nu t_\nu^{(\omega)}. \quad (3.50)$$

Then the statement that the reflection coefficients furnish a representation π_a of \mathfrak{C}_ω simply means that

$$R_{\Lambda\bar{\Lambda};\Omega}^A = \pi_a(t_\Lambda^{(\omega)}) \quad (3.51)$$

for all allowed labels Λ .

Owing to factorization arguments, the classifying algebra \mathfrak{C}_ω can be expected to be an associative algebra over \mathbb{C} . Further properties of \mathfrak{C}_ω follow from our general picture above which relates the classifying algebra to the properties of chiral blocks with automorphism type ω . In particular, analogously as in the case of the fusion rule algebra we expect that \mathfrak{C}_ω is commutative and that the specific generator $t_\Omega^{(\omega)}$ that is associated to the vacuum sector is a unit element (in this regard, it is a nice consistency check that Ω is an allowed label for *every* automorphism type – this would no longer be true if we would deal with arbitrary automorphisms of the chiral algebra in place of fusion rule automorphisms). Moreover, the evaluation at the unit element should provide a conjugation (involutive automorphism) of the algebra, which in turn together with the other properties implies that the algebra is semi-simple. When the theory is rational, \mathfrak{C}_ω is finite-dimensional, and therefore commutativity and semi-simplicity imply that it has only one-dimensional irreducible representations, as many inequivalent ones as its dimension.

Taking these properties for granted, without loss of generality we should also be allowed to require that the (equivalence classes of) irreducible representations of \mathfrak{C}_ω are already

²⁶ Sometimes a different normalization convention for the reflection coefficients $R_{\Lambda\bar{\Lambda};\Omega}^A$ is chosen and also the factor N_Ω^{AA} (3.39) is included. The so obtained reflection coefficients have the serious disadvantage that they do not furnish a representation of a classifying algebra any more.

exhausted by those representations which according to (3.51) are provided by the various possible Chan–Paton types a . In other words, the allowed boundary conditions for given automorphism type are *precisely* the one-dimensional irreducible representations of the classifying algebra, so that in particular for a rational theory their number (which by other methods would be quite difficult to determine) is

$$|\mathcal{I}_\omega| \equiv |\{A = (\omega, a) \mid \omega \text{ fixed}\}| = \dim \mathfrak{C}_\omega. \quad (3.52)$$

In short, up to the information about the explicit form of the boundary blocks, a boundary (or D-brane) state is nothing but a mnemonic for some definite irreducible representation of the classifying algebra.

A less obvious property of the classifying algebra concerns the integrality and positivity of the structure constants ${}^{(\omega)}\mathcal{N}_{\lambda\mu}{}^\nu$ of the classifying algebra. In the case of the fusion rule algebra, the structure constants are non-negative integers because they count the dimensions of spaces of chiral blocks. This needs no longer be true for non-Neumann automorphism types ω . In the general case we would rather expect that, at least when ω is implementable in the sense of subsection 3.4.3, then in place of dimensions the structure constants will correspond to traces on the spaces of chiral blocks. This generalizes the structure of the classifying algebra that was found in [75]. We expect that these traces on the spaces of chiral blocks are related to twisted traces in the sectors \mathcal{H}_Λ (or in other words, generalized character valued indices) similar to those that were studied in [81], namely to traces of operators that involve also the maps $\theta_\omega^{(\lambda)}$ (3.17). The relation between the traces on the sectors and the traces on the spaces of chiral blocks should generalize the Verlinde formula which relates the modular transformation properties of the ordinary characters to the dimensions of the spaces of ordinary chiral blocks (which are special examples of traces, namely of the unit matrix). In the context of open conformal field theory, twisted traces have been considered in [31] (compare equation (4.28) of [31]). When the implementing maps (3.17) have order two (as is e.g. the case for the classifying algebra that was obtained in [75]), then the structure constants ${}^{(\omega)}\mathcal{N}_{\lambda\mu}{}^\nu$ will still be integers, though they are allowed to be negative; but for general order they even need not be integers any more.

As a final comment we point out that the description of boundary conditions in terms of representations of some algebraic object is actually in nice correspondence to the geometric description, say for type IIB compactifications of the superstring on a Calabi–Yau manifold \mathcal{M} . (For some background about boundary conditions in type II superstring theories see [89].) In this case, specifying Dirichlet boundary conditions amounts to the specification of a coherent sheaf on \mathcal{M} . Coherent sheaves, however, are just certain modules of the structure sheaf of the manifold. Now the structure sheaf of \mathcal{M} , i.e. the sheaf of local germs of holomorphic functions on \mathcal{M} , can be thought of as a precursor of the chiral algebra ‘before quantization’, so that also in the geometric approach the possible boundary conditions are determined by the representation theory of an appropriate algebraic object.

3.8.3 Examples

Unfortunately, though computable in principle, the quantities from which one can calculate the structure constants ${}^{(\omega)}\mathcal{N}_{\lambda\mu}{}^\nu$ in an operator framework, such as the fusing and braiding matrices and the operator product coefficients, are so far not available for a generic conformal field theory. They have only been worked out for $\mathfrak{su}(2)$ WZW theories (both with the diagonal and with non-diagonal modular invariants) and for Virasoro minimal models. It has been conjectured by Cardy [9] that (expressed with the help of the notions introduced above) the classifying algebra for a rational conformal field theory with charge conjugation modular invariant and boundary conditions of Neumann automorphism type just coincides with the fusion algebra of the chiral theory. The only case where this proposal has been verified in an explicit calculation is for $\mathfrak{su}(2)$ WZW theories [14,15], in which case as just mentioned the relevant data of the chiral conformal field theory are fully known.

But even when these data are not known explicitly, our concept of classifying algebra turns out to be very fruitful. For instance, Cardy's conjecture follows as an immediate consequence of our general picture, because for blocks of Neumann automorphism type – that is, in particular, for ordinary blocks in the case when the torus partition function is given by charge conjugation – the classifying algebra is nothing but the fusion rule algebra. Furthermore, by the same token, we also get the result that the classifying algebra for blocks of charge conjugation automorphism type is the fusion rule algebra, too, as soon as the torus partition function is the diagonal one.

As a check on the ideas presented above, we discuss a few more examples. We start with the theory of a single uncompactified free boson. Then the chiral sectors \mathcal{H}_q are labelled by the real numbers, $q \in \mathbb{R}$, and the fusion product just realizes charge conservation, $q_1 \star q_2 = q_1 + q_2$. In the torus partition function every diagonal combination (q, q) of sectors appears precisely once. There are two automorphism types which we denote by N and D; they correspond to Neumann respectively Dirichlet boundary conditions for the free boson X . As seen in subsection 3.6 there is only a single N-boundary block $|B\rangle \equiv |B_0\rangle$, and as a consequence only a single N-boundary condition. In contrast, there is a D-boundary block $|D_q\rangle$ for every $q \in \mathbb{R}$; accordingly, the ‘D-brane states’ are of the form

$$|\mathcal{D}^a\rangle = \int_{-\infty}^{\infty} dq R_{q\bar{q};\Omega}^a |D_q\rangle. \quad (3.53)$$

According to our discussion above, the classifying algebra should be the fusion algebra. Hence we have to realize the relations $R_{q_1\bar{q}_1;\Omega}^a R_{q_2\bar{q}_2;\Omega}^a = R_{q_1+q_2\bar{q}_1+\bar{q}_2;\Omega}^a$; their solutions read

$$R_{q\bar{q};\Omega}^a = e^{-iaq} \quad \text{with } a \in \mathbb{R}. \quad (3.54)$$

As we will see below, the label $a \in \mathbb{R}$ that characterizes the boundary condition can be interpreted as the position of the D-brane. By comparison with the remarks at the end of subsection 3.4.6, this means that we have recovered the last missing geometrical datum of the D-brane.

The free boson compactified on a circle of radius R can be studied in a similar way. The choice of a compactification radius is equivalent to a choice of a torus partition function for

the boson; infinite radius (i.e., the uncompactified case) corresponds to the diagonal partition function, and multiplication of the partition function with the charge conjugation matrix amounts to going from the radius R to the T-dual compactification with radius $2/R$. In all cases the fusion product, which is a chiral concept, is the same as for the diagonal partition function, i.e. just expresses charge conservation. At radius R , the fields that occur in the torus partition function have charges $(q_L, q_R) = (n/R + 2mR, n/R - 2mR)$, where the momentum number n and the winding number m take their values in the integers. It follows that there are infinitely many boundary blocks of Neumann type, since only the momentum number is required to vanish, $n = 0$, whereas the winding is arbitrary. For the case of Dirichlet boundary conditions the situation is reversed; the winding number must vanish, $m = 0$, but the brane can carry arbitrary momentum n . In short, the Neumann blocks are $|B_{2mR}\rangle$ with $m \in \mathbb{Z}$ and the Dirichlet blocks are $|D_{n/R}\rangle$ with $n \in \mathbb{Z}$, where $|B_q\rangle$ and $|D_q\rangle$ are as defined in (3.37) and (3.38), respectively. This is compatible, of course, with T-duality which interchanges momentum and winding states as well as Dirichlet and Neumann boundary conditions.

This time the classifying algebra is just the restriction of the fusion rule algebra (which is the group algebra of $\mathbb{Z} \times \mathbb{Z}$) to the allowed sectors, i.e. both in the Neumann and in the Dirichlet case \mathfrak{C} is the group algebra of \mathbb{Z} . We then obtain the formula

$$R_{nn;\Omega}^a = e^{-ian/R} \quad (3.55)$$

for the reflection coefficients; since n takes its values in the integers, a can now be restricted to lie in $\mathbb{R} \bmod 2\pi R\mathbb{Z}$. The interpretation of a (and n) depends on whether we deal with Dirichlet or Neumann boundary conditions; in the case of Neumann boundary conditions a is interpreted to come from a U(1) background gauge field, while in the case of Dirichlet boundary conditions a is identified with the position of the D-brane. This follows [90,91] from the relation

$$e^{iqX} |\mathcal{D}^a\rangle \equiv e^{iqX} \sum_{n \in \mathbb{Z}} e^{-ian/R} |D_{n/R}\rangle = \sum_{n \in \mathbb{Z}} e^{-ian/R} |D_{q+n/R}\rangle = e^{-iaq} |\mathcal{D}^a\rangle. \quad (3.56)$$

Notice that it is consistent to give q the dimension of a momentum, i.e. the inverse of a length. Equation (3.56) only makes sense if the momentum q takes its values in \mathbb{Z}/R . This is precisely the quantization of momentum on a circle of radius R .

When the square of the compactification radius is a rational number, $R^2 = 2r/s$ with r and s coprime, the free boson theory possesses further symmetries so that it becomes a rational conformal field theory, with $2rs$ sectors. One can then impose the additional requirement that the boundary conditions preserve also these new symmetries. We can show that this amounts to restricting a to be an rs -th root of unity. This can be interpreted geometrically as restricting the positions of the D-branes to the vertices of a regular rs -gon. In other words, imposing on the boundary also the rational symmetries restricts the D-brane moduli to take their values only in a subset of ‘rational’ points. This pattern is familiar from the bulk theories where typically a conformal field theory is rational only at isolated points of its moduli space.

In the more complicated case of d bosons compactified on a torus T^d (see subsection 3.4.6), the classifying algebra has to be defined on just one ‘half’ of the lattice $\Gamma(g, B, M)$

(3.30), either the left moving or the right moving part (of course, via multiplication by M both are isomorphic). For a given background g, B , this range depends on the choice of the automorphism type M . As usual, the representations of the respective classifying algebras will give rise to additional continuous moduli, the Chan–Paton types.

Note that in some of the previous examples we have successfully applied the concept of a classifying algebra successfully even to theories for which the underlying relation (3.49) was not derived originally: we used it also for (generalized) Dirichlet boundary conditions, and we have applied it also to theories that are not rational. Another extension holds for the case of non-trivial torus partition functions π . Using the explicit form of the operator product coefficients and the fusing matrices for $\mathfrak{su}(2)$ WZW theories, a classifying algebra for the case where $\omega = \omega_C$ and where the modular invariant π is of D_{odd} -type has been computed in [14, 15]. A classifying algebra for $\omega = \omega_C$ and general automorphism modular invariants π of simple current type was presented in [75]. In this case, the structure constants of the classifying algebra are still integers, but negative integers occur as well.

4 Strings and branes

This paper is mainly concerned with the structure of (open) conformal field theory. But of course, once we have established various new features of these theories, we can also draw conclusions for string theory, for which conformal field theory plays the role of describing consistent vacuum configurations. As an illustration, we present in this section a few simple applications of our results, namely scattering amplitudes in brane backgrounds, the annulus amplitude, and comments concerning the properties that must be satisfied by boundary conditions.

4.1 Tree level amplitudes in a brane background

To compute scattering amplitudes for a string theory, the general recipe is to take the correlators of the underlying conformal field theory, impose the BRST cohomology and then integrate over the moduli. We briefly illustrate this prescription for the case of the scattering amplitude for n_o open and n_c closed bosonic strings at ‘tree’ level, i.e. on the disk C . This amplitude reads

$$A(n_o, n_c) = c_{n_o, n_c} \int d\mu_x \int d\mu_z \left\langle \prod_{p=1}^{n_o} \Psi_{\mu_p}^{A_p B_p}(x_p) \prod_{q=1}^{n_c} \phi_{\lambda_q, \tilde{\lambda}_q}(z_q) \right\rangle. \quad (4.1)$$

Here the various quantities have the following meaning:

- The prefactor c_{n_o, n_c} is a normalization constant. For general conformal field theories, the complete calculation of this normalization will be a difficult task. Some details about the value of the normalization constant in the case of free bosons can be found in [29].
- $\int d\mu_x$ is an integral over the positions $x_p \in \partial C$ of the open string insertions. It already implements an ordering of these positions; e.g. when the world sheet is taken to be the

upper half-plane, it is given by

$$d\mu_x = \prod_{p=1}^{n_o} dx_p \theta(x_{p+1} - x_p), \quad (4.2)$$

while when the world sheet is the unit disk, one has to translate this formula to the corresponding coordinates on the disk via the map

$$z \mapsto \frac{i - z}{i + z}, \quad (4.3)$$

which sends e.g. $0 \mapsto 1$, $\infty \mapsto -1$, $\pm 1 \mapsto \pm i$.

■ $\int d\mu_z$ is an integral over the positions z_q of the closed string insertions, corresponding to the covering surface of the world sheet and with the Möbius invariance taken into account properly. Thus

$$d\mu_z = (\text{Vol}(\text{Möbius}))^{-1} \prod_{q=1}^{n_c} d^2 z_q. \quad (4.4)$$

■ The expectation value $\langle \dots \rangle$ is the *correlation function* of the relevant primary fields, i.e. of the boundary fields $\Psi_{\mu_p}^{A_p B_p}$ ²⁷ that correspond to the relevant on-shell states of the open strings and the bulk fields $\phi_{\lambda_q, \tilde{\lambda}_q}$ that correspond to the relevant on-shell states of the closed strings. Radial ordering of the bulk fields is implicit (and also an ordering of the boundary fields, but that is already taken care of by the measure $d\mu_x$).

Using the results of the previous sections, we can express the conformal field theory correlation function as a linear combination of chiral blocks. The correlation function depends, of course, on the set $\{(A_p, B_p)\}$ of chosen boundary conditions. More precisely, the blocks themselves depend on the chosen automorphism type ω , while the appropriate linear combination depends on the Chan–Paton type $a \in \mathcal{I}_\omega$, which is implemented by including the corresponding prefactors that consist of normalizations and reflection coefficients, analogously to (3.42). In the following we just write one of these blocks; the summation over the relevant allowed intermediate sectors (including multiplicities, which we suppress as well; compare the remarks after (2.4)) must be restored at the end. In terms of the fields, this means in particular that we have to express the bulk fields through chiral vertex operators as in (3.2), i.e. we write $\prod_{q=1}^{n_c} \phi_{\lambda_q, \tilde{\lambda}_q}(z_q) = \prod_{q=1}^{n_c} \varphi_{\lambda_q}(z_q) \odot \prod_{q=1}^{n_c} \varphi_{\tilde{\lambda}_q}(Iz_q)$. We may also specify the basis of blocks, e.g. the one for which the arguments are radially ordered. In that case the ordering in the first product is ‘opposite’ to the one in the second product, analogously as in the formula (3.13).

To evaluate the amplitude further, one may also use the Möbius transformations to fix one of the insertion points, say p_1 to $z_1 = 0$ (and hence $I(z_1)$ to ∞), so that one is left with the product of $2(n_c - 1)$ chiral vertex operators sandwiched between highest weight vectors $|\psi_{\tilde{\lambda}_1}\rangle$ and $|\psi_{\lambda_1}\rangle$. Moreover, by imposing the intertwining property

$$\begin{pmatrix} \nu & \mu \\ \Lambda \end{pmatrix}(z) = \zeta^{L_0 - 1} \begin{pmatrix} \nu & \mu \\ \Lambda \end{pmatrix}(z/\zeta) \zeta^{-L_0} \quad (4.5)$$

²⁷ The boundary labels A_p and B_p are of course not independent, but satisfy $B_p = A_{p+1}$ for $p = 1, 2, \dots, n_o - 1$ and $B_{n_o} = A_1$.

for Virasoro-primary chiral vertex operators of conformal weight $\Delta = 1$ one may scale all other positions by $1/z_{n_c}$. In the case of Neumann automorphism type and $n_o = 0$ (i.e. no boundary insertions), e.g., the chiral block then reads

$$(z_{n_c} I(z_{n_c}))^{-(n_c-2)} \langle \psi_{\tilde{\lambda}_1} | \left(\prod_{q=2}^{n_c} \varphi_{\tilde{\lambda}_q} \left(I\left(\frac{z_q}{z_{n_c}}\right) \right) \right) \left((I(z_{n_c}))^{-L_0} (z_{n_c})^{L_0} \right) \left(\prod_{q=2}^{n_c} \varphi_{\lambda_q} \left(\frac{z_q}{z_{n_c}} \right) \right) | \psi_{\lambda_1} \rangle. \quad (4.6)$$

Finally one may separate the three parts of the operator appearing here by using the state-field correspondence so as to insert twice a summation over a complete set of states. This displays nicely the over-all structure of the chiral block. Of course, when one is dealing with a generic conformal field theory, then it is a difficult task to write down the chiral block, and thereby the string scattering amplitude, more explicitly. In contrast, when one specializes to the theory of free bosons, a lot of simplifications occur, by which one can reduce the results above to the corresponding formulas for free bosons which can be found in [29].

4.2 The annulus amplitude

As a particular example, we consider the annulus vacuum-to-vacuum amplitude with Neumann automorphism type on both boundary components. This can be calculated as

$$A^{ab} = \int_0^\infty \frac{dt}{t^2} A^{ab}(t) = \int_0^\infty \frac{dt}{t^2} \sum_{\mu, \nu \in \Xi} A_\mu^{ab} S_{\mu\nu} \chi_\nu\left(\frac{i}{2t}\right) = \frac{1}{2} \int_0^\infty du \sum_{\mu, \nu \in \Xi} A_\mu^{ab} S_{\mu\nu} \chi_\nu(iu) \quad (4.7)$$

with A_μ^{ab} as defined in (3.44). This amplitude, in contrast to closed string amplitudes, is *not* finite by itself, since massless or tachyonic states lead to contributions to the characters over which the moduli integral diverges. The common strategy is then the following. After choosing a standard normalization of the moduli integrals, one adds up the integrands of the various amplitudes for all surfaces (both oriented and unoriented) that have the same Euler number, where one allows in addition for certain multiplicities, the so-called *Chan–Paton* multiplicities.²⁸ More precisely, essentially one counts each boundary condition a not just once, but N_a times, which amounts to multiplying the annulus coefficients A_μ^{ab} by factors of $N_a N_b$. One then requires that after inclusion of these multiplicities the sum of the integrands is an integrable function on the moduli space. By imposing this cancellation of the ‘tadpoles’ one determines (at least partly) the values of the Chan–Paton multiplicities. This constitutes one of the few known ways for determining which sectors have to be included in a consistent string theory. Let us point out that the determination of Chan–Paton multiplicities is a problem of string theory and cannot even be formulated in pure conformal field theory terms. (Also note that there is a priori no reason why a consistent solution should exist at all.)

²⁸ The Chan–Paton multiplicities are responsible for the gauge symmetries in open string theories, see e.g. [92, 93]. The reader may be accustomed to having only a single type of Chan–Paton label; this corresponds to the situation of uncompactified free bosons (i.e., with diagonal torus partition function). In contrast, in the generic case there will be an (essentially [26] independent) Chan–Paton multiplicity for each Chan–Paton type a , that is, once the automorphism type has been fixed, one for each allowed boundary condition.

4.3 Boundary conditions

Our next remark concerns the allowed boundary conditions. In the previous sections, we have always implicitly assumed that the boundary conditions preserve the full symmetries in the bulk (possibly in a twisted way), which was reflected by the fact that we used the full chiral algebra to define the boundary blocks. This is actually a very strong condition, and for specific applications it might be necessary to relax it. Indeed, in the application of open conformal field theory to two-dimensional critical phenomena there is typically no reason to require that the boundary preserves more symmetries than just the Virasoro algebra, and in special situations it may even be possible to dispense of the preservation of the full Virasoro algebra.

In string theory, it is usually argued that the boundary should preserve the symmetry that is gauged, i.e. the Virasoro algebra in the bosonic string, respectively its corresponding super extensions for the various types of superstrings. It seems to us, however, that this requirement is a bit too restrictive. Namely, when we work in the covariant description of string theory, we must supplement the ‘internal’ conformal field theory by a ghost system and take a BRST cohomology on it. Accordingly, the boundary blocks have to be complemented by boundary blocks in the ghost theory. When doing so, boundary states for which the (super-)Virasoro algebra is preserved only up to BRST-exact terms seem to be perfectly admissible as well. (It has been suggested [6] that this happens in the presence of non-trivial Ramond–Ramond background charges.) A similar phenomenon arises in the description of the fixed point sectors in gauged WZW theories. These sectors exist due to certain selection rules. In view of this fact, it is tempting to conjecture that a similar algebraic structure underlies the D-branes that resolve the conifold singularities; recall that at the conifold point a subset of the ‘perturbative’ states becomes infinitely massive and decouples, leading to an effective selection rule.

5 Outlook

In this paper we have established the structures in open conformal field theory that are relevant to the description of D-branes for arbitrary conformal field theory backgrounds. We do not repeat any of our results here; rather, we comment on several open problems and propose further lines of research. We start with questions on the level of chiral conformal field theory. Clearly, the system of D-brane blocks deserves further study. First, one should try to find a still more explicit characterization of the fusion rule automorphisms that lead to consistent automorphism types. Moreover, one should explore in detail to what extent the properties of ordinary chiral blocks are realized for the system of D-brane blocks as well. In particular, one should establish the existence of a Knizhnik–Zamolodchikov connection, and factorization rules as well as an analogue of the Verlinde formula should be formulated and proven. In this context we remark that in factorization constraints typically blocks of several different automorphism types are involved. It is also worth stressing that the Knizhnik–Zamolodchikov connection is a central piece of structure for chiral conformal field theory. From the existence of a generalized Knizhnik–Zamolodchikov connection one could in particular derive the existence of analogues of braiding

and fusing matrices for general automorphism types. The latter will be a crucial input in the proof of the existence of a classifying algebras for arbitrary automorphism type.

On the level of full conformal field theory, the structure of the set \mathcal{I}_ω of Chan–Paton types that are allowed for a given automorphism type ω remains to be clarified. As was argued in [27, 14], this set is in particular endowed with a conjugation; this conjugation is not unique, which leads to different possibilities for Chan–Paton groups in the corresponding string theories. A more detailed description of the set of Chan–Paton types would hopefully lead to a natural characterization of the consistent conjugations. In this context, it is a striking observation [14, 75] that in the case of non-trivial torus partition functions structures in the space of boundary conditions show up which closely resemble the ones implied by modular invariance in closed conformal field theory. Finally, also the description of D-branes that wrap a cycle more than once should be made more explicit. Recall that an n -fold wrapping of a D-brane corresponds to a vector bundle of rank n over its world volume. In particular, as we have seen, the description of those vector bundles which do not split into a direct sum of line bundles requires a careful analysis of the embedding of (subsectors of) chiral conformal field theories into some other chiral conformal field theory.

Let us also comment on a few aspects on the level of string theory. The conformal field theory structures established in this paper provide us with a description of the usual Dirichlet p -branes. Clearly, other intriguing questions are raised by the study of the other types of branes that are believed to be present in string theories, like e.g. the Neveu–Schwarz five-brane. For some recent attempts to describe open strings in the background of this brane, we refer to [94, 95].

Finally, it would be important to find a better procedure than tadpole cancellation to determine Chan–Paton multiplicities. Several theories are known [96, 97] in which these conditions do not allow for any solution, but which otherwise seem to be consistent. This problem is connected to the problem of finding a clearly formulated principle which tells us which sectors have to be included in a string theory with which multiplicities. So far, this type of question is answered mostly by invoking auxiliary arguments which cannot be formulated in a purely stringy way, like e.g. anomaly freeness of some effective field theory. Such a principle would also be a necessary prerequisite for a proof of string dualities beyond the BPS level.

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