

CERN-TH/97-357

CPH-TH-S586.1297

RI-12-97

hep-th/9712084

M-THEORY AND U-DUALITY ON T^d WITH GAUGE BACKGROUNDSN.A. Obers^a, B. Pioline^{a,b} and E. Rabinovici^{a,c}^a*Theory Division, CERN, 1211 Geneva 23, Switzerland*^b*Centre de Physique Théorique, Ecole Polytechnique,† F-91128 Palaiseau, France*^c*Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel***Abstract**

The full U-duality symmetry of toroidally compactified M-theory can only be displayed by allowing non-rectangular tori with expectation values of the gauge fields. We construct an $E_d(\mathbf{Z})$ U-duality invariant mass formula incorporating non-vanishing gauge backgrounds of the M-theory three-form \mathcal{C} . We interpret this mass formula from the point of view of the Matrix gauge theory, and identify the coupling of the three-form to the gauge theory as a topological theta term, in agreement with earlier conjectures. We give a derivation of this fact from D-brane analysis, and obtain the Matrix gauge theory description of other gauge backgrounds allowed by the Discrete Light-Cone Quantization. We further show that the conjectured extended U-duality symmetry of Matrix theory on T^d in the Discrete Light-Cone Quantization has an implementation as an action of $E_{d+1}(\mathbf{Z})$ on the BPS spectrum. Some implications for the proper interpretation of the rank N of the Matrix gauge theory are discussed.

†Laboratoire Propre du CNRS UPR A.0014.

CERN-TH/97-357, CPTH-S586.1297, RI-12-97

December 1997, revised April 1998

1. Introduction

It was suggested that toroidal compactification of M-theory [1] in the infinite momentum frame is described by a Matrix gauge theory on the T-dual torus [2, 3, 4]. This gauge theory ought to reduce to Supersymmetric Yang-Mills (SYM) theory with 16 supercharges for up to three compact directions. When more directions are compactified, several suggestions have been made on how to supplement SYM with new degrees of freedom at short distances, still avoiding the coupling to gravity [5, 6]. Non-perturbative dualities of this supersymmetric gauge theory, together with the mapping class group of the torus on which the gauge theory lives, account for the U-dualities [7] of the corresponding maximally supersymmetric type II theories. For instance, the U-duality groups $Sl(2, \mathbf{Z}) \times Sl(3, \mathbf{Z})$ of M-theory compactified on a three-torus (type II compactified on a two-torus) correspond to the electric-magnetic duality of SYM in 1+3 dimensions and the reparametrizations of the dual three-torus respectively [4] (see [8] for a relation of this duality to the membrane-fivebrane duality).

For higher dimensional compactifications, it has been shown that the electric-magnetic duality \mathbf{Z}_2 on the $T^3 \subset T^d$ fibres, together with the permutations \mathcal{S}_d of the torus directions, generates a finite group, the Weyl group¹ $\mathcal{W}(E_d) = \mathbf{Z}_2 \times \mathcal{S}_d$ of the Cremmer-Julia hidden symmetry $E_d(\mathbb{R})$ [9]. This Weyl group is the subgroup of U-duality preserving the rectangular shape of the torus and the vanishing expectation value of the M-theory gauge three-form \mathcal{C}_{IJK} . The U-duality group includes $Sl(d, \mathbf{Z}) \times SO(d-1, d-1, \mathbf{Z})$, corresponding to the mapping class group of the M-theory compactification together with the perturbative string T-duality; in the following, we shall refer to this product as $E_d(\mathbf{Z})$, although it is not known whether this subgroup is sufficient² to generate the U-duality group $E_d(\mathbb{R}) \cap Sp(28, \mathbf{Z})$ conjectured by Hull and Townsend [7] when $d \leq 7$. In particular, $E_d(\mathbf{Z})$ contains elements shifting the three-form potential \mathcal{C}_{IJK} by an integer, or acting as

¹We denote by $G \times G'$ the group obtained by taking the generators of G and G' together.

²We thank B. Julia for pointing this out.

modular transformations on the torus. The full U-duality symmetry can only be displayed by allowing such skew tori with arbitrary uniform value of the gauge potential.

In Section 2, we will extend the analysis of Ref. [9] to determine U-duality invariant mass formulae for the 1/2 BPS states of M-theory compactified on general tori with non-vanishing gauge background. Our strategy will be to first construct a T-duality invariant mass formula valid in the presence of an arbitrary $B_{ij} = \mathcal{C}_{sij}$ field, derive the action of the spectral flow $B_{ij} \rightarrow B_{ij} + \Delta B_{ij}$, and subsequently covariantize this flow to include the generators $\mathcal{C}_{IJK} \rightarrow \mathcal{C}_{IJK} + \Delta \mathcal{C}_{IJK}$ as well as the additional Borel generators that appear for $d \geq 6$. We will point out the relation with results previously obtained in the context of instanton corrections to type II string theory couplings [10].

The Matrix gauge theory corresponding to compactification on skew tori is simply a supersymmetric gauge theory defined on the dual skew tori. As already proposed in Ref. [11], the expectation value of the gauge potential on the other hand turns into a set of topological couplings on the dual torus. In the particular example of M-theory on a three-torus, this is simply the theta-angle $\theta = \mathcal{C}_{123}$, which extends the electric-magnetic duality from \mathbf{Z}_2 to $Sl(2, \mathbf{Z})$. These couplings have, however, been inferred rather than derived, and a recent argument [12, 13] allows a more systematic derivation from the D-brane action, at least for M-theory backgrounds corresponding to Ramond-Ramond (RR) potentials in the type IIA string description. In Section 3, we will translate the M-theory mass formula into the Matrix gauge theory language and show the agreement with the coupling derived from the D-brane analysis.

M(atr)ix theory still lacks a proof of eleven-dimensional Lorentz covariance to shorten its name to M-theory. In the original conjecture [2], this feature was credited to the large- N infinite-momentum limit. The much stronger Discrete Light Cone (DLC) conjecture [14], if correct, allows Lorentz invariance to be checked at finite N – or rather at finite N 's, since the non-manifest Lorentz generators mix distinct N superselection sectors. In particular,

M(atrrix) theory on T^d in the DLC should exhibit a U-duality $E_{d+1}(\mathbf{Z})$, if one assumes that U-duality is unaffected by light-like compactifications. In Section 4, we shall show that the promotion of the rank N to an ordinary charge [15] allows the existence of an $E_{d+1}(\mathbf{Z})$ action on the spectrum of BPS states. Related results have been obtained in Refs. [22, 23, 24, 25, 26].

2. M-theory BPS states and Invariant Mass Formulae

The authors of Ref. [9] have investigated the $\mathcal{W}(E_d)$ orbits of two BPS states that are required to exist in the Matrix gauge theory reducing to SYM in the infrared³ : the quantum of *flux*, with energy $P_F^- = g^2 s_I^2 / (NV_s)$, and the quantum of *momentum*, with energy $P_M^- = 1/s_I$. From the M-theory point of view, they correspond to a Kaluza-Klein excitation with mass $\mathcal{M}_F = \sqrt{P^+ P^-} = 1/R_I$, and to a membrane wrapped on a circle of the torus times the light-cone direction, yielding a particle with mass $\mathcal{M}_M = R_I R_I / l_p^3$. The generalization to skew tori is immediate:

$$\mathcal{M}_F^2 = m_I g^{IJ} m_J, \quad \mathcal{M}_M^2 = \frac{R_I^2}{l_p^6} n^I g_{IJ} n^J. \quad (2.1)$$

Here m_I describes the KK momentum, while n^I labels the cycle of T^d on which the membrane wraps. In the following, we shall describe how these mass formulae can be extended to include all states of these two U-duality multiplets and the dependence on all M-theory moduli.

³We only deviate from the notations used in Ref. [9] in that the light-cone compact radius R_{11} is now R_I ; l_p is therefore the eleven-dimensional Planck length, R_I the radii of the compactification torus, s_I the radii of the gauge theory torus, V_s its volume, g the gauge coupling.

2.1 The flux multiplet

Under electric-magnetic duality on three of the directions of the Yang-Mills torus, it has been shown that the flux quantum turns into a set of states with masses

$$\frac{1}{R_I}, \frac{R_I R_J}{l_p^3}, \frac{R_I R_J R_K R_L R_M}{l_p^6}, \frac{R_I^2 R_J R_K R_L R_M R_N R_P}{l_p^9}, \frac{R_I^2 R_J^2 R_K^2 R_L R_M R_N R_P R_Q}{l_p^{12}},$$

$$\frac{R_I^2 R_J^2 R_K^2 R_L^2 R_M^2 R_N^2 R_P R_Q}{l_p^{15}}, \frac{R_I^3 R_J^2 R_K^2 R_L^2 R_M^2 R_N^2 R_P^2 R_Q^2}{l_p^{18}},$$

starting to appear for $d = 1, 2, 5, 7, 8, 8, 8$ respectively (indices I, J , etc., are distinct). The charges labelling superposition of these states can therefore be cast into integer tensors $m_I, m^{IJ}, m^{IJKLM}, m^{I;JKLMNPQ}, m^{IJK;LMNPQRST}$, etc. where the groups of indices separated by a semi-colon are antisymmetric and no symmetry across a semi-colon is assumed. In short, the flux multiplet is described by a set of integer charges

$$m_1, m^2, m^5, m^{1;7}, m^{3;8}, m^{6;8}, m^{1;8;8},$$

where the integers label the number of indices. This yields the correct number of charges to make up the representations of E_d U-duality groups. The contribution of a given charge tensor to the total square mass is simply given by its square norm induced by the torus metric g_{IJ} , with the appropriate symmetry factor and power of l_p :

$$\mathcal{M}_F^2 = m_I g^{IJ} m_J + \frac{1}{2! l_p^6} m^{IJ} g_{IK} g_{JL} m^{KL} + \frac{1}{5! l_p^{12}} m^{IJKLM} g_{IN} g_{JP} g_{KQ} g_{LR} g_{MS} m^{NPQRS} + \dots$$

(2.2)

The mass formula is compatible with the interpretation of the m_I charge as the KK momentum along the I -th direction of the transverse torus, m^{IJ} as the wrapping number of the M-theory membrane on a two-cycle of the same torus, and m^{IJKLM} as the wrapping number of the M-theory five-brane on a five-cycle. The charge $m^{1;7}$ yields a tension of the form R_I^2/l_p^9 , corresponding to Taub-NUT gravitational monopole on the R_I direction. The higher charges are not understood at present. As in Ref. [9] we draw consequences from symmetry arguments in the hope that dynamical issues will be resolved.

T-duality invariant mass formulae

As it stands, the mass formula (2.2) is invariant under $Sl(d, \mathbf{Z})$, but not under $SO(d - 1, d - 1, \mathbf{Z})$ T-duality: it only holds when the background gauge fields vanish. In order to reinstate the dependence on \mathcal{C}_{IJK} , we first decompose the flux multiplet as a sum of T-duality irreducible representations, and couple them to the NS two-form $B_{ij} = \mathcal{C}_{sij}$. For that purpose, we choose a direction $s \in \{1, \dots, d\}$ on T^d and rewrite the mass formula (2.2) in terms of the type IIA string theory variables (g_s, l_s) , related to the M-theory variables by $R_s = l_s g_s, l_p = l_s g_s^{1/3}$:

$$\begin{aligned} \mathcal{M}_F^2 = & \left[\frac{m_s^2}{g_s^2} + (m_1)^2 \right] + \left[(m^{s1})^2 + \frac{(m^2)^2}{g_s^2} \right] + \left[\frac{(m^{s4})^2}{g_s^2} + \frac{(m^5)^2}{g_s^4} \right] \\ & + \left[\frac{(m^{s;s6})^2}{g_s^2} + \frac{(m^{s;7})^2 + (m^{1;s6})^2}{g_s^4} + \frac{(m^{1;7})^2}{g_s^6} \right] \\ & + \left[\frac{(m^{s2;s7})^2}{g_s^4} + \frac{(m^{3;s7})^2}{g_s^6} \right] + \left[\frac{(m^{s5;s7})^2}{g_s^6} + \frac{(m^{6;s7})^2}{g_s^8} \right] + \left[\frac{(m^{s;s7;s7})^2}{g_s^6} + \frac{(m^{1;s7;s7})^2}{g_s^8} \right], \end{aligned} \quad (2.3)$$

where we retained only the powers of the string coupling and the index structure. T-duality commutes with the grading in powers of g_s , so we learn that the flux multiplet decomposes as a sum of five representations:

$$\begin{aligned} V = (m_1, m^{s1}), \quad S_B = (m_s, m^2, m^{s4}, m^{s;s6}), \quad T = (m^5, m^{1;s6}, m^{s;7}, m^{s2;s7}), \\ S_A = (m^{1;7}, m^{3;s7}, m^{s5;s7}, m^{s;s7;s7}), \quad V' = (m^{6;s7}, m^{1;s7;s7}). \end{aligned}$$

The irrep V is merely a vectorial representation of $SO(d - 1, d - 1, \mathbf{Z})$, for which the mass formula is known from the the usual tori partition functions [18]:

$$\mathcal{M}_V^2 = (m_i + B_{ji} m^{sj}) g^{ik} (m_k + B_{lk} m^{sl}) + m^{si} g_{ij} m^{sj}. \quad (2.4)$$

The irrep S_B on the other hand already arose in Ref. [10] as the set of type IIB D-brane charges. It is well known that the type II RR gauge fields transform as a spinorial representation of $SO(d - 1, d - 1, \mathbf{R})$, the Clifford algebra being generated by inner and wedge

products with the torus first cohomology, and the chirality depending on the type A or B [10]. The corresponding charges therefore transform as a (conjugate) spinor, hence the notation S_B . Note that this does not imply that the states in S_B correspond to the D-branes of type II string theory, but simply that they transform in the same way. The T-duality invariant mass formula comes as a by-product of the analysis of Ref. [10]:

$$\begin{aligned} \mathcal{M}_{S_B}^2 = & \left(m_s + \frac{1}{2} B_{ij} m^{ij} + \frac{1}{2 \cdot 2^2} B_{ij} B_{kl} m^{sijkl} + \frac{1}{3! \cdot 2^3} B_{ij} B_{kl} B_{mn} m^{s;ijklmn} + \dots \right)^2 \\ & + \frac{1}{2} \left(m^{ij} + \frac{1}{2} B_{kl} m^{sklij} + \frac{1}{2 \cdot 2^2} B_{kl} B_{mn} m^{s;sklmnij} + \dots \right)^2 \\ & + \frac{1}{4!} \left(m^{sijkl} + \frac{1}{2} B_{mn} m^{s;smnijkl} + \dots \right)^2 + \frac{1}{6!} \left(m^{s;ijklmn} + \dots \right)^2 + \dots \end{aligned} \quad (2.5)$$

In the above equation, we have again dropped the metric contractions and the powers of l_s . The dots include the higher even forms arising in the reduction of the spinor of $SO(d-1, d-1, \mathbb{R})$ to antisymmetric forms of $Sl(d-1, \mathbb{R})$, but are irrelevant for $d \leq 8$. The representation T reduces to a singlet at $d-1 = 5$, when it starts appearing, and to a vector V when $d-1 = 6$ (upon dualization of the **5** and **6** indices). For $d-1 = 7$, it extends to an $SO(d-1, d-1, \mathbb{R})$ two-form together with a singlet, as is easily seen by dualizing on T^7 to $(m_2, m_1^{1s}, m^s, m^{s2;s})$. The mass formula is then obtained⁴ by tensor product from Eq. (2.4):

$$\begin{aligned} \mathcal{M}_T^2 = & \frac{1}{5!} \left(m^{ijklm} + B_{np} \left(\frac{1}{2} m^{s;npijklm} - m^{n;spijklm} \right) + \frac{1}{2} B_{np} B_{qr} m^{snq;sprijklm} \right)^2 \\ & + \frac{1}{6!} \left(m^{p;sijklmn} - B_{qr} m^{sqp;srijklmn} \right)^2 + \frac{1}{7!} \left(m^{s;ijklmnp} + \frac{1}{2} B_{qr} m^{sqr;sijklmnp} \right)^2 \\ & + \frac{1}{2 \cdot 7!} \left(m^{sqr;sijklmnp} \right)^2 . \end{aligned} \quad (2.6)$$

Finally, the irreps S_A and V' only arise for $d-1 = 7$. S_A is, after dualizing the **7** indices, a sum of odd forms of $Sl(d-1, \mathbb{R})$, and therefore a spinor representation of $SO(d-1, d-1, \mathbb{Z})$

⁴This requires a precise identification of the T-duality singlet among m^s and $\text{Tr } m_1^{1s}$.

with chirality opposite to S_B :

$$\begin{aligned} \mathcal{M}_{S_A}^2 = & \left(m^{i;7} + \frac{1}{2} B_{jk} m^{jki;s7} + \frac{1}{8} B_{jk} B_{lm} m^{sjklmi;s7} + \dots \right)^2 \\ & + \frac{1}{3!} \left(m^{ijk;s7} + \frac{1}{2} B_{lm} m^{slmijk;s7} + \dots \right)^2 + \frac{1}{5!} (m^{sijklm;s7} + \dots)^2 + (m^{s;s7;s7})^2 + \dots, \end{aligned} \quad (2.7)$$

while V' reduces to a representation V after dualizing the **6** and **7** indices.

T-duality spectral flows

Adding $\mathcal{M}_{\{V,S_B,T,S_A,V'\}}$ together, we obtain the T-duality invariant flux multiplet mass formula, which is still of the form in Eq.(2.3) but for replacing the m charges with shifted charges \tilde{m} incorporating the effect of the B field, e.g.

$$\tilde{m}_s = m_s + \frac{1}{2} B_2 m^2 + \frac{1}{8} B_2^2 m^{s4} + \frac{1}{48} B_2^3 m^{s;s6}. \quad (2.8)$$

The mass spectrum is thus globally invariant under the integer shift $B_{ij} \rightarrow B_{ij} + \Delta B_{ij}$, even though the latter induces a *spectral flow* within each T-duality multiplet:

$$\begin{aligned} V : \quad & m_i \rightarrow m_i + \Delta B_{ji} m^{sj}, \quad m^{si} \rightarrow m^{si}, \\ S_B : \quad & m_s \rightarrow m_s + \frac{1}{2} \Delta B_{ij} m^{ij}, \quad m^{ij} \rightarrow m^{ij} + \frac{1}{2} \Delta B_{kl} m^{sklij}, \\ & m^{sijkl} \rightarrow m^{sijkl} + \frac{1}{2} \Delta B_{mn} m^{s;smnijkl}, \quad m^{s;sijklmn} \rightarrow m^{s;sijklmn}, \\ T : \quad & m^{ijklm} \rightarrow m^{ijklm} + \Delta B_{np} \left(\frac{1}{2} m^{s;npijklm} - m^{n;spijklm} \right), \\ & m^{p;sijklmn} \rightarrow m^{p;sijklmn} - \Delta B_{qr} m^{sqr;srijklmn}, \\ & m^{s;sijklmnp} \rightarrow m^{s;sijklmnp} + \frac{1}{2} \Delta B_{qr} m^{sqr;sijklmnp}, \\ & m^{sqr;sijklmnp} \rightarrow m^{sqr;sijklmnp}, \\ S_A : \quad & m^{i;jklmnpq} \rightarrow m^{i;jklmnpq} + \frac{1}{2} \Delta B_{rt} m^{rti;sijklmnpq}, \\ & m^{ijk;slmnpqrt} \rightarrow m^{ijk;slmnpqrt} + \frac{1}{2} \Delta B_{uv} m^{suvijk;slmnpqrt}, \\ & m^{sijklm;snpqrtuv} \rightarrow m^{sijklm;snpqrtuv} + \frac{1}{2} \Delta B_{wx} m^{s;swxijklm;snpqrtuv}, \\ & m^{s;s7;s7} \rightarrow m^{s;s7;s7}, \\ V' : \quad & m^{ijklmn;spqrtuvw} \rightarrow m^{ijklmn;spqrtuvw} - \frac{1}{2} \Delta B_{xy} m^{x;syijklmn;spqrtuvw}, \\ & m^{1;s7;s7} \rightarrow m^{1;s7;s7} \end{aligned} \quad (2.9)$$

The flow indeed acts as an automorphism on the charge lattice; note that, except for the highest weight ($m^{1;8;8}$ in $d = 8$), the charges cannot be restricted to positive integers. This fact will be of use in Section 4.

Alternatively, the above spectral flow can be recast into a system of differential equations for the shifted charges \tilde{m} , e.g.

$$S_B : \begin{aligned} \frac{\partial \tilde{m}_s}{\partial B_{ij}} &= \frac{1}{2} \tilde{m}^{ij} , & \frac{\partial \tilde{m}^{ij}}{\partial B_{kl}} &= \frac{1}{2} \tilde{m}^{ijkl} , \\ \frac{\partial \tilde{m}^{ijkl}}{\partial B_{mn}} &= \frac{1}{2} \tilde{m}^{s;ijklmn} , & \frac{\partial \tilde{m}^{s;ijklmn}}{\partial B_{pq}} &= 0 , \end{aligned} \quad (2.10)$$

which can be integrated to yield the mass formula; the constants of integration correspond to the integer charges m . The integrability of this system of differential equations follows from the commutativity of the spectral flow.

2.2 U-duality spectral flows

The mass formula obtained so far is invariant under T-duality and holds for vanishing values of RR gauge backgrounds. In order to obtain a U-duality invariant mass formula, we have to allow expectation values of the M-theory gauge three-form \mathcal{C}_{IJK} , which extends the NS two-form $B_{ij} = \mathcal{C}_{sij}$; the expectation value of the RR one-form is already incorporated as the off-diagonal component $\mathcal{A}_i = g_{si}/R_s^2 \neq 0$ of the metric in Eq.(2.2). For $d \geq 6$, one should also allow expectation values of the six-form \mathcal{E}_{IJKLMN} Poincaré-dual to \mathcal{C}_{IJK} in eleven dimensions: in the string theory language, it corresponds to the RR five-form \mathcal{E}_{s5} together with the NS six-form dual to $B_{\mu\nu}$ in ten dimensions. For $d = 8$, the eight KK gauge fields $g_{\mu I}$ in three space-time dimensions are dual to eight scalars \mathcal{K}_I , which, together with g_{IJ}, \mathcal{C}_3 and \mathcal{E}_6 , span the $E_8/SO(16)$ scalar manifold. \mathcal{K}_I may alternatively be thought of as the form $\mathcal{K}_{1;8}$. $\mathcal{K}_{s;s7}$ is then nothing but the expectation value of the RR seven-form on the string theory seven-torus.

Together with the Teichmüller transformations $\gamma_I \rightarrow \gamma_I + \gamma_J$ on the cycles γ_I of the compactification torus, the integer shifts of the gauge potential expectation values provide

the necessary *Borel* generators to extend the finite Weyl group $\mathcal{W}(E_d)$ to the full $E_d(\mathbf{Z})$ U-duality group. These two sets of generators are actually conjugated under T-duality, since a skew torus turns into a torus with non-vanishing B_{ij} field under T-duality.

In order to reinstate the \mathcal{C}_{IJK} dependence in the mass formula, we covariantize the $B_{ij} = \mathcal{C}_{sij}$ spectral flow (2.9) under $Sl(d, \mathbf{Z})$. This yields

$$\begin{aligned}
m_I &\rightarrow m_I + \frac{1}{2} \Delta \mathcal{C}_{JKI} m^{JK} \\
m^{IJ} &\rightarrow m^{IJ} + \frac{1}{6} \Delta \mathcal{C}_{KLM} m^{KLMIJ} \\
m^{IJKLM} &\rightarrow m^{IJKLM} + \frac{1}{2} \Delta \mathcal{C}_{NPQ} m^{N;PQIJKLM} \\
m^{I;JKLMNPQ} &\rightarrow m^{I;JKLMNPQ} + \frac{1}{2} \Delta \mathcal{C}_{RST} m^{RSI;TJKLMNPQ} \\
m^{IJK;8} &\rightarrow m^{IJK;8} + \frac{1}{6} \Delta \mathcal{C}_{LMN} m^{LMNIJK;8} \\
m^{IJKLMN;8} &\rightarrow m^{IJKLMN;8} + \frac{1}{2} \Delta \mathcal{C}_{PQR} m^{P;QRIJKLMN;8} \\
m^{1;8;8} &\rightarrow m^{1;8;8}
\end{aligned} \tag{2.11}$$

Here however, the \mathcal{C} spectral flow turns out to be *non-integrable*. Indeed, denoting by ∇^{IJK} the flow induced by the shift $\mathcal{C}_{IJK} \rightarrow \mathcal{C}_{IJK} + \Delta \mathcal{C}_{IJK}$, we find

$$[\nabla^{IJK}, \nabla^{LMN}] = 20 \nabla^{IJKLMN} \tag{2.12}$$

where ∇^{IJKLMN} is the flow induced by the shift $\mathcal{E}_{IJKLMN} \rightarrow \mathcal{E}_{IJKLMN} + \Delta \mathcal{E}_{IJKLMN}$:

$$\begin{aligned}
m_I &\rightarrow m_I + \frac{1}{5!} \Delta \mathcal{E}_{JKLMNI} m^{JKLMN} \\
m^{IJ} &\rightarrow m^{IJ} + \frac{1}{5!} \Delta \mathcal{E}_{KLMNPQ} m^{K;LMNPQIJ} \\
m^{IJKLM} &\rightarrow m^{IJKLM} + \frac{1}{(3!)^2} \Delta \mathcal{E}_{NPQRST} m^{NPQ;RSTIJKLM} \\
m^{I;JKLMNPQ} &\rightarrow m^{I;JKLMNPQ} + \frac{1}{5!} \Delta \mathcal{E}_{RSTUVW} m^{RSTUVI;WJKLMNPQ} \\
m^{IJK;8} &\rightarrow m^{IJK;8} + \frac{1}{5!} \Delta \mathcal{E}_{LMNPQR} m^{L;MNPQR IJK;8} \\
m^{6;8} &\rightarrow m^{6;8} \\
m^{1;8;8} &\rightarrow m^{1;8;8}
\end{aligned} \tag{2.13}$$

For $d \leq 7$, Eq.(2.12) is the only non-zero commutation relation, while for $d = 8$ the two flows ∇^{IJK} and ∇^{IJKLMN} close on a $\mathcal{K}_{1,8}$ flow. We shall, however, restrict ourselves to

the case $d \leq 7$ for simplicity. This non-commutativity does not come as a surprise if one considers successive application of the transformations that shift the values of the background fields \mathcal{C}_{IJK} by integers. The reason is that a point in the homogeneous moduli space $E_d(\mathbb{R})/K_d(\mathbb{R})$, where $K_d(\mathbb{R})$ is the maximal compact subgroup of E_d , can be parametrized by a coset representative $g \in E_d(\mathbb{R})$; the latter can be represented according to the Iwasawa decomposition

$$g \in E_d(\mathbb{R}) = K_d(\mathbb{R}) \cdot A_d(\mathbb{R}) \cdot N_d(\mathbb{R}) \quad (2.14)$$

into compact K_d , abelian A_d and nilpotent N_d factors; $N_d(\mathbb{R})$ can be thought of as the group of upper triangular matrices with 1's on the diagonal, whereas the compact factor is modded out in the quotient. The gauge potentials (and the off-diagonal metric) enter into the $N_d(\mathbb{R})$ factor, whereas the (diagonal part of the) metric enters in the abelian factor $A_d(\mathbb{R})$. Spectral flows act on g from the right as elements of $N_d(\mathbb{Z})$, and correspond to isometries of the scalar manifold. They can be reabsorbed into a left action on the integer charge vector m , so that the mass formula

$$\mathcal{M}^2 = m^t \cdot g^t g \cdot m \quad (2.15)$$

is invariant. The nilpotent matrices $N_d(\mathbb{Z})$ exhibit commutation relations graded by the distance away from the diagonal, thus implying non-commutativity for the spectral flows. In the case at hand, the \mathcal{C}_3 , \mathcal{E}_6 and $\mathcal{K}_{1;8}$ gauge potentials then parametrize the first, second and third diagonal rows respectively above the main diagonal of $N_d(\mathbb{R})$.

The non-integrability can be evaded by combining the $\Delta\mathcal{C}_3$ shift with a $\Delta\mathcal{E}_6$ shift,

$$\frac{1}{5!} \Delta\mathcal{E}_{IJKLMN} = \frac{1}{12} \mathcal{C}_{[IJK} \Delta\mathcal{C}_{LMN]} \quad (2.16)$$

such that the resulting flow

$$\partial^{IJK} = \nabla^{IJK} - 10C_{KLM} \nabla^{KLM IJK} \quad (2.17)$$

becomes integrable. The extra shift (2.16) is invisible in the type IIA picture for zero RR potentials since it does not contribute to the T-duality spectral flow. We emphasize again that these terms are generated as a consequence of integrability of the flow, which we take as a guide for reconstructing the covariantized flow. Note also that this flow does *not* preserve the integer lattice of charges anymore, and consequently does not deserve the name of spectral flow; equivalently, it does not correspond to an isometry of the scalar manifold E_d/K_d . Instead, the correct isometry is obtained by accompanying the \mathcal{C}_3 shift (2.17) by a compensating \mathcal{E}_6 shift opposite to Eq.(2.16), and induces the true spectral flow (2.11).

The flow (2.17) however allows us to integrate the corresponding system of differential equations

$$\begin{aligned}
\partial^{JKL}\tilde{m}_I &= \frac{1}{2}\tilde{m}^{JK}\delta_I^L & \nabla^{JKLMNP}\tilde{m}_I &= \frac{1}{5!}\tilde{m}^{JKLMN}\delta_I^P \\
\partial^{KLM}\tilde{m}^{IJ} &= \frac{1}{6}\tilde{m}^{KLMIJ} & \nabla^{KLMNPQ}\tilde{m}^{IJ} &= \frac{1}{5!}\tilde{m}^{K;LMNPQIJ} \\
\partial^{NPQ}\tilde{m}^{IJKLM} &= \frac{1}{2}\tilde{m}^{N;PQIJKLM} & \nabla^{NPQRST}\tilde{m}^{IJKLM} &= 0 \\
\partial^{RST}\tilde{m}^{I;JKLMNPQ} &= 0 & \nabla^{RSTUVW}\tilde{m}^{I;JKLMNPQ} &= 0
\end{aligned} \tag{2.18}$$

to obtain the U-duality invariant mass formula for the flux multiplet in $d \leq 7$,

$$\mathcal{M}_F^2 = (\tilde{m}_1)^2 + \frac{1}{2! l_p^6} (\tilde{m}^2)^2 + \frac{1}{5! l_p^{12}} (\tilde{m}^5)^2 + \frac{1}{7! l_p^{18}} (\tilde{m}^{1;7})^2 \tag{2.19}$$

where the shifted charges read

$$\begin{aligned}
\tilde{m}_I &= m_I + \frac{1}{2}\mathcal{C}_{JKI}m^{JK} + \left(\frac{1}{4!}\mathcal{C}_{JKL}\mathcal{C}_{MNI} + \frac{1}{5!}\mathcal{E}_{JKLMNI}\right)m^{JKLMN} \\
&\quad + \left(\frac{1}{3!4!}\mathcal{C}_{JKL}\mathcal{C}_{MNP}\mathcal{C}_{QRI} + \frac{1}{2\cdot 5!}\mathcal{C}_{JKL}\mathcal{E}_{MNPQRI}\right)m^{J;KLMNPQR} \\
\tilde{m}^{IJ} &= m^{IJ} + \frac{1}{3!}\mathcal{C}_{KLM}m^{KLMIJ} + \left(\frac{1}{4!}\mathcal{C}_{KLM}\mathcal{C}_{NPQ} + \frac{1}{5!}\mathcal{E}_{KLMNPQ}\right)m^{K;LMNPQIJ} \\
\tilde{m}^{IJKLM} &= m^{IJKLM} + \frac{1}{2}\mathcal{C}_{NPQ}m^{N;PQIJKLM} \\
\tilde{m}^{I;JKLMNPQ} &= m^{I;JKLMNPQ}
\end{aligned} \tag{2.20}$$

This formula is written for the case $d = 7$ and is invariant under $E_7(\mathbb{Z})$. It reduces to the

exact mass formulae in $d < 7$ by simply dropping the forms with more than d antisymmetric indices.

As an illustration of the T-duality invariance, we display the shift in the T-duality vector charge m^{s^1} implied by the above equation:

$$\begin{aligned} \tilde{m}^{s^1} + \mathcal{A}_1 \tilde{m}^2 = & m^{s^1} + [\mathcal{A}_1 m^2 + (\mathcal{C}_3 + \mathcal{A}_1 B_2) m^{s^4} + (\mathcal{E}_{s^5} + \mathcal{C}_3 B_2 + \mathcal{A}_1 B_2 B_2) m^{s;s^6}] \\ & + [\mathcal{A}_1 \mathcal{C}_3 m^5 + (\mathcal{E}_6 + \mathcal{C}_3^2 + \mathcal{A}_1 \mathcal{E}_{s^5} + \mathcal{A}_1 B_2 \mathcal{C}_3) m^{1;s^6}] . \end{aligned} \quad (2.21)$$

The first bracket in this expression precisely involves the tensor product of the *charge* spinor representation S_B with the *RR moduli* spinor representation. Indeed, the multiplet $(\mathcal{A}, \mathcal{C} + \mathcal{A}B, \mathcal{E} + \mathcal{C}B + \mathcal{A}B^2)$ transforms as a spinor multiplet, since it appears in the expansion of the T-invariant D-brane coupling $e^{B+F} \mathcal{R}$ in powers of F . The combination of moduli $(\mathcal{A}_1 \mathcal{C}_3, \mathcal{E}_6 + \mathcal{C}_3^2 + \mathcal{A}_1 \mathcal{E}_{s^5} + \mathcal{A}_1 B_2 \mathcal{C}_3)$ on the other hand should transform as part of a second order tensor under T-duality.

2.3 The momentum multiplet

Having obtained the full U-duality invariant mass formula for the flux multiplet, we now briefly discuss the case of the momentum multiplet. As shown in Ref. [9], applying U-duality on a state of mass $\mathcal{M} = R_I R_I / l_p^3$ generates masses

$$\frac{R_I R_I R_J R_K R_L}{l_p^6}, \frac{R_I R_I^2 R_J R_K R_L R_M R_N}{l_p^9}, \frac{R_I R_I^2 R_J^2 R_K^2 R_L R_M R_N R_P}{l_p^{12}}, \frac{R_I R_I^2 R_J^2 R_K^2 R_L^2 R_M^2 R_N^2 R_P}{l_p^{15}}, \dots$$

The dots stand for many extra contributions occurring when $d \geq 8$. For simplicity, we shall restrict ourselves to $d \leq 7$. The integer charges corresponding to these states can be written as a set of integer forms

$$n^1, n^4, n^{1;6}, n^{3,7}, n^{6,7} .$$

Decomposing these representations according to the g_s^2 grading as in Eq. (2.3), we find that they combine under T-duality as

$$S = (n^s), \quad S_A = (n^1, n^{s3}, n^{s;s5}), \quad T = (n^4, n^{s;6}, n^{1;s5}, n^{s2;s6}),$$

$$S'_A = (n^{1;6}, n^{3;s6}, n^{s5;s6}), \quad S' = (m^{6;s6}).$$

The singlet S exists in any dimension, while in $d - 1 = 6$, T contains an antisymmetric $SO(6, 6)$ two-form and a singlet. The spinor representation S'_A and the singlet S' only exists in $d - 1 = 6$. Applying the same reasoning as for the flux multiplet, we obtain the $E_6(\mathbf{Z})$ -invariant mass formula for the momentum multiplet in the case $d = 6$:

$$\mathcal{M}_M^2 = R_l^2 \left[\frac{1}{l_p^6} (\tilde{n}^1)^2 + \frac{1}{l_p^{12}} (\tilde{n}^4)^2 + \frac{1}{l_p^{18}} (\tilde{n}^{1;6})^2 \right] \quad (2.22)$$

where the shifted charges are given by

$$\begin{aligned} \tilde{m}^I &= m^I + \frac{1}{3!} \mathcal{C}_{JKLM} m^{JKLI} + \left(\frac{1}{4!} \mathcal{C}_{JKLM} \mathcal{C}_{MNP} + \frac{1}{5!} \mathcal{E}_{JKLMNP} \right) m^{J;KLMNPI} \\ \tilde{m}^{IJKL} &= m^{IJKL} + \frac{1}{2} \mathcal{C}_{MNP} m^{M;NPIJKL} \\ \tilde{m}^{I;JKLMNP} &= m^{I;JKLMNP} \end{aligned} \quad (2.23)$$

As the overall factor R_l^2 in Eq. (2.22) shows, the momentum multiplet describes extended objects with one world-volume direction wrapped on the longitudinal (light-like) circle⁵. States with n^I charge correspond to membranes wrapped on R_l and a transverse radius R_I , and states with n^{IJKL} charge correspond to five-branes wrapped on four transverse directions besides the longitudinal direction. The last charge $n^{1;6}$ corresponds to Taub-NUT gravitational monopoles. The mass formula (2.23) can be extended to $d = 7, 8$ although the index structure soon becomes intricate.

⁵Without mention of DLCQ, one could also understand the momentum multiplet as the *multiplet of strings* of M-theory, with tension \mathcal{M}_M/R_l .

2.4 Solitons and instantons

As an illustration of the momentum multiplet mass formula, we display the $d \leq 5$ case, where only n^1 and n^4 contribute:

$$\begin{aligned} \mathcal{M}_M^2 = \frac{R_l^2}{l_p^6} & \left[\left(n^I + \frac{1}{3!} n^{IJKL} C_{JKL} \right) g_{IM} \left(n^M + \frac{1}{3!} n^{MNPQ} C_{NPQ} \right) \right. \\ & \left. + \frac{1}{4!} \frac{1}{l_p^6} n^{IJKL} g_{IM} g_{JN} g_{KP} g_{LQ} n^{MNPQ} \right]. \end{aligned} \quad (2.24)$$

This is precisely the U-duality invariant quantity obtained in the study of instanton corrections to R^4 couplings in type II theories [19, 10], where it was found that in order to obtain an $SO(5, 5, \mathbf{Z})$ -invariant result, one should include, in addition to the D0-branes (described by n^1) and the D2-branes (described by n^{s3}), extra states with a four-form charge n^4 [10]⁶. It was further noticed that these states would give e^{-1/g_s^2} effects, which came as a surprise since T^4 compactifications of type II string do not seem to allow for NS five-brane instantons. In the present framework, n^1 and n^4 naturally appear as membranes and five-branes wrapped on the longitudinal direction in addition to one or four transverse directions, giving solitons in the remaining six-dimensional theory. One should therefore think of the non-perturbative threshold obtained in Ref. [10] as a sum of *soliton loops* rather than of instanton effects. This conclusion should however be taken with care, since we have not been able to show that the $SO(5, 5, \mathbf{Z})$ Eisenstein series obtained from Eq. (2.24) contains the correct one-loop R^4 coupling.

3. Gauge backgrounds in Matrix theory

Gauge backgrounds of M-theory should have a counterpart as couplings in the Matrix gauge theory. In this Section, we will translate the mass formulae of the M-theory BPS

⁶The charge n_s in Ref. [10] was associated to the integer dual (in the sense of Poisson resummation) to the number of D-branes bound together.

states obtained in the previous Section into the gauge theory language, and show that they arise from topological couplings in the gauge theory. We will determine these couplings from D-brane analysis.

3.1 BPS states of Matrix gauge theory

As already emphasized in Ref. [9], the translation from the M-theory mass to the light-cone energy, equated to the energy in the Yang-Mills theory, differs for the flux and momentum multiplets. The general formula for bound states of flux and momenta states (*i.e.* having non zero values of both the m and n charges) reads

$$E_{YM} = \frac{\mathcal{M}_F^2}{P^+} + \sqrt{\mathcal{M}_M^2} \quad (3.1)$$

where \mathcal{M}_F and \mathcal{M}_M are the masses of the flux and momentum multiplet given in (2.19) and (2.22) respectively, and $P^+ = N/R_l$ is the quantized light-cone momentum. Expressing (l_p, R_s, g_{IJ}) in terms of the Yang-Mills parameters (g^2, \tilde{g}_{IJ}) , and restricting for simplicity to $d \leq 7$, we obtain:

$$E_{YM} = \frac{g^2}{NV_s} \left[(\tilde{m}^1)^2 + \left(\frac{V_s}{g^2}\right)^2 (\tilde{m}_2)^2 + \left(\frac{V_s}{g^2}\right)^4 (\tilde{m}_5)^2 + \left(\frac{V_s}{g^2}\right)^6 (\tilde{m}_{1;7})^2 \right] \\ + \sqrt{(\tilde{n}_1)^2 + \left(\frac{V_s}{g^2}\right)^2 (\tilde{n}_4)^2 + \left(\frac{V_s}{g^2}\right)^4 (\tilde{n}_{1;6})^2 + \left(\frac{V_s}{g^2}\right)^6 (\tilde{n}_{3;7})^2 + \left(\frac{V_s}{g^2}\right)^8 (\tilde{n}_{6;7})^2}, \quad (3.2)$$

where the index contractions are now performed with the dual metric $\tilde{g}_{IJ} = g^{IJ} l_p^6 / R_s^2$. All upper indices in the M-theory picture are turned into lower indices in the Matrix gauge theory picture. For $d \leq 3$, Eq. (3.2) reduces to

$$E_{YM} = \frac{g^2}{NV_s} \left(m^I + \frac{1}{2} \mathcal{C}^{IJK} m_{JK} \right) \tilde{g}_{IL} \left(m^L + \frac{1}{2} \mathcal{C}^{LMN} m_{MN} \right) \\ + \frac{V_s}{Ng^2} (m_{IJ} \tilde{g}^{IK} \tilde{g}^{JL} m_{KL}) + \sqrt{n_I \tilde{g}^{IJ} n_J}. \quad (3.3)$$

This includes the energy of the electric flux m^I (*i.e.* the momentum conjugate to $\int F_{0I}$) and the magnetic flux $m_{IJ} = \int F_{IJ}$ in the diagonal Abelian subgroup of $U(N)$, together with the energy of a massless excitation with quantized momentum n_I . The shift of the electric flux m^I in the presence of a \mathcal{C}_3 gauge field background is the manifestation of the Witten phenomenon [20] and indicates that the coupling of \mathcal{C}_3 to the gauge theory occurs through a topological term $\int \mathcal{C}^{IJK} F_{0I} F_{JK}$. Indeed, the only effect of such a coupling is to shift the momentum conjugate to $\partial_0 A_I$ by a quantity $\mathcal{C}^{IJK} \int F_{JK}$. In the next Subsection, we will derive the existence of this coupling from the D-brane action.

When $d = 4$, an extra charge n_4 appears in the momentum contribution, which can be interpreted as the momentum along a (dynamically generated) fifth dimension of radius g^2 [5]. We can indeed rewrite Eq. (3.2) in a U-duality ($Sl(5, \mathbf{Z})$)-invariant way as

$$E_{YM} = \frac{1}{NV_5} m^{AB} \tilde{g}_{AC} \tilde{g}_{BD} m^{CD} + \sqrt{n_A \tilde{g}^{AB} n_B}, \quad (3.4)$$

where A, B, \dots , now run from 1 to 5, and $V_5 = V_s g^2$ is the volume of the five-dimensional torus. One may now interpret m^{AB} as the quantized flux (in the diagonal Abelian group) conjugate to a $U(N)$ two-form gauge field B_{AB} living on the 1+5 world volume. Note that the dependence of E_{YM} on the volume of the five-dimensional volume is through a global factor $V_5^{-1/5}$. This agrees nicely with the scale invariance of the conjectured 1+5-dimensional gauge theory [5].

3.2 D-brane gauge couplings and Matrix theory

The Matrix theory prescription for M-theory compactifications may be recovered by viewing the DLC light-like compactification as an infinitely boosted space-like compactification described by weakly coupled type IIA string theory⁷. The Matrix gauge theory is then identified as the gauge theory on the world-volume of N Dd branes, obtained by a maximal

⁷Subtleties may hide behind this formal equivalence [17].

T-duality from the N D0-branes [12, 13]. Whereas so far the prescription was only applied for compactifications with vanishing gauge field expectation values, one may extend this argument to find the couplings that these VEVs induce in the gauge theory, from the well-known gauge couplings of D-branes.

The Dd -brane T-dual to the N D0-brane interacts with the RR fields through a topological Wess-Zumino term [16]

$$S_{\text{RR}} = \int dt \int_{\tilde{T}^d} \text{Str} e^{F+B} \wedge \mathcal{R}, \quad (3.5)$$

where \tilde{T}^d is the dual torus and the integral picks up the contribution of $d + 1$ forms in the integrand. F is the $U(N)$ field strength and $\mathcal{R} = \sum_p \mathcal{R}^{(p)}$ is the total RR potential. The symmetrized trace is taken in the adjoint representation of $U(N)$ and will be omitted henceforth. The NS two-form B couples to the Abelian diagonal part of the $U(N)$ field strength F ; it would appear after T-duality if we were considering DLCQ of M-theory in the presence of a background value of \mathcal{C}_{-IJ} , where the minus sign denotes the compact light-cone coordinate. This case has been addressed recently in Ref.[22] and seems to require drastic changes in the compactification prescription. We will therefore restrict our attention to $B = 0$, in which case the metric on the dual torus is the inverse of the M-theory metric.

For $d \leq 8$, the topological coupling truncates to

$$S_{\text{RR}} = \int dt \int_{\tilde{T}^d} \left[\mathcal{R}^{(d+1)} + F \mathcal{R}^{(d-1)} + \frac{1}{2} F^2 \mathcal{R}^{(d-3)} + \frac{1}{3!} F^3 \mathcal{R}^{(d-5)} + \frac{1}{4!} F^4 \mathcal{R}^{(d-7)} \right], \quad (3.6)$$

where we have omitted the wedge products for notational simplicity. The fields $\mathcal{R}^{(p)}$ are pulled back from the target space onto the Dd -brane world volume, with the embedding coordinates $X^\mu(\sigma)$. We will work in the static gauge in which the target space coordinates of the torus coincide with the world-volume coordinates of the D-brane.

The fields $\mathcal{R}^{(p)}$ are related by a T-duality, on the d spatial directions of the Dd -brane world volume, to the RR fields in the original D0-brane picture. The action of this maximal

T-duality on a $q \leq d + 1$ RR form is

$$\mathcal{R}_{0i_1 \dots i_{q-1}}^{(q)} \rightarrow \mathcal{R}_{0i_q \dots i_d}^{(d+2-q)}, \quad \mathcal{R}_{i_1 \dots i_q}^{(q)} \rightarrow \mathcal{R}_{i_{q+1} \dots i_d}^{(d-q)}, \quad (3.7)$$

where we have distinguished two cases, depending on whether or not the RR field has a component in the time direction or not. Ignoring for a moment the transverse fluctuations around the Dd -brane background, the Matrix model action in the D0-brane picture is:

$$\begin{aligned} S_{\text{RR}} = \int dt \int_{\bar{T}^d} & \left[\mathcal{R}_0^{(1)} + F_{0i} \mathcal{R}_i^{(1)} + F_{ij} \mathcal{R}_{0ij}^{(3)} + F_{0i} F_{jk} \mathcal{R}_{ijk}^{(3)} + F_{ij} F_{kl} \mathcal{R}_{0ijkl}^{(5)} \right. \\ & + F_{0i} F_{jk} F_{lm} \mathcal{R}_{ijklm}^{(5)} + F_{ij} F_{kl} F_{mn} \mathcal{R}_{0ijklmn}^{(7)} \\ & \left. + F_{0i} F_{jk} F_{lm} F_{nr} \mathcal{R}_{ijklmnr}^{(7)} + F_{ij} F_{kl} F_{mn} F_{rs} \mathcal{R}_{0ijklmnr}^{(9)} \right], \end{aligned} \quad (3.8)$$

where $\mathcal{R}^{(1)} = \mathcal{A}$ is the type IIA RR one-form, $\mathcal{R}^{(3)} = \mathcal{C}$ is the RR three-form, etc. The time component \mathcal{A}_0 can be gauge transformed to zero since the time coordinate is non-compact. The type IIA gauge fields arise under reduction of the M-theory metric and (dual) gauge fields on the space-like radius of radius R_s , as discussed below Eq. (2.7). After a large boost of rapidity $\beta = 1 - (R_s/R_l)^2$, this circle becomes quasi-lightlike and the metric takes the form⁸

$$ds^2 = dx^+ (dx^- + \mathcal{A}_i dx^i) + dx^i g_{ij} dx^j. \quad (3.9)$$

We can therefore identify \mathcal{A}_i with g_{+I}/R_l^2 , where we promoted the string theory spatial index i to the M-theory transverse direction I . At the same time, \mathcal{C}_{ijk} is identified with the M-theory transverse three-form \mathcal{C}_{IJK} , whereas the NS two-form B_{ij} would turn into C_{-IJ} , as already anticipated at the beginning of this section. $\mathcal{R}_{0ijkl}^{(5)}$ turns into \mathcal{E}_{-+IJKL} , while $\mathcal{R}_{ijklmnp}^{(7)}$ and $\mathcal{R}_{0ijklmn}^{(7)}$ become the components $\mathcal{K}_{-;-IJKLMNOP}$ and $\mathcal{K}_{-;-+IJKLMNOP}$ of the M-theory $\mathcal{K}_{1,8}$ form in the DLC. The nine-form $\mathcal{R}^{(9)}$ is associated to a type IIA cosmological constant term and will be discarded below.

⁸The coordinates x^- and $x^{i=1 \dots d}$ are compact variables with radius R_l and R_i respectively, while x^+ and the spacetime coordinates x^μ are non compact.

Using these identifications we may then immediately read off the coupling of the supersymmetric gauge theory to the M-theory backgrounds:⁹

$$S_{\text{Matrix}} = \int dt \int_{\tilde{T}^d} [F_{0I}g^{+I} + F_{IJ}\mathcal{C}^{+IJ} + F_{0I}F_{JK}\mathcal{C}^{IJK} + F_{IJ}F_{KL}\mathcal{E}^{-+IJKL} \quad (3.10)$$

$$+ F_{0I}F_{JK}F_{LM}\mathcal{E}^{-IJKLM} + F_{IJ}F_{KL}F_{MN}\mathcal{K}^{-;-+IJKLMN} + F_{0I}F_{JK}F_{LM}F_{NP}\mathcal{K}^{-;-IJKLMNP}] .$$

The only term involving an eleven-dimensional Lorentz scalar, and therefore a genuine modulus of M-theory compactification, is the third term. As a consequence we find that the expectation value of the three-form induces the following topological coupling in the Matrix gauge theory :

$$S_c = \mathcal{C}^{IJK} \int dt \int_{\tilde{T}^d} F_{0I}F_{JK} , \quad (3.11)$$

as inferred from the gauge theory energy (3.3). This also agrees with the conjecture in Ref. [11]. Some of the remaining terms in Eq. (3.10) were observed in Ref. [22] and in the supermembrane context [26].

We next turn to the possible effects of terms containing transverse fluctuations on the Dd -brane. These will arise through the expansion of the q -forms,

$$\mathcal{R}^{(q)} = \sum_{p=0}^q \mathcal{R}_{i_1 \dots i_{q-p} \mu_1 \dots \mu_p} D_{i_{q-p+1}} X^{\mu_1} \wedge \dots \wedge D_{i_q} X^{\mu_p} . \quad (3.12)$$

For any non-zero value of p , these couplings will always involve forms with at least one component in the spacetime directions, transverse to the brane. Since the maximal T-duality does not affect the transverse space, the dual RR forms still involve transverse indices. Consequently, such terms cannot generate couplings of the M-theory moduli to the gauge theory.

This reasoning does not touch upon the couplings of the $\mathcal{E}_6, \mathcal{K}_{1;8}$ moduli (related to \mathcal{C}_3 under U-dualities when $d \geq 6$) to the Matrix gauge theory. Such fields correspond to

⁹As we noted below Eq. (3.2), the lower indices of the M-theory fields turn into upper indices in the Matrix gauge theory.

NS gauge potentials in the D-brane picture, and are obviously not incorporated in the topological coupling Eq. (3.5). The partial picture obtained for $d \geq 6$ may well be related to the difficulties in defining the Matrix gauge theory in these dimensions.

4. Nahm-type duality and eleven-dimensional Lorentz invariance

In the last two Sections, we discussed the occurrence of the $E_d(\mathbf{Z})$ U-duality symmetry both from the point of view of M-theory and its Matrix gauge theory DLCQ description. However, eleven-dimensional Lorentz invariance implies that this symmetry should extend to an $E_{d+1}(\mathbf{Z})$ action on the M-theory BPS spectrum, to which we now turn.

As already noticed in Ref. [9], many of the states of the flux multiplet, describing various branes wrapped on k transverse directions, have a counterpart in the momentum multiplet as the same brane wrapping $k-1$ transverse directions and the IMF (or light-cone) compact direction. Indeed, comparing the two mass formulae (2.19) and (2.22), we see that we can interpret the $E_d(\mathbf{Z})$ flux m and momentum n charges as charges of a *flux* multiplet M of $E_{d+1}(\mathbf{Z})$:

$$\begin{aligned}
 m_1 &= M_1 & m^{1;7} &= M^{1;7} , \quad n^{1;6} = M^{1;6} \\
 m^2 &= M^2 , \quad n^1 = M^{l1} & m^{3,8} &= M^{3;8} , \quad n^{3,7} = M^{3;l7} \\
 m^5 &= M^5 , \quad n^4 = M^{l4} & m^{6,8} &= M^{6,8} , \quad n^{6,7} = M^{6;l7}
 \end{aligned}
 \tag{4.1}$$

where we now denote the light-cone direction by an index l to avoid confusion. A notable exception is the transverse KK state m_1 of the flux multiplet, with mass $1/R_l$, which does not correspond to any state with mass $1/R_l$ in the momentum multiplet. The reason is clear: the longitudinal momentum is fixed in a given Matrix gauge theory to equal the rank of the $U(N)$ gauge group. Following the suggestion in Ref. [14], we regard the tensor product of all gauge theories for all values of N as defining a M(eta) theory on which the eleven-dimensional Lorentz symmetry is represented. N would then appear as an additional

charge M_l necessary to label the vacuum of M(eta) theory. When $d \geq 6$, the singlet

$$N = M_l \tag{4.2}$$

should be accompanied by

$$N^{2;7} \equiv M^{l2;l7}, \quad N^6 \equiv M^{l;l6}, \quad N^{5;7} \equiv M^{l5;l7}, \quad N^{1;7;7} \equiv M^{1;l7;l7} \tag{4.3}$$

and, when $d = 7$, by two extra singlets

$$N^7 = M^{l;l7}, \quad N^{7;7} = M^{l;l7;l7}. \tag{4.4}$$

The charges in Eq. (4.3) label a new U-duality multiplet that transforms as a **56** of $E_7(\mathbf{Z})$ (as is easily seen by dualizing the **6** and **7** indices to $N^2, N_1, N^5, N^{1;7}$). For $d = 6$, it simply reduces to a singlet of $E_6(\mathbf{Z})$. We shall hereafter refer to these new charges and N as forming the (reducible) *rank multiplet*. The dimension of the three U-duality multiplets for $1 \leq d \leq 8$, as well as the U-duality group and the dimension of the corresponding scalar manifold, are listed in the table below.

d		1	2	3	4	5	6	7	8
$E_d(\mathbf{Z})$		1	$Sl(2)$	$Sl(3) \times Sl(2)$	$Sl(5)$	$SO(5, 5)$	E_6	E_7	E_8
scalars		1	3	7	14	25	42	70	128
Flux	$\{m\}$	1	3	(3, 2)	10	16	27	56	248
Momentum	$\{n\}$	1	2	(3, 1)	5	10	27	133	3875
Rank	$\{N\}$	1	1	1	1	1	1 + 1	56 + 1 + 1 + 1	∞
Total	$\{M\}$	3	6	10	16	27	56	248	∞

The extra generators of $E_{d+1}(\mathbf{Z})$ correspond to an extra *Weyl* generator exchanging the light-cone direction with a chosen direction I on T^d ($R_l \leftrightarrow R_I$ for the case of a rectangular

torus), and a *Borel* generator, corresponding to the spectral flow $\mathcal{C}_{IJK} \rightarrow \mathcal{C}_{IJK} + 1$ for some directions J, K on the torus.

(i) As is obvious from the derivation in Sec. 2.2 of Ref. [9], the addition of the Weyl transformation $R_l \leftrightarrow R_I$ to the Weyl group of E_d enhances the latter to $\mathcal{W}(E_{d+1})$. Note in particular that for $d = 8$, this is the (infinite) Weyl group of the affine Lie algebra E_9 , which implies the appearance of an infinite set of multiplets in addition to the flux, momentum and rank multiplet of $E_8(\mathbf{Z})$; for $d = 9$, this is the Weyl group of the hyperbolic algebra E_{10} . We shall refrain from diving in these waters and restrict to $d \leq 7$. The action of this Weyl transformation on the parameters (for rectangular tori) is by definition $R_l \leftrightarrow R_I$, while leaving the other R_J 's and l_p invariant. In particular, the Newton constant in $11 - (d + 1)$ dimensions

$$\frac{1}{\kappa^2} = \frac{V_R R_l}{l_p^9} = R_l^{(d-7)/2} \frac{V_s^{(d-5)/2}}{g^{d-3}} \quad (4.5)$$

is invariant under U-duality¹⁰. In terms of the M(atr)ix theory, this means

$$g^2 \rightarrow \left(\frac{R_l}{R_I} \right)^{d-4} g^2, \quad s_I \rightarrow s_I, \quad s_{J \neq I} \rightarrow \left(\frac{R_l}{R_I} \right) s_J. \quad (4.6)$$

Note that the transformed parameters depend on the original ones *and on* R_l . On the other hand, the only dependence of the gauge theory on R_l should be through a multiplicative factor in the Hamiltonian, since R_l can be rescaled by a Lorentz boost. This leaves open the question of how the M(eta) theory itself depends on R_l . The action on the charges M simply follows from the exchange of the I and l indices. In terms of flux, momentum and

¹⁰This is the same $E_d(\mathbf{Z})$ -invariant combination as appeared in Ref. [9], up to powers of R_l , which anyway does not transform under $E_d(\mathbf{Z})$.

rank charges, this means

$$\begin{aligned}
 N &\leftrightarrow m_I \\
 n^1 &\leftrightarrow m^{I1} & N^{2;7} &\leftrightarrow m^{I2,I7} \\
 n^4 &\leftrightarrow m^{I4} & N^6 &\leftrightarrow m^{I,I6} \\
 n^{1;6} &\leftrightarrow m^{1;I6} & N^{5;7} &\leftrightarrow m^{I5,I7} \\
 n^{3;7} &\leftrightarrow m^{3;I7} & N^{1;7;7} &\leftrightarrow m^{1;I7,I7} \\
 n^{6;7} &\leftrightarrow m^{6;I7} & N^{7;7} &\leftrightarrow m^{I,I7,I7} \\
 N^7 &\leftrightarrow n^{I;6}
 \end{aligned} \tag{4.7}$$

In particular, the rank N of the gauge group is exchanged with the electric flux m_I , whereas the momenta are exchanged with magnetic fluxes. This is reminiscent of Nahm duality, relating (at the classical level) a $U(N)$ gauge theory on T^2 with background flux m to a $U(m)$ gauge theory on the dual torus with background flux N [21]. There is, however, no proof at this stage that this duality survives quantum corrections and dimensional oxidation. This may eventually be proved by a stringy argumentation.

(ii) The Borel generator $\mathcal{C}_{lJK} \rightarrow \mathcal{C}_{lJK} + \Delta\mathcal{C}_{lJK}$ is obtained from the usual $E_d(\mathbf{Z})$ shifts by conjugation under Nahm-type duality. It is therefore not an independent generator, but still gives a spectral flow on the BPS spectrum:

$$\begin{aligned}
 N &\rightarrow N + \Delta\mathcal{C}_{l2} m^2 \\
 m_1 &\rightarrow m_1 + \Delta\mathcal{C}_{l2} n^1 \\
 m^2 &\rightarrow m^2 + \Delta\mathcal{C}_{l2} n^4 \\
 m^5 &\rightarrow m^5 + \Delta\mathcal{C}_{l2} n^{1;6} \\
 m^{1;7} &\rightarrow m^{1;7} + \Delta\mathcal{C}_{l2} n^{3;7} ,
 \end{aligned} \tag{4.8}$$

the other charges being non-affected. In particular, this implies that states with negative N need to be incorporated in the M(eta) theory if it is to be $E_{d+1}(\mathbf{Z})$ -invariant. This is somewhat surprising since the DLC quantization selects $N > 0$, and seems to require a revision both of the interpretation of N as the rank of a gauge theory and of the relation

between N and the light-cone momentum P^{+11} . The resolution of the first point may come from the conjecture, made in Ref. [22], that the Yang-Mills theory should be replaced, in the presence of a background value for \mathcal{C}_{IJ} , by a gauge theory on a non-commutative torus. Rather than being a liability, this may actually turn into an asset by carrying the non-commutative geometry constructions into the quantum realm.

Acknowledgements : This work was supported in part by the EEC under the TMR contract ERBFMRX-CT96-0090, the Israel Academy of Sciences and Humanities - Centers of Excellence Programme, and the American-Israel Bi-National Science Foundation.

References

- [1] E. Witten, Nucl. Phys. **B443** (1995) 85, hep-th/9503124.
- [2] T. Banks, W. Fischler, S. Shenker and L. Susskind, Phys. Rev. **D55** (1997) 5112, hep-th/9610043. For recent reviews, see A. Bilal, hep-th/9710136, and T. Banks, hep-th/9710231.
- [3] W. Taylor IV, Phys. Lett. **B394** (1997) 283, hep-th/9611042.
- [4] L. Susskind, hep-th/9611164; O. Ganor, S. Ramgoolam and W. Taylor IV, Nucl. Phys. **B492** (1997) 191, hep-th/9611202; W. Fischler, E. Halyo, A. Rajaraman and L. Susskind, Nucl. Phys. **B501** (1997) 409, hep-th/9703102.
- [5] M. Rozali, Phys.Lett. **B400** (1997) 260, hep-th/9702136.
- [6] M. Berkooz, M. Rozali and N. Seiberg, Phys. Lett. **B408** (1997) 105, hep-th/9704089; M. Berkooz and M. Rozali, hep-th/9705175; N. Seiberg, Phys. Lett. **B408** (1997) 98, hep-th/9705221; O. Aharony, M. Berkooz, S. Kachru, N. Seiberg and E. Silverstein, Adv. Theor. Math. Phys. **1** (1998) 148, hep-th/9707079; A. Hanany and G.

¹¹See Ref. [27] for a discussion of DLCQ with negative light-cone momentum.

- Lifschytz, hep-th/9708037; A. Losev, G. Moore and S.L. Shatashvili, hep-th/9707250; O. Aharony, M. Berkooz, S. Kachru and E. Silverstein, hep-th/9709118; O. Ganor, hep-th/9709139.
- [7] C.M. Hull and P.K. Townsend, Nucl. Phys. **B438** (1995) 109, hep-th/9410167.
- [8] O. Aharony, Nucl. Phys. **B476** (1996) 470, hep-th/9604103.
- [9] S. Elitzur, A. Giveon, D. Kutasov and E. Rabinovici, Nucl. Phys. **B509** (1998) 122, hep-th/9707217.
- [10] B. Pioline and E. Kiritsis, Phys. Lett. **B418** (1998) 61, hep-th/9710078.
- [11] D. Berenstein, R. Corrado and J. Distler, Nucl. Phys. **B503** (1997) 239, hep-th/9704087.
- [12] A. Sen, hep-th/9709220.
- [13] N. Seiberg, Phys. Rev. Lett. **79** (1997) 3577, hep-th/9710009.
- [14] L. Susskind, hep-th/9704080.
- [15] F. Hacquebord and H. Verlinde, Nucl. Phys. **B508** (1997) 609, hep-th/9707179.
- [16] M.R. Douglas, hep-th/9512077.
- [17] S. Hellerman and J. Polchinski, hep-th/9711037.
- [18] For a review, see for instance: A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. **244** (1994) 77.
- [19] M.B. Green and M. Gutperle, Phys. Lett. **B398** (1997) 69, hep-th/9701093; M.B. Green and P. Vanhove, Phys. Lett. **B408** (1997) 122, hep-th/9704145; M.B. Green, M. Gutperle and P. Vanhove, Phys. Lett. **B409** (1997) 177, hep-th/9706175; E. Kiritsis and B. Pioline, to appear in Nucl. Phys. B., hep-th/9707018.

- [20] E. Witten, Phys. Lett. **86B** (1979) 283.
- [21] W. Nahm, Proceedings of the Monopole Meeting, Trieste, ed. N. Craigie, 1982; P.J. Braam and P. van Baal, Commun. Math. Phys. **122** (1989) 267.
- [22] A. Connes, M. Douglas and A. Schwarz, hep-th/9711162.
- [23] C. M. Hull, hep-th/9711179.
- [24] M. Blau and M. O'Loughlin, hep-th/9712047.
- [25] C. M. Hull, hep-th/9712075.
- [26] B. de Wit, K. Peeters and J.C. Plefka, hep-th/9712082.
- [27] S-J. Rey, hep-th/9712055.