# Infrared singularities in the null-plane bound-state equation when going to 1+1 dimensions

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## Abstract

In this paper we first consider the null-plane bound-state equation for a $q\bar{q}$ pair in 1+3 dimensions and in the lowest-order Tamm-Dancoff approximation. Light-cone gauge is chosen with a causal prescription for the gauge pole in the propagator. Then we show that this equation, when dimensionally reduced to 1+1 dimensions, becomes 't Hooft's bound-state equation, which is characterized by an $x^+$- instantaneous interaction. The deep reasons for this coincidence are carefully discussed.

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# I. INTRODUCTION

One of the challenging problems confronting gauge theories is the transition from theories defined in the usual 1+3 dimensions to 1+1 dimensional theories. In turn 1+1 dimensional theories are interesting as sometimes they are solvable, or, at least, they provide useful insights in non-perturbative phenomena.

A central role is played by the choice of the light-cone gauge, owing to its natural partonic interpretation. On the other hand this gauge, at least in perturbative treatments, exhibits more severe infrared (IR) singularities.

Yang-Mills theories in light-cone gauge were first quantized on a null plane (light-front quantization [1]). In this procedure the gauge pole in the polarization tensor occurring in the free propagator is treated according to the Cauchy principal value (CPV) prescription, which has the merit of being "real", namely not to contribute to the propagator absorptive part. However, in so doing, a conflict is induced with the usual "Feynman" pole, which, on physical grounds, in 1+3 dimensions must be prescribed in a causal way. This conflict can for instance be seen as the occurrence of extra unwanted terms when perturbative integrals undergo a Wick rotation [2].

To remedy this situation, a causal prescription (ML) was proposed in refs. [3], [4] for the gauge pole; this prescription was in turn derived by equal-time canonical quantization in ref. [5] and shown to be mandatory in 1+3 dimensions for a consistent renormalization [2].

When $x^+$-ordered perturbation theory is used, more severe IR singularities occur, which often have been regularized by means of artificial cutoffs. On the other hand, the ML prescription cannot be easily implemented. This difficulty is carefully explained in ref. [6] in which the bound-state equation for a $q\bar{q}$ pair is considered in the lowest-order Tamm-Dancoff approximation [7]. The relevance of using a causal prescription for the gauge pole is fully recognized and a concrete solution for implementing the ML prescription is proposed.

The situation drastically changes in 1+1 dimensions. Here ultraviolet (UV) singularities no longer occur, hence there is no need of renormalization. Both equal-time and null-plane

quantization seem a priori viable [8]. The latter indeed does no longer conflict with causality as no vector degrees of freedom propagate, the gauge field only providing an "instantaneous" potential between fermions. Canonical quantization suggests the CPV prescription on (both) Feynman and gauge poles. [1]

A celebrated example of this theory in the large-$N$ approximation is 't Hooft's bound-state equation [9]. From it a beautiful physical picture emerges with meson bound states lying on rising Regge trajectories. The counterpart of this equation in equal-time quantization was proposed by Wu [10], a quite difficult two-variable integral equation, whose (approximate) solution for particular values of external parameters has been obtained only very recently [11]. The resulting physical picture is quite different from 't Hooft's; in particular no rising Regge trajectories are found.

On the other hand, if the 1+1 theory is to be considered as the limit of a theory in higher dimensions, then the equal-time formulation (with related causal prescription) seems unavoidable. This is also the conclusion one reaches when considering a perturbative Wilson loop calculation at $\mathcal{O}(g^4)$ [12]: Feynman and light-cone gauges provide the same result, even in the limit $d \to 2$, only when canonically quantized at equal time. This result in turn is quite different from the one derived using the instantaneous potential coming from null-plane quantization.

Two different theories thus seem to exist in 1+1 dimensions, one being the limit of theories in higher dimensions, the other being simpler and endowed with nice physical consequences. We would like to stress that the difference is not in technical details: the two formulations have a different content of degrees of freedom [2].

Still, we show that the bound-state equation, in the lowest order Tamm-Dancoff approximation and with a causal prescription on the gauge pole, when dimensionally reduced to

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[1]Of course "Feynman" and gauge pole have to be treated coherently; we remind the reader that the product $\frac{1}{[q^+]_{CPV}} \frac{1}{[q^+]_{ML}}$ does not define a distribution.

1+1 dimensions, coincides with 't Hooft's equation, in spite of the fact that interaction is here described by an x$^+$ instantaneous potential. As a consequence, in this particular instance, the prescription on the poles turns out to be irrelevant. This phenomenon is rooted in the cancellation of IR singularities between "real" and "virtual" contributions [13].

The above considerations motivate the present work. We start from the concrete lowest-order Tamm-Dancoff approximation of ref. [6] in 1+3 dimensions (Sect. 2) and then, in Sect. 3, dimensionally reduce it to the 1+1 dimensional case: starting from the "causal" formulation of the bound-state equation, we show that it eventually coincides with the one in which the interaction is mediated by an x$^+$-instantaneous potential, namely with 't Hooft's equation, in spite of the seemingly different physical inputs. The reason for this coincidence as well as further considerations are drawn in Sect. 4.

## II. THE BETHE-SALPETER EQUATION IN 1+3 DIMENSIONS

In this section we recall concepts and results developed in ref. [6], which the reader is invited to consult. We follow the notation used there.

The integral equation for a bound state in the $q\bar{q}$ channel is considered in the null-plane formulation, $x^+ \equiv (x^0 + x^3)/\sqrt{2}$ playing the role of time. The idea behind this framework is that partons cannot pop up spontaneously from the vacuum, when the theory is quantized in a "physical" gauge; one usually chooses the light-cone gauge $A_- \equiv (A_0 - A_3)/\sqrt{2} = 0$. Then a truncation on the number of partons allowed in the wave function (Tamm-Dancoff approximation [7]) becomes viable.

For a deeper insight as well as physical motivations, the reader should consult the abundant literature on the subject (see references in [6]).

In light-front calculations, singularities occur in the IR region of $p^+$, which require a suitable prescription to be handled. The situation becomes worse in the gauge $A_- = 0$, as gauge-dependent singularities conspire with the previous "Feynman" ones and must be treated together in a consistent way.

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We consider a meson with momentum

$$P^\mu = \left(P^+, \frac{\mathbf{P}^2 + M^2}{2P^+}, \mathbf{P}\right), \tag{1}$$

which is composed of a quark and an antiquark. The meson state vector is normalized by

$$\langle P^+, \mathbf{P}|\hat{P}^+, \hat{\mathbf{P}}\rangle = (2\pi)^3 2P^+ \delta(P^+ - \hat{P}^+)\delta(\mathbf{P} - \hat{\mathbf{P}}). \tag{2}$$

Next we consider the Bethe-Salpeter wave function for the meson at $\mathbf{P} = 0$

$$\Phi(p)_{\alpha\beta} = \int d^4\mathbf{x} e^{ipx} \langle 0|\psi_\alpha(\mathbf{x})\bar{\psi}_\beta(0)|P^+, \mathbf{0}\rangle, \tag{3}$$

where $\psi_\alpha$ is the quark field. From eq. (3) one can project the null-plane wave function $\psi$ [6]

$$\psi(x, \mathbf{p}; s_1, s_2) = \frac{1}{2P^+} \int \frac{dp^-}{2\pi} \bar{u}(xP^+, \mathbf{p}; s_1)\gamma^+ \Phi(p)\gamma^+ v((1-x)P^+, -\mathbf{p}; s_2), \tag{4}$$

where $x = p^+/P^+$, normalized as

$$1 = (2\pi)^{-3} \int_0^1 \frac{dx}{2x(1-x)} \int d\mathbf{p} \sum_{ss'} |\psi(x, \mathbf{p}; s, s')|^2. \tag{5}$$

The spinors $u$ and $v$ are normalized to

$$\bar{u}_\alpha(p^+, \mathbf{p}; s)\gamma^+ u_\alpha(p^+, \mathbf{p}; s') = \bar{v}_\alpha(p^+, \mathbf{p}; s)\gamma^+ v_\alpha(p^+, \mathbf{p}; s') = 2p^+\delta_{ss'}. \tag{6}$$

If we denote by $S(p) = [-i(p\cdot\gamma - m)]^{-1}$ the free fermion propagator and by $\Sigma(p)$ the fermion self-energy, the Bethe-Salpeter equation takes the form

$$\Phi(p)_{\alpha\beta} = S(p)_{\alpha\alpha'}S(p-P)_{\beta'\beta} \int \frac{d^4k}{(2\pi)^4} T(p,k)_{\alpha'\alpha''\beta'\beta''} \Phi(k)_{\alpha''\beta''}$$
$$+ S(p)_{\alpha\alpha'}[-i\Sigma(p)_{\alpha'\alpha''}]\Phi(p)_{\alpha''\beta} + \Phi(p)_{\alpha\beta''}[-i\Sigma(p-P)_{\beta''\beta'}]S(p-P)_{\beta'\beta}$$
$$- S(p)_{\alpha\alpha'}[-i\Sigma(p)_{\alpha'\alpha''}]\Phi(p)_{\alpha''\beta''}[-i\Sigma(p-P)_{\beta''\beta'}]S(p-P)_{\beta'\beta}, \tag{7}$$

where summation over repeated indices is understood.

In the first term, $T(p,k)$ represents the Bethe-Salpeter kernel, consisting of all two-particle irreducible diagrams. We shall consider for it the first perturbative approximation, namely one-gluon exchange. Similarly the self-energy will be replaced by its one-loop approximation $\Sigma_1$. Renormalization in the $\overline{\text{MS}}$ scheme is understood.

## A. The "real" diagram contribution

Let us begin by considering the first term in eq. (7), in the approximation we have just mentioned:

$$\Phi(p)_{1GE} = iC_F g^2 S(p) \int \frac{d^4 k}{(2\pi)^4} \gamma_\alpha \Phi(k) \gamma_\beta$$

$$\times \frac{N^{\alpha\beta}(k-p)}{[(k-p) \cdot n][(k-p)^2 + i\epsilon]} S(p-P), \tag{8}$$

$C_F$ being the Casimir constant of the fundamental representation and

$$N^{\alpha\beta}(q) = -q \cdot n \, g^{\alpha\beta} + q^\alpha n^\beta + q^\beta n^\alpha$$

the numerator of the gluon propagator. The gauge fixing null vector $n^\mu = (0,1,0,0), n \cdot A = 0$ appears also in the denominator and gives rise to the mentioned gauge dependent singularity at $(k-p) \cdot n = 0$.

A simple algebra now gives

$$\psi_{1GE}(x, \mathbf{p}; s_1, s_2) = -iC_F g^2 \frac{x(1-x)P^+}{\pi} \frac{\bar{u}(xP^+, \mathbf{p}; s_1)}{p^2 - m^2 + i\epsilon} \int dp^- \int \frac{d^4 k}{(2\pi)^4} \gamma_\alpha \Phi(k) \gamma_\beta$$

$$\times \frac{N^{\alpha\beta}(k-p)}{[(k-p) \cdot n][(k-p)^2 + i\epsilon]} \frac{v((1-x)P^+, -\mathbf{p}; s_2)}{(P-p)^2 - m^2 + i\epsilon}. \tag{9}$$

In ref. [6] it is carefully explained how the Tamm-Dancoff approximation allows the above quantity $\psi_{1GE}$ to be expressed in terms of the null-plane wave function $\psi$. We are not going to repeat the argument and simply quote the result:

$$\psi_{1GE}(x, \mathbf{p}; s_1, s_2) = -\int \frac{d^2 \mathbf{k}}{(2\pi)^3} \int dy \int \frac{dk^-}{2\pi} \int \frac{dp^-}{2\pi} F(x, y, k^- - p^-, \mathbf{k}, \mathbf{p})$$

$$\times \left[ (k^- - \omega(y, \mathbf{k}^2) + i\epsilon \, \text{sign}(y))^{-1} - (k^- - \mathcal{E} + \omega(1-y, \mathbf{k}^2) - i\epsilon \, \text{sign}(1-y))^{-1} \right]$$

$$\times \left[ (p^- - \omega(x, \mathbf{p}^2) + i\epsilon \, \text{sign}(x))^{-1} - (p^- - \mathcal{E} + \omega(1-x, \mathbf{p}^2) - i\epsilon \, \text{sign}(1-x))^{-1} \right]$$

$$\times \left[ P^+(y-x) \right]^{-1} \left[ 2P^+(y-x)(k^- - p^-) - (\mathbf{k} - \mathbf{p})^2 + i\epsilon \right]^{-1}, \tag{10}$$

where $y = \frac{k^+}{P^+}$, $\omega(y, \mathbf{k}^2) = \frac{\mathbf{k}^2 + m^2}{2yP^+}$, $\mathcal{E} = \frac{M^2}{2P^+}$ and $F$ is a short-hand notation for the quantity

$$F(x, y, k^- - p^-, \mathbf{k}, \mathbf{p}) = -\frac{C_F g^2}{4P^+ y(1-y)} \sum_{s_1' s_2'} \psi(y, \mathbf{k}; s_1', s_2') \bar{u}(xP^+, \mathbf{p}; s_1) \gamma_\alpha u(yP^+, \mathbf{k}; s_1')$$

$$\frac{N^{\alpha\beta}(k-p)}{\mathcal{E} - \omega(1-x, \mathbf{p}^2) - \omega(x, \mathbf{p}^2) + i\epsilon} \bar{v}((1-y)P^+, -\mathbf{k}; s_2') \gamma_\beta v((1-x)P^+, -\mathbf{p}; s_2). \tag{11}$$

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Null-plane perturbation theory is recovered by performing the integrations over $k^-$ and $p^-$. The function $F$ depends on them linearly; therefore these integrations would be simple, were the gauge singularity $P^+(y - x) = 0$ in the denominator prescribed in such a way as not to involve minus components. Then contour integrations would lead to the result [6]

$$
\psi_{1GE}(x, \mathbf{p}; s_1, s_2) = \theta(1 - x)\theta(x) \int \frac{d^2\mathbf{k}}{(2\pi)^3}
$$
$$
\times \left\{ \int_x^1 dy \left[ P^+(y - x) \right]^{-1} \frac{F(x, y, \mathcal{E} - \omega(1 - y, \mathbf{k}^2) - \omega(x, \mathbf{p}^2), \mathbf{k}, \mathbf{p})}{2(y - x)P^+[\mathcal{E} - \omega(1 - y, \mathbf{k}^2) - \omega(x, \mathbf{p}^2)] - (\mathbf{k} - \mathbf{p})^2 + i\epsilon} \right.
$$
$$
\left. + \int_0^x dy \left[ P^+(y - x) \right]^{-1} \frac{F(x, y, -\mathcal{E} + \omega(y, \mathbf{k}^2) + \omega(1 - x, \mathbf{p}^2), \mathbf{k}, \mathbf{p})}{2(x - y)P^+[\mathcal{E} - \omega(y, \mathbf{k}^2) - \omega(1 - x, \mathbf{p}^2)] - (\mathbf{k} - \mathbf{p})^2 + i\epsilon} \right\}, \quad (12)
$$

where the support of the function $\psi$ has been explicitly exhibited.

Unfortunately the above expression is meaningless as there are manifest singularities at the extrema of integration. In other words the gauge singularity calls for a prescription before integrating over the minus components.

In refs. [3], [4] and [5], arguments are presented in favour of the *causal* prescription (ML)

$$
\frac{1}{[q^+]_{ML}} \equiv \frac{1}{q^+ + i\epsilon \, \mathrm{sign}(q^-)} = \frac{q^-}{q^+ q^- + i\epsilon}, \quad (13)
$$

which would not conflict with the (causal) "Feynman" poles, allowing for a Wick's rotation without extra contributions. This would not be the case for the CPV prescription, suggested in [1].

A problem then arises in eq. (10), as the integrations over the minus components can no longer be done in a straightforward way.

The solution proposed in ref. [6] is to perform a subtraction, using the identity (see also ref. [14])

$$
\int_{-\infty}^{+\infty} dy \frac{1}{[P^+(y - x)]_{ML}} \frac{1}{2(y - x)P^+[k^- - p^-] - (\mathbf{k} - \mathbf{p})^2 + i\epsilon} = 0. \quad (14)
$$

We stress that this identity holds only if the ML prescription is chosen.

By this subtraction the gauge pole is "sterilized"; the integrations over the minus components can be performed, now leading to the result

$$\psi_{1GE}(x, \mathbf{p}; s_1, s_2) = \theta(1 - x)\theta(x) \int \frac{d^2\mathbf{k}}{(2\pi)^3}$$

$$\times \left\{ \int_x^\infty dy \left[ P^+(y - x) \right]^{-1} \left( \frac{F(x, y, \mathcal{E} - \omega(1 - y, \mathbf{k}^2) - \omega(x, \mathbf{p}^2), \mathbf{k}, \mathbf{p})\theta(1 - y)}{2(y - x)P^+[\mathcal{E} - \omega(1 - y, \mathbf{k}^2) - \omega(x, \mathbf{p}^2)] - (\mathbf{k} - \mathbf{p})^2 + i\epsilon} \right. \right.$$

$$- \frac{F(x, x, \mathcal{E} - \omega(1 - x, \mathbf{k}^2) - \omega(x, \mathbf{p}^2), \mathbf{k}, \mathbf{p})}{2(y - x)P^+[\mathcal{E} - \omega(1 - x, \mathbf{k}^2) - \omega(x, \mathbf{p}^2)] - (\mathbf{k} - \mathbf{p})^2 + i\epsilon} \right)$$

$$+ \int_{-\infty}^x dy \left[ P^+(y - x) \right]^{-1} \left( \frac{F(x, y, -\mathcal{E} + \omega(y, \mathbf{k}^2) + \omega(1 - x, \mathbf{p}^2), \mathbf{k}, \mathbf{p})\theta(y)}{2(x - y)P^+[\mathcal{E} - \omega(y, \mathbf{k}^2) - \omega(1 - x, \mathbf{p}^2)] - (\mathbf{k} - \mathbf{p})^2 + i\epsilon} \right.$$

$$\left. \left. - \frac{F(x, x, -\mathcal{E} + \omega(x, \mathbf{k}^2) + \omega(1 - x, \mathbf{p}^2), \mathbf{k}, \mathbf{p})}{2(x - y)P^+[\mathcal{E} - \omega(x, \mathbf{k}^2) - \omega(1 - x, \mathbf{p}^2)] - (\mathbf{k} - \mathbf{p})^2 + i\epsilon} \right) \right\}. \tag{15}$$

No end-point singularities are left after this procedure.

## B. The self-energy contributions

Now we turn our attention to the other three terms in eq. (7) involving the self-energy in which we will retain, coherently with the approximation done on the "exchange" graph, only the one-loop contribution. One can have a self-energy insertion on the quark line, on the antiquark line, or on both.

The terms involving the quark and the antiquark self-energy are, respectively

$$\Phi_{SE1}(p) = -iS(p)\Sigma_1(p)\Phi(p)$$

and

$$\Phi_{SE2}(p) = -i\Phi(p)\Sigma_1(p - P)S(p - P).$$

Here $\Sigma_1$ is the one-loop self-energy, renormalized in the $\overline{\text{MS}}$ scheme. The corresponding contributions to the null-plane wave function are

$$\psi_{SE1}(x, \mathbf{p}; s_1, s_2) = \int \frac{xdp^-}{2\pi} \frac{\bar{u}(xP^+, \mathbf{p}; s_1)\Sigma_1(p)\Phi(p)\gamma^+ v((1 - x)P^+, -\mathbf{p}; s_2)}{p^2 - m^2 + i\epsilon} \tag{16}$$

and

$$\psi_{SE2}(x, \mathbf{p}; s_1, s_2) = -\int \frac{(1 - x)dp^-}{2\pi} \frac{\bar{u}(xP^+, \mathbf{p}; s_1)\gamma^+ \Phi(p)\Sigma_1(p - P)v((1 - x)P^+, -\mathbf{p}; s_2)}{(p - P)^2 - m^2 + i\epsilon}, \tag{17}$$

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respectively.

In ref. [6] it is carefully explained how the contribution from the self-energy insertion on both quark and antiquark lines can be split into two pieces, one that will cancel part of $\Phi_{SE1}$ and another that will cancel part of $\Phi_{SE2}$. These cancellations are part of the Tamm-Dancoff approximation we are considering. We are thereby left with the following two self-energy contributions:

$$
\begin{aligned}
\psi_{SE1}(x, \mathbf{p}; s_1, s_2) = \frac{1}{2xP^+} \sum_{s_1'} \psi(x, \mathbf{p}; s_1', s_2) \\
\times \frac{\bar{u}(xP^+, \mathbf{p}; s_1)\Sigma_1(xP^+, \mathcal{E} - \omega(1-x, \mathbf{p}^2), \mathbf{p})u(xP^+, \mathbf{p}; s_1')}{\mathcal{E} - \omega(1-x, \mathbf{p}^2) - \omega(x, \mathbf{p}^2) + i\epsilon}
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
\psi_{SE2}(x, -\mathbf{p}; s_1, s_2) = -\frac{1}{2(1-x)P^+} \sum_{s_2'} \psi(x, -\mathbf{p}; s_1, s_2') \\
\times \frac{\bar{v}((1-x)P^+, -\mathbf{p}; s_2')\Sigma_1(-(1-x)P^+, -\mathcal{E} + \omega(x, \mathbf{p}^2), \mathbf{p})v((1-x)P^+, -\mathbf{p}; s_2)}{\mathcal{E} - \omega(1-x, \mathbf{p}^2) - \omega(x, \mathbf{p}^2) + i\epsilon}.
\end{aligned}
\tag{19}
$$

The unrenormalized quark self-energy in the one-loop approximation is given by

$$
\Sigma_1^\varepsilon(p) = g^2 C_F \mu^{2\varepsilon} \int \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}} \gamma_\alpha S(k) \gamma_\beta \frac{N^{\alpha\beta}(q)}{[q \cdot n]_{ML}(q^2 + i\epsilon)},
\tag{20}
$$

where $k^\alpha$ is the quark momentum and $q^\alpha = p^\alpha - k^\alpha$ the gluon momentum. We use dimensional regularization, the coupling constant g is dimensionless and $\mu$ is the running mass scale.

This equation can be rewritten as

$$
-i\Sigma_1^\varepsilon(p) = g^2 C_F \mu^{2\varepsilon} \int \frac{d^{2-2\varepsilon}\mathbf{k}}{(2\pi)^{2-2\varepsilon}} \left[ A + B^\mu \gamma_\mu + C^\mu \gamma_\mu \right],
\tag{21}
$$

where $B^\mu$ is the only term that receives a contribution from the gauge pole. The quantities $A$, $B^\mu$ and $C^\mu$ are given by

$$
A = \int \frac{dk^+ dk^-}{(2\pi)^2} \frac{-2m(1-\varepsilon)}{(p-k)^2 + i\epsilon} \frac{1}{k^2 - m^2 + i\epsilon},
\tag{22}
$$

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$$B^\mu = \int \frac{dk^+ dk^-}{(2\pi)^2} \frac{1}{[p^+ - k^+]_{ML}} \frac{1}{2(p^+ - k^+)(p^- - k^-) - (\mathbf{p} - \mathbf{k})^2 + i\epsilon}$$
$$\times \frac{\mathcal{B}^\mu(k^+, k^-, \mathbf{k}; p^+, p^-, \mathbf{p})}{2k^+ k^- - \mathbf{k}^2 - m^2 + i\epsilon}, \tag{23}$$

with

$$\mathcal{B}^+ = 0,$$

$$\mathcal{B}^- = 4k^+(p^- - k^-) - 2\mathbf{k} \cdot (\mathbf{p} - \mathbf{k}),$$

$$\mathcal{B}^j = 2k^+(p^j - k^j) - 2(p^+ - k^+)k^j$$

and

$$C^\mu = \int \frac{dk^+ dk^-}{(2\pi)^2} \frac{2k^\mu(1-\varepsilon)}{(p-k)^2 + i\epsilon} \frac{1}{k^2 - m^2 + i\epsilon}. \tag{24}$$

The gauge singularity in eq. (23), being prescribed according to (ML), does not spoil the convergence of the integrals. In other words no singularity of IR type occurs, thanks to the prescription, while UV singularities in eq. (21) are cured by dimensional regularization. In passing we stress that this procedure has the merit of clearly disentangling possible IR and UV singularities.

Now the gauge pole can be "sterilized" by a suitable subtraction, in the same way as we did for the exchange term, thereby allowing us to perform the integration over $k^-$; we obtain

$$B^\mu = \frac{-i}{4\pi p^+} \int_{-\infty}^1 \frac{dx}{1-x} \left\{ \frac{\theta(x)\mathcal{B}^\mu(xp^+, \frac{\mathbf{k}^2 + m^2}{2xp^+}, \mathbf{k}; p^+, p^-, \mathbf{p})}{2x(1-x)p^+ p^- - (1-x)(\mathbf{k}^2 + m^2) - x(\mathbf{p} - \mathbf{k})^2 + i\epsilon} \right.$$
$$\left. - \frac{\mathcal{B}^\mu(p^+, \frac{\mathbf{k}^2 + m^2}{2p^+}, \mathbf{k}; p^+, p^-, \mathbf{p})}{2(1-x)p^+ p^- - (1-x)(\mathbf{k}^2 + m^2) - (\mathbf{p} - \mathbf{k})^2 + i\epsilon} \right\}. \tag{25}$$

We notice that the potential singularity at $x = 1$ is cancelled by the subtraction. We also notice that the two subtractions, the one in the exchange term and the one in the self-energy expressions, although dictated by a similar philosophy, have nothing to do with each other.

The null-plane wave function $\psi$ is eventually obtained as

$$\psi(x, \mathbf{p}; s_1, s_2) = \psi_{1GE}(x, \mathbf{p}; s_1, s_2) + \psi_{SE1}(x, \mathbf{p}; s_1, s_2) + \psi_{SE2}(x, \mathbf{p}; s_1, s_2). \tag{26}$$

At this point we have recalled from ref. [6] all the concepts we need to develop our argument in the next section.

## III. THE BETHE-SALPETER EQUATION IN 1+1 DIMENSIONS

When going to 1+1 dimensions, UV singularities will no longer show up; in turn the IR behaviour is worsened. Subtleties occur in this dimensional reduction.

We start from unrenormalized, dimensionally regularized quantities. First, in 1+1 dimensions, the coupling constant acquires the dimension of a mass; this is automatically provided by the factor $\mu^{2\varepsilon}$. But, in this case, the meaning of such a mass completely changes: it is no longer a running mass scale, but rather it tunes the dimensionful coupling, which is a free parameter characterizing the theory.

Second, the quantities $A$ and $C^\mu$ in eqs. (22),(24) vanish, in strictly 1+1 dimensions, as a consequence of the Dirac algebra. However, if the calculation is performed in $4 - 2\varepsilon$ dimensions and the loop integration over transverse momenta is carried on, the $\varepsilon - 1$ zero coming from the polarization factor, is fully compensated by a pole, leading eventually in the limit $\varepsilon \to 1$ to the non vanishing expressions

$$-i\Sigma_A = -\frac{iC_F g^2}{2\pi} \frac{m}{p^2 - m^2 + i\epsilon} \tag{27}$$

and

$$-i\Sigma_C = -\frac{iC_F g^2}{2\pi} \frac{p^\mu \gamma_\mu}{p^2 - m^2 + i\epsilon}, \tag{28}$$

where we have again denoted by $g$ the coupling constant of the theory, which differs from the coupling constant of the previous Section by the factor $\mu$.

We stress that the above quantities are sensitive to the way in which the transition to 1+1 dimensions is performed. This anomaly-type phenomenon is reminiscent of an analogous effect we found in perturbative Wilson loop calculations [12] and is worthy of further study; it points towards a discontinuity of the theory in the limit $\varepsilon \to 1$ [8].

The terms $B^\mu$, which are the ones affected by the gauge pole, are instead insensitive to the way in which the reduction is performed: the same result is indeed obtained just ignoring transverse degrees of freedom or taking the limit $\varepsilon \to 1$ at the very end of the calculation.

There is then the problem of formulating the gluon exchange contribution in $4-2\varepsilon$ dimensions. To this purpose one should consider unrenormalized quantities, which are expected to produce singularities of UV nature just as poles at some integer values of dimensions. Unfortunately, in order to decide whether the limit $\varepsilon \to 1$ is smooth, one should solve the integral equation for a generic value of $\varepsilon$, or, at least, to have a control on its behavior with respect to transverse momentum.

We leave to a future investigation the interesting problem of studying the limit $\varepsilon \to 1$. In the sequel we adopt the attitude of working directly in 1+1 dimensions, "freezing" the transverse degrees of freedom. We drop everywhere the transverse-momentum dependence in eq. (10). This procedure turns the simple pole at $P^+(y - x) = 0$ into a double pole. Integration over this double pole is perfectly prescribed, though; thanks to the ML recipe, both singularities lie on the same side of the integration contour. In other words no pinch occurs when dropping transverse momenta. Nevertheless a double pole would require *two* subtractions to be sterilized. We would like to stress again that this "sterilization" is *not* required to give the integrals a meaning (they are indeed already perfectly defined), but motivated by the desire to perform first the integration over minus components in order to recover the null-plane perturbative formulation.

We might operate subtractions also in this case, repeating the treatment of the previous section; however, as it will become apparent that subtractions are not needed in 1+1 dimensions, we shall recover null-plane perturbation theory by following a slightly different procedure.

In 1+1 dimensions, eq. (10) becomes

$$
\begin{aligned}
\psi_{1GE}(x) = &- \int \frac{dy}{2\pi} \int \frac{dk^-}{2\pi} \int \frac{dp^-}{2\pi} F(x, y, k^- - p^-) \\
&\times \left[ (k^- - \omega(y) + i\epsilon \, \mathrm{sign}(y))^{-1} - (k^- - \mathcal{E} + \omega(1 - y) - i\epsilon \, \mathrm{sign}(1 - y))^{-1} \right]
\end{aligned}
$$

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$$\times \left[ (p^- - \omega(x) + i\epsilon \, \text{sign}(x))^{-1} - (p^- - \mathcal{E} + \omega(1-x) - i\epsilon \, \text{sign}(1-x))^{-1} \right]$$

$$\times \frac{1}{\left[ P^+(y-x) \right]_{ML}} \left[ 2P^+(y-x)(k^- - p^-) + i\epsilon \right]^{-1}, \tag{29}$$

with

$$F(x, y, k^- - p^-) = -\frac{C_F g^2}{4P^+ y(1-y)} \psi(y) \bar{u}(xP^+) \gamma^+ u(yP^+)$$

$$\frac{2(k^- - p^-)}{\mathcal{E} - \omega(1-x) - \omega(x) + i\epsilon} \bar{v}((1-y)P^+) \gamma^+ v((1-x)P^+). \tag{30}$$

Taking the detailed expressions of the light-cone spinors into account [15], eq. (30) can be written as

$$F(x, y, k^- - p^-) = -2C_F g^2 P^+ \psi(y) \frac{\sqrt{x(1-x)}}{\sqrt{y(1-y)}} \frac{(k^- - p^-)}{\mathcal{E} - \omega(1-x) - \omega(x) + i\epsilon}. \tag{31}$$

Equation (29) in turn becomes

$$\psi_{1GE}(x) = \frac{C_F g^2 P^+}{\mathcal{E} - \omega(1-x) - \omega(x) + i\epsilon} \int \frac{dy}{2\pi} \int \frac{dk^-}{2\pi} \int \frac{dp^-}{2\pi} \psi(y) \frac{\sqrt{x(1-x)}}{\sqrt{y(1-y)}}$$

$$\times \left[ (k^- - \omega(y) + i\epsilon \, \text{sign}(y))^{-1} - (k^- - \mathcal{E} + \omega(1-y) - i\epsilon \, \text{sign}(1-y))^{-1} \right]$$

$$\times \left[ (p^- - \omega(x) + i\epsilon \, \text{sign}(x))^{-1} - (p^- - \mathcal{E} + \omega(1-x) - i\epsilon \, \text{sign}(1-x))^{-1} \right]$$

$$\times \left[ P^+(y-x) \right]_{ML}^{-2}. \tag{32}$$

We have now reached a complete symmetry between gauge and "Feynman" pole. This pole should be prescribed causally in equal-time quantization; this is certainly mandatory when propagating transverse degrees of freedom are present, *i.e.* in higher dimensions. Its causal prescription forces the gauge pole to be causal too, for consistency. On the other hand, the causal option follows from equal-time quantization [12].

Let us now go back to eq. (13) and consider the identity

$$\frac{1}{[q^+]_{ML}} \equiv \frac{1}{q^+ + i\epsilon \, \text{sign}(q^-)} = \frac{1}{[q^+]_{CPV}} - i\pi \, \text{sign}(q^-) \delta(q^+), \tag{33}$$

which, after differentiation with respect to $q^+$, becomes

$$\frac{1}{[q^+]_{ML}^2} = \frac{1}{[q^+]_{CPV}^2} + i\pi \, \text{sign}(q^-) \delta'(q^+). \tag{34}$$

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At this point it is convenient to change the normalization of the function $\psi$, by defining

$$\psi(x) = \phi(x)\sqrt{x(1-x)}.$$

Introducing eq. (34) in eq. (32), we obtain

$$\phi_{1GE}(x) = \phi_{1GE}^{CPV}(x) + \phi_{1GE}^{(s)}, \tag{35}$$

with

$$
\begin{aligned}
\phi_{1GE}^{CPV}(x) = {} & \frac{C_F g^2 P^+}{\mathcal{E} - \omega(1-x) - \omega(x) + i\epsilon} \int \frac{dy}{2\pi} \int \frac{dk^-}{2\pi} \int \frac{dp^-}{2\pi} \phi(y) \\
& \times \left[ (k^- - \omega(y) + i\epsilon\,\mathrm{sign}(y))^{-1} - (k^- - \mathcal{E} + \omega(1-y) - i\epsilon\,\mathrm{sign}(1-y))^{-1} \right] \\
& \times \left[ (p^- - \omega(x) + i\epsilon\,\mathrm{sign}(x))^{-1} - (p^- - \mathcal{E} + \omega(1-x) - i\epsilon\,\mathrm{sign}(1-x))^{-1} \right] \\
& \times \left[ P^+(y-x) \right]_{CPV}^{-2}
\end{aligned}
\tag{36}
$$

and

$$
\begin{aligned}
\phi_{1GE}^{(s)}(x) = {} & \frac{i\pi C_F g^2 P^+}{\mathcal{E} - \omega(1-x) - \omega(x) + i\epsilon} \int \frac{dy}{2\pi} \int \frac{dk^-}{2\pi} \int \frac{dp^-}{2\pi} \phi(y) \\
& \times \left[ (k^- - \omega(y) + i\epsilon\,\mathrm{sign}(y))^{-1} - (k^- - \mathcal{E} + \omega(1-y) - i\epsilon\,\mathrm{sign}(1-y))^{-1} \right] \\
& \times \left[ (p^- - \omega(x) + i\epsilon\,\mathrm{sign}(x))^{-1} - (p^- - \mathcal{E} + \omega(1-x) - i\epsilon\,\mathrm{sign}(1-x))^{-1} \right] \\
& \times \ \mathrm{sign}(k^- - p^-)\delta'(P^+(y-x)) \ .
\end{aligned}
\tag{37}
$$

In eq. (36) the integrations over the minus components of the momenta can be easily performed, leading to the expression

$$\phi_{1GE}^{CPV}(x) = -\frac{C_F g^2}{P^+[\mathcal{E} - \omega(1-x) - \omega(x) + i\epsilon]} \int_0^1 \frac{dy}{2\pi} \phi(y) \left[ (y-x) \right]_{CPV}^{-2} \ . \tag{38}$$

In turn, eq. (37) becomes

$$
\begin{aligned}
\phi_{1GE}^{(s)}(x) = {} & -\frac{i\pi C_F g^2}{\mathcal{E} - \omega(1-x) - \omega(x) + i\epsilon} \int \frac{dy}{2\pi} \int \frac{dk^-}{2\pi} \int \frac{dp^-}{2\pi} \delta(y-x)\,\mathrm{sign}(k^- - p^-) \\
& \times \left[ (p^- - \omega(x) + i\epsilon\,\mathrm{sign}(x))^{-1} - (p^- - \mathcal{E} + \omega(1-x) - i\epsilon\,\mathrm{sign}(1-x))^{-1} \right] \\
& \times \Big( \ \phi'(y)\left[ (k^- - \omega(y) + i\epsilon\,\mathrm{sign}(y))^{-1} - (k^- - \mathcal{E} + \omega(1-y) - i\epsilon\,\mathrm{sign}(1-y))^{-1} \right] \\
& + \phi(y)\frac{d}{dy}\left[ (k^- - \omega(y) + i\epsilon\,\mathrm{sign}(y))^{-1} - (k^- - \mathcal{E} + \omega(1-y) - i\epsilon\,\mathrm{sign}(1-y))^{-1} \right] \Big) \ . \tag{39}
\end{aligned}
$$

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Now integrations over the minus components can be done; the first term vanishes for symmetry reasons; the second one, after some algebra, taking the expression for $\omega$ into account, becomes

$$\phi_{1GE}^{(s)}(x) = \frac{g^2 m^2 C_F \phi(x)}{4\pi(P^+)^2} \frac{1}{(\mathcal{E} - \omega(x) - \omega(1-x) + i\epsilon)^2} [x^{-2} + (1-x)^{-2}]. \tag{40}$$

Then we repeat the treatment in the expressions concerning the self-energy contributions. Let us therefore go back to eq. (23), which, in 1+1 dimensions, becomes

$$B^- = \int \frac{2k^+ dk^+ dk^-}{(2\pi)^2} \frac{1}{[p^+ - k^+]_{ML}^2} \frac{1}{2k^+ k^- m^2 + i\epsilon} = \frac{ip^-}{\pi(p^2 - m^2 + i\epsilon)}. \tag{41}$$

Using the identity (34), we obtain the splitting

$$B^- = \frac{i}{2\pi p^+} + \frac{im^2}{\pi(2xP^+)^2} \frac{1}{\mathcal{E} - \omega(x) - \omega(1-x) + i\epsilon} \tag{42}$$

and, correspondingly,

$$\phi_{SE1}(x) = \phi_{SE1}^{CPV}(x) + \phi_{SE1}^{(s)}(x) \tag{43}$$

with

$$\phi_{SE1}^{CPV}(x) = -\frac{g^2 C_F}{2\pi x P^+} \phi(x) \frac{1}{\mathcal{E} - \omega(1-x) - \omega(x) + i\epsilon} \tag{44}$$

and

$$\phi_{SE1}^{(s)}(x) = -\frac{g^2 m^2 C_F}{\pi(2xP^+)^2} \phi(x) \frac{1}{(\mathcal{E} - \omega(1-x) - \omega(x) + i\epsilon)^2}. \tag{45}$$

Similarly, for the second self-energy contribution we get

$$\phi_{SE2}(x) = \phi_{SE2}^{CPV}(x) + \phi_{SE2}^{(s)}(x), \tag{46}$$

with

$$\phi_{SE2}^{CPV}(x) = -\frac{g^2 C_F}{2\pi(1-x)P^+} \phi(x) \frac{1}{\mathcal{E} - \omega(1-x) - \omega(x) + i\epsilon} \tag{47}$$

and

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$$\phi_{SE2}^{(s)}(x) = -\frac{g^2 m^2 C_F}{\pi(2(1-x)P^+)^2}\phi(x)\frac{1}{(\mathcal{E} - \omega(1-x) - \omega(x) + i\epsilon)^2} \ . \tag{48}$$

Summing everything together, we find that all $\phi^{(s)}$'s cancel and we are left with:

$$\phi(x) = -\frac{C_F g^2}{2\pi P^+[\mathcal{E} - \omega(1-x) - \omega(x) + i\epsilon]}$$
$$\times \Big[\frac{\phi(x)}{x(1-x)} + \int_0^1 dy\phi(y)\big[(y-x)\big]_{CPV}^{-2}\Big] \ . \tag{49}$$

ML and CPV are completely equivalent in this case!

We remark that eq. (49) is nothing but 't Hooft's equation [9], in spite of the seemingly different physical inputs.

## IV. FINAL REMARKS

We started by considering a "causal" formulation of the bound-state integral equation in the lowest-order Tamm-Dancoff approximation, in particular by considering only one-loop contributions to the self-energy, and then, after a suitable dimensional reduction, we ended up with 't Hooft's equation in which all planar diagrams are summed (large-$N$ approximation) with an "instantaneous" potential between quarks. How did it happen?

The reason why "causal" and "instantaneous" interactions lead to the same answer in this case has already been anticipated; it is rooted in the cancellations occurring in 1+1 dimensions thanks to one-loop unitarity. Those cancellations had already been noticed [16], although in a different context and with a different technique.

In turn the reason why the Tamm-Dancoff approximation reproduces 't Hooft's full planar summation is due to the dynamical circumstance that in 't Hooft's formulation the exact solution for the self-energy coincides with its $\mathcal{O}(g^2)$ expression.

This fact also explains why we did not recover Wu's equation, when considering the "causal" formulation. As a matter of fact, in Wu's treatment the exact solution for the self-energy exhibits a quite involved analytical structure; in particular, it does not generally

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match, in the relevant Ward identity, the expression used for the vertex in the bound-state equation.

Since at large $N$ the same set of diagrams, the planar ones, are summed in both formulations, we envisage a potential conflict, beyond the one-loop approximation, between planarity and "causal" formulation in 1+1 dimensions. This crucial issue in our opinion deserves further study.

In dimensions higher than 2, causality looks mandatory and only one formulation (the "causal" one) can reasonably survive.

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