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Two-loop Correction to the Leptonic Decay of Quarkonium

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Abstract

Applying asymptotic expansions at threshold, we compute the two-loop QCD correction to the short-distance coefficient that governs the leptonic decay $\psi \rightarrow l^+l^-$ of a S -wave quarkonium state and discuss its impact on the relation between the quarkonium non-relativistic wave function at the origin and the quarkonium decay constant in full QCD.

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Quarkonium decays played an important role in establishing Quantum Chromodynamics (QCD) as a weakly interacting theory at short distances. Calculations of heavy quarkonium decays usually proceed under the assumption that the heavy quark-antiquark bound state is non-relativistic and that the decay amplitude factorizes into the bound state wave function at the origin and a short-distance quark-antiquark annihilation amplitude. One can then explain the small width of the S -wave spin-triplet charmonium state J/ψ , because it can decay only through electro-magnetic annihilation or annihilation into at least three gluons [1]. Today's understanding of quarkonium bound states has refined this picture and allows us to calculate relativistic corrections systematically at the expense of introducing further non-perturbative parameters that characterize the bound state. Such calculations can be done most transparently in the framework of a non-relativistic effective field theory (NRQCD) [2,3] that implements the factorization of contributions to the (partial) decay widths from different length scales. Besides potentially large relativistic corrections, the size of radiative corrections to the quark-antiquark annihilation amplitudes has always been a matter of concern. The one-loop radiative corrections to the decays $\psi \rightarrow l^+l^-$ (where $l = e, \mu$) [4], $\psi \rightarrow$ light hadrons, $\psi \rightarrow \gamma +$ light hadrons [5] are large and question the practicability of factorization for charmonium and, perhaps, even bottomonium. [Here and in the following we use ψ as a label for any S -wave spin-triplet state, i.e. J/ψ or ψ' for charmonium and $\Upsilon(nS)$ for bottomonium.]

In this Letter we address this question and report on the calculation of two-loop short-distance corrections to the leptonic decay $\psi \rightarrow l^+l^-$. This is the first (and also 'easiest') two-loop matching calculation for quarkonium decays and it can also be used to connect the decay constant of the ψ meson defined in QCD with the non-relativistic wave function at the origin, which appears in NRQCD and potential models. In turn, this wave function is an important input parameter for the prediction of other quarkonium decays and also quarkonium production cross sections.

We recall that ψ decays leptonically through interaction with the electro-magnetic current. The partial decay rate, neglecting the tiny lepton masses, is exactly given by

$$\Gamma(\psi \rightarrow l^+l^-) = \frac{4\pi e_Q^2 \alpha_{em}^2 f_\psi^2}{3M_\psi}, \quad (1)$$

where M_ψ is the mass of ψ , α_{em} the fine structure constant and e_Q the electric charge of the heavy quark in units of the electron charge. The ψ decay constant f_ψ is defined through the following matrix element of the electro-magnetic current:

$$\langle \psi(p) | \bar{Q} \gamma_\mu Q | 0 \rangle = (-i) f_\psi M_\psi \epsilon_\mu^*(p). \quad (2)$$

[$\epsilon_\mu(p)$ is the ψ polarization vector and p the ψ momentum.] The leptonic decay rates are known experimentally [6]. For J/ψ we find $f_{J/\psi} = (405 \pm 15)$ MeV.

The decay constant parametrizes the strong interaction effects and contains long- and short-distance contributions. For quarkonium the short-distance scale is $1/M_\psi$ and the long-distance (bound state) scales are $1/(M_\psi v)$ and $1/(M_\psi v^2)$, where v is the (small)

characteristic velocity of the ψ 's quark constituents. The short-distance contributions can be isolated, and calculated in perturbation theory, by matching the vector current in QCD onto a series of operators in NRQCD. Up to corrections of order v^4 , the matching relation is given by

$$\begin{aligned} \langle \psi(p) | \bar{Q} \gamma^\mu Q | 0 \rangle &= \Lambda^{\mu i}(p) \left[C_0 \left(\alpha_s, \frac{m_Q}{\mu} \right) \langle \psi | \psi^\dagger \sigma_i \chi | 0 \rangle(\mu) \right. \\ &\quad \left. + \frac{C_1 \left(\alpha_s, \frac{m_Q}{\mu} \right)}{6m_Q^2} \langle \psi | \psi^\dagger \vec{D}^2 \sigma_i \chi | 0 \rangle(\mu) + \mathcal{O}(v^4) \right], \end{aligned} \quad (3)$$

where ψ (not to be confused with the ψ meson) and χ denote non-relativistic two-spinors, \vec{D} the spatial covariant derivative and m_Q the heavy quark mass. [More details on notation and NRQCD can be found in Ref. [3]. Note, however, that we use a relativistic normalization of states also for the matrix elements in NRQCD.] The matrix elements on the right-hand side are defined in the ψ rest frame and $\Lambda(p)$ is the matrix that performs the Lorentz boost into this frame. The matching coefficients C_0 and C_1 are expressed as series in the strong coupling α_s and account for the short-distance QCD corrections. The matching coefficients and matrix elements in NRQCD individually depend on the factorization scale μ . C_1 is defined (as in Ref. [7]) such that $C_1 = 1 + \mathcal{O}(\alpha_s)$ and [4]

$$C_0 \left(\alpha_s, \frac{m_Q}{\mu} \right) = 1 - \frac{2C_F \alpha_s(m_Q)}{\pi} + c_2(m_Q/\mu) \left(\frac{\alpha_s}{\pi} \right)^2 + \dots, \quad (4)$$

where $C_F = (N_c^2 - 1)/(2N_c)$ and $N_c = 3$ is the number of colours. We now discuss the calculation of the two-loop matching coefficient c_2 .

Since the matching coefficient contains only short-distance effects, it can be obtained by replacing the quarkonium state ψ on both sides of Eq. (3) by a free quark-antiquark pair of on-shell quarks at small relative velocity. In terms of this on-shell matrix element, the matching equation can be rewritten as

$$Z_{2,\text{QCD}} \Gamma_{\text{QCD}} = C_0 Z_{2,\text{NRQCD}} Z_J^{-1} \Gamma_{\text{NRQCD}} + \mathcal{O}(v^2), \quad (5)$$

where Z_2 are the on-shell wave function renormalization constants in QCD and NRQCD and Γ the amputated, bare electro-magnetic annihilation vertices in QCD and NRQCD. The Feynman diagrams for Γ_{QCD} at two loops are shown in Fig. 1. Since the current $J = \psi^\dagger \sigma_i \chi$ need not be conserved in NRQCD, we allowed for its renormalization, $J_{\text{bare}} = Z_J J_{\text{ren}}$, on the right-hand side. We then obtain C_0 by calculating all other quantities in Eq. (5) in dimensional regularization and using the modified minimal subtraction scheme ($\overline{\text{MS}}$ scheme) [8].

The matching calculation is considerably simplified, if one uses the threshold expansion of Ref. [9] to compute Γ_{QCD} directly as an expansion in v^2 . The threshold expansion is obtained by writing down contributions corresponding to hard ($l \sim m_Q$), soft ($l \sim m_Q v$), potential ($l_0 \sim m_Q v^2$, $l_i \sim m_Q v$) and ultrasoft ($l \sim m_Q v^2$) regions. The

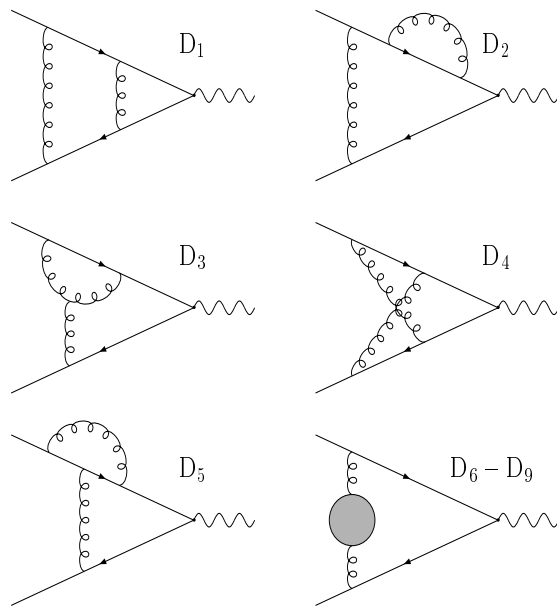


Figure 1: Diagrams that contribute to Γ_{QCD} . Symmetric diagrams exist for $D_{2,3,5}$. The last diagram summarizes vacuum polarization contributions from massless fermions (D_6), gluons (D_7), ghosts (D_8) and the massive fermion with mass m_Q (D_9).

contributions from soft, potential and ultrasoft loop momenta can all be identified with diagrams in NRQCD that appear in the calculation of Γ_{NRQCD} . Hence, they drop out of the matching relation Eq. (5) and it suffices to compute the contribution to the threshold expansion of the diagrams in Fig. 1, where all loop momenta are hard. [The threshold expansion is not only convenient; it also provides an implicit definition of NRQCD in dimensional regularization. This is necessary, because dimensionally regularized NRQCD is not given by the dimensionally regularized Feynman integrals constructed from the vertices and propagators of NRQCD. In order to avoid that the cut-off for the effective theory be treated as larger than m_Q , dimensionally regularized NRQCD has to be supplemented by a prescription for expanding the Feynman *integrands*. This prescription is provided to all orders as part of the diagrammatic threshold expansion method [9].]

We briefly describe the calculation of the hard contributions to Γ_{QCD} . [Details of the calculational method and a solution to the recurrence algorithm for the two-loop integrals will be given in a long write-up of this Letter.] The spinor structure of the on-shell matrix element in QCD is conventionally parametrized by two form factors, F_1 and F_2 , of which only the combination $F_1 + F_2$ is required here. Since terms of order v^2 are not needed to determine C_0 [see Eq. (5)], we may set the relative momentum to zero and compute the form factors directly at threshold. The form factors have Coulomb singularities at threshold and diverge as $1/v^2$. However, these singularities appear only in the soft, potential and ultrasoft contributions, and the hard contribution is well-defined directly

	Colour factor	$\frac{1}{\epsilon^2}$	$\frac{1}{\epsilon}$	finite
D_1	C_F^2	$\frac{9}{32}$	$-\frac{27}{64} - \frac{5\pi^2}{24}$	$-\frac{81}{128} - \frac{133\pi^2}{96} - \frac{5\pi^2 \ln 2}{12} - \frac{35\zeta(3)}{8}$
D_2	C_F^2	$-\frac{3}{16}$	$-\frac{43}{32}$	$\frac{733}{192} + \frac{971\pi^2}{576}$
D_3	$C_F C_A$	$\frac{15}{32}$	$-\frac{5}{64} - \frac{\pi^2}{16}$	$\frac{715}{384} - \frac{319\pi^2}{576} - \frac{\pi^2 \ln 2}{8} - \frac{21\zeta(3)}{16}$
D_4	$C_F (C_A - 2C_F)$	0	$\frac{3}{16} - \frac{\pi^2}{16}$	$-\frac{39}{32} - \frac{251\pi^2}{1152} - \frac{3\pi^2 \ln 2}{8} - \frac{31\zeta(3)}{16}$
D_5	$C_F (C_A - 2C_F)$	$-\frac{9}{32}$	$-\frac{19}{64}$	$\frac{761}{384} + \frac{1157\pi^2}{1152} + \frac{\pi^2 \ln 2}{6} - \frac{3\zeta(3)}{4}$
D_6	$C_F T_F n_f$	$-\frac{1}{8}$	$\frac{5}{48}$	$-\frac{355}{288} - \frac{5\pi^2}{48}$
D_7	$C_F C_A$	$\frac{19}{128}$	$-\frac{53}{768}$	$\frac{6787}{4608} + \frac{95\pi^2}{768}$
D_8	$C_F C_A$	$\frac{1}{128}$	$\frac{1}{768}$	$\frac{361}{4608} + \frac{5\pi^2}{768}$
D_9	$C_F T_F$	$-\frac{1}{4}$	$\frac{13}{48}$	$-\frac{145}{96} + \frac{5\pi^2}{72}$
Sum	C_F^2	$\frac{21}{32}$	$-\frac{99}{64} - \frac{\pi^2}{12}$	$\frac{637}{384} - \frac{733\pi^2}{576} + \zeta(3)$
	$C_F C_A$	$\frac{11}{32}$	$-\frac{49}{192} - \frac{\pi^2}{8}$	$\frac{4811}{1152} + \frac{209\pi^2}{576} - \frac{\pi^2 \ln 2}{3} - 4\zeta(3)$
	$C_F T_F n_f$	$-\frac{1}{8}$	$\frac{5}{48}$	$-\frac{355}{288} - \frac{5\pi^2}{48}$
	$C_F T_F$	$-\frac{1}{4}$	$\frac{13}{48}$	$-\frac{145}{96} + \frac{5}{72}$

Table 1: Coefficient of $(\alpha_s/\pi)^2 (e^{\gamma_E} m_Q^2/(4\pi\mu^2))^{-2\epsilon}$ for the hard contribution to the diagrams of Fig. 1 evaluated at threshold $q^2 = 4m^2$. $D_{2,3,5}$ include a factor of 2 to account for the corresponding symmetric diagrams. For $SU(3)$ the colour factors are $C_F = 4/3$, $C_A = 3$, $T_F = 1/2$. The number of light (massless) quark flavours is denoted by n_f .

at threshold in dimensional regularization. The loop integrals simplify considerably, once the relative momentum is set to zero, since they then depend only on a single scale. We then project on the form factor $F_1 + F_2$ and reduce all integrals to integrals without numerators. These integrals can be further reduced to ‘simple’ integrals and two non-trivial two-loop integrals by means of recurrence relations derived from integration by parts in the loop momenta [10]. The solution to the recurrence relations is the difficult part of the calculation. The remaining non-trivial two-loop integrals can be calculated explicitly using standard Feynman parameters. The results obtained for the diagrams of Fig. 1 are listed in Table 1.

After summing all the diagrams, multiplying by the two-loop QCD on-shell wave function renormalization constant [11], and performing (one-loop) coupling and mass renormalization the result still contains poles in $\epsilon = (4 - d)/2$ (where d is the space-time dimension). Since the wave function renormalization constant $Z_{2,\text{NRQCD}}$ in NRQCD equals 1 up to higher-order terms in v^2 , not needed here, these poles are attributed to

an anomalous dimension of J , which first arises at the two-loop order. As a consequence the matrix element $\langle\psi|\psi^\dagger\sigma_i\chi|0\rangle(\mu)$ is factorization scale-dependent. We define it in the $\overline{\text{MS}}$ scheme and obtain the anomalous dimension for the NRQCD vector current J :

$$\gamma_J = \frac{d \ln Z_J}{d \ln \mu} = -C_F (2C_F + 3C_A) \frac{\pi^2}{6} \left(\frac{\alpha_s}{\pi}\right)^2 + \mathcal{O}(\alpha_s^3). \quad (6)$$

The scale dependence is compensated by the scale dependence of the two-loop matching coefficient $c_2(m_Q/\mu)$ in Eq. (4). Separating the different colour group factors, the final result for $c_2(m_Q/\mu)$ in the $\overline{\text{MS}}$ scheme is:

$$c_2(m_Q/\mu) = C_F^2 c_{2,A} + C_F C_A c_{2,NA} + C_F T_F n_f c_{2,L} + C_F T_F c_{2,H}, \quad (7)$$

$$c_{2,A} = \pi^2 \left[\frac{1}{6} \ln \left(\frac{m_Q^2}{\mu^2} \right) - \frac{79}{36} + \ln 2 \right] + \frac{23}{8} - \frac{\zeta(3)}{2}, \quad (8)$$

$$c_{2,NA} = \pi^2 \left[\frac{1}{4} \ln \left(\frac{m_Q^2}{\mu^2} \right) + \frac{89}{144} - \frac{5}{6} \ln 2 \right] - \frac{151}{72} - \frac{13\zeta(3)}{4}, \quad (9)$$

$$c_{2,L} = \frac{11}{18}, \quad (10)$$

$$c_{2,H} = -\frac{2\pi^2}{9} + \frac{22}{9}. \quad (11)$$

Here $\zeta(3) = 1.202\dots$ and we have taken all fermions with masses less than m_Q as massless, which is a good approximation even for $m_Q = m_b$, the bottom quark mass, in which case we neglect m_c , the charm quark mass. The coefficient $c_{2,L}$ proportional to n_f , the number of light fermions, has been recently obtained by Braaten and Chen [7]. Note also that the C_F^2 -term of the form factors $F_{1,2}$ close to threshold has been calculated by Hoang [12], using the absorptive parts of the form factors obtained in Ref. [13]. Because the result in Ref. [12] contains hard and soft (potential, ultrasoft) contributions, it is not possible to extract the matching coefficient $c_{2,A}$ from Ref. [12]. [The structures in $c_{2,A}$ that cannot arise from the small loop momentum regions agree with Hoang's result.]

The size of the two-loop correction to C_0 [Eq. (4)] given by Eqs. (7)–(11) is enormous. We define the (scale-dependent) non-relativistic decay constant as

$$\langle\psi|\psi^\dagger\sigma_i\chi|0\rangle(\mu) = (-i)f_\psi^{\text{NR}}(\mu)M_\psi\epsilon_i^*, \quad (12)$$

in analogy with Eq. (2). The non-relativistic decay constant is related to the ψ wave function at the origin by $M_\psi(f_\psi^{\text{NR}})^2 = 12|\Psi(0)|^2$. Using Eq. (3), we obtain

$$f_\psi = \left(1 - \frac{8\alpha_s(m_Q)}{3\pi} - (44.55 - 0.41n_f) \left(\frac{\alpha_s}{\pi}\right)^2 + \dots \right) f_\psi^{\text{NR}}(m_Q). \quad (13)$$

With $\alpha_s(m_c) \approx 0.35$ and $\alpha_s(m_b) \approx 0.21$ the second-order correction exceeds the first-order correction even for the bottomonium states. For charmonium, the second-order

term is almost twice as large as the already sizeable first-order correction. [Note that the BLM estimate of the two-loop correction [7] is far off the exact two-loop result.] Perturbative matching at the scale $\mu = m_Q$ does not seem to work. Can the factorization of short- and long-distance effects still be useful?

A novel, and perhaps unexpected, aspect at the two-loop level is the factorization scale dependence of the non-relativistic decay constant and, hence, the quarkonium wave function at the origin. The scale dependence is large, especially due to the non-abelian term in Eq. (6). This scale dependence of the wave function indicates the limitation of the non-relativistic potential model approach already at leading order in v^2 , since the wave functions obtained from solving the Schrödinger equation are scale-independent. Despite this shortcoming it may be argued that the wave function/non-relativistic decay constant obtained from potential models corresponds – if to anything – to the wave function evaluated at a scale typical for the bound state and not m_Q . This point of view is also evident if the wave function is computed non-perturbatively using lattice NRQCD, in which case the ultraviolet cut-off/factorization scale is also much smaller than m_Q . We consider $\mu = 1 \text{ GeV}$ as an adequate bound state scale for bottomonium and charmonium, as the applicability of perturbation theory prevents us from taking yet smaller scales. We then find, for bottomonium ($m_b = 5 \text{ GeV}$, $n_f = 4$):

$$f_{\Upsilon(nS)} = \left(1 - \frac{8\alpha_s(m_b)}{3\pi} - 1.74 \left(\frac{\alpha_s(m_b)}{\pi} \right)^2 + \dots \right) f_{\Upsilon(nS)}^{\text{NR}}(1 \text{ GeV}) \approx 0.81 f_{\Upsilon(nS)}^{\text{NR}}(1 \text{ GeV}). \quad (14)$$

The numerical factor 0.81 is stable against variations of the scale of the coupling at fixed factorization scale 1 GeV. At this factorization scale the second-order correction is numerically insignificant. Although we do not know whether the three-loop correction would also be small at the low factorization scale, we tend to consider Eq. (14) as an accurate prediction. This prediction could be tested, if $f_{\Upsilon(nS)}^{\text{NR}}(1 \text{ GeV})$ were accurately known, for example from NRQCD lattice simulations. The large scale dependence raises the theoretical question (the answer to which we postpone to the long write-up) as to whether it is necessary to resum the logarithms $\ln(m_Q^2/\mu^2)$ to all orders.

The situation is less favourable for charmonium states. Since $m_c \approx 1.5 \text{ GeV}$, the size of the second-order correction is altered little for scales $\mu > 1 \text{ GeV}$. We have not succeeded to find a trustworthy interpretation of Eq. (13) and conclude that the factorization of short-distance and long-distance effects may not be useful in practice for charmonium. Since the leptonic decay is the simplest conceivable decay, this puts into question the possibility to obtain universal relations between various charmonium decays and production processes through the use of NRQCD [3,14]. This pessimistic conclusion may be biased by our use of the $\overline{\text{MS}}$ factorization scheme. It is conceivable that other factorization schemes or relations between physical observables, from which the wave function at the origin is eliminated, exhibit better convergence properties of their perturbative series. A definitive conclusion on this issue can be obtained only once a second quarkonium decay is computed to two-loop order.

While this paper was being written, Czarnecki and Melnikov [15] also considered the two-loop matching of the electro-magnetic current, also using the asymptotic expansion method of [9]. After correction of a trivial normalization error (A. Czarnecki and K. Melnikov, private communication), their result agrees with ours.

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