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MASS PERTURBATIONS IN TWISTED $N = 4$ SUPERSYMMETRIC GAUGE THEORIES

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ABSTRACT

Mass perturbations of the twisted $N = 4$ supersymmetric gauge theory considered by Vafa and Witten to test S-duality are studied for the case of Kahler four-manifolds. It is shown that the resulting mass-perturbed theory can be regarded as an equivariant extension associated to a $U(1)$ symmetry of the twisted theory, which is only present for Kahler manifolds. In addition, it is shown that the partition function, the only topological invariant of the theory, remains invariant under the perturbation.

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1. Introduction

Topological quantum field theory [1] has become a very useful framework to make predictions in differential topology, and to test some of the recent ideas emerged in the context of duality as a symmetry of field theories with extended supersymmetry [2, 3]. The most celebrated examples are the prediction made by Witten [4] stating that the Donaldson invariants of four-manifolds can be expressed in terms of the Seiberg-Witten invariants, and the strong-coupling test of S-duality carried out by Vafa and Witten [5] making use of a twisted four-dimensional $N = 4$ supersymmetric gauge theory.

The topological quantum field theory leading to the Donaldson invariants can be regarded as a twisted version of the $N = 2$ supersymmetric pure gauge theory. This theory, now known as the Donaldson-Witten theory, possesses observables whose correlation functions correspond to those invariants. These quantities are independent of the coupling constant of the theory and can thus be studied in both the weak and the strong coupling limits. By going to the weak coupling limit, it can be shown that these correlation functions do in fact correspond to the Donaldson polynomials of four-manifolds. These are basically intersection numbers on classical instanton moduli spaces, which are sensitive to the differentiable structure of the four-manifold. However, while the weak-coupling analysis provides an astonishing link to the Donaldson theory, it is not possible to perform explicit calculations without using the standard methods inherent in the Donaldson theory. A natural way around is to exploit the coupling constant independence of the theory to study it in the strong-coupling limit. However, this analysis requires a precise knowledge of the infrared behaviour of the $N = 2$ supersymmetric gauge theory, and this was out of reach until the explicit solution of Seiberg and Witten [2, 3]. The understanding of the strong-coupling dynamics of the $N = 2$ supersymmetric gauge theory triggered a major breakthrough, by turning the problem of calculating correlation functions in a twisted supersymmetric gauge theory into one of counting solutions of Witten's Abelian monopole equations [4]. This approach makes possible an explicit calculation of the Donaldson polynomials in terms of Seiberg-Witten invariants. A similar structure has been proposed for a generalization of the Donaldson-Witten theory known as the non-Abelian monopole theory [6]. Recently, these results have been reviewed in [7], and they have been extended and rederived in a more general framework in [8].

There is, however, a complementary approach due to Witten [9] (sometimes referred to as the "abstract" approach, as opposite to the "concrete" approach described in the previous paragraph), which works only on Kahler manifolds and relies heavily on standard results on $N = 1$ supersymmetric gauge theories, such as gluino condensation and chiral symmetry breaking. But this is doubly as good, for the agreement found between the proposed formulas in the topological field theory and previously known mathematical results gives support to the conjectured picture in the physical theory. The same idea has subsequently been applied to other $N = 2$ supersymmetric gauge theories, as in [10], to obtain explicit results for the topological invariants associated to non-Abelian monopole theory, and also to one of the twisted $N = 4$ supersymmetric gauge theories [5], to make an explicit computation of the partition function of the theory on Kahler manifolds.

The way the construction works is the following. When formulated on Kahler manifolds, the number of BRST charges of a topological quantum field theory is doubled, in such a way that, for example, the Donaldson-Witten theory has an enhanced $N_T = 2$ topological symmetry on Kahler manifolds, while the Vafa-Witten theory has $N_T = 4$ topological symmetry. In either case, one of the BRST charges comes from the underlying $N = 1$ subalgebra which corresponds to the formulation of the physical theory in $N = 1$ superspace. By suitably adding mass terms for some of the chiral superfields in the theory, one can break the extended ($N = 2$ or $N = 4$) supersymmetry of the physical theory down to $N = 1$. For the reason sketched above, the corresponding twisted massive theory on Kahler manifolds should still retain at least one topological symmetry. One now exploits the metric independence of the topological theory. By scaling up the metric in the topological theory, $g_{\mu\nu} \to t g_{\mu\nu}$, one can take the limit $t \to \infty$. In this limit, the metric on X becomes nearly flat. As the twisted and the physical theories coincide on flat and hyper-Kahler manifolds, this means that in the $t \to \infty$ limit the predictions of the perturbed topological theory should coincide with those of the physical (massive) theory. But the $t \to \infty$ limit also corresponds to the infrared limit of the physical $N = 1$ supersymmetric gauge theory, in which the massive superfields can be integrated out, so one is left with an effective massless $N = 1$ supersymmetric gauge theory (possibly) coupled to $N = 1$ supersymmetric matter, whose infrared behaviour is –hopefully– easier to deal with. In this way, the computations in the topological field theory can be reduced to the analysis of contributions from the vacua of the associated $N = 1$ supersymmetric gauge theory.

There is, however, an obvious drawback to this construction. The introduction of a mass perturbation may (and in general will) distort the original topological field theory. This poses no problem in the case of the Donaldson-Witten theory, as Witten was able to prove that the perturbation is topologically trivial, in the sense that it affects the theory in an important but controllable way [9]. However, the arguments presented there do not carry over to other, more general situations, so one has to repeat the analysis case by case. In the case of the Vafa-Witten theory, the required perturbation gives rise to an a priori different theory, in fact an equivariant extension of the original theory with respect to a $U(1)$ action on the moduli space, which is present only on Kahler manifolds. We do not know whether the theories are actually different or not. But in any case, we are primarily interested in calculating the partition function of the theory which, as we will argue below, is actually invariant under the perturbation.

The main purpose of this paper is to show that, as assumed in [5], the abstract approach can be applied successfully to the Vafa-Witten theory on Kahler manifolds. In the process we find that the mass-perturbed theory involved in this approach can be regarded as an equivariant extension associated to a certain $U(1)$ symmetry.

The paper is organized as follows. In sect. 2 we review the twisting procedure involved in $N = 4$ supersymmetric gauge theories, and formulate the Vafa-Witten theory on Kahler manifolds. In sect. 3 we analyse the possible mass perturbations of the theory, and show that the partition function associated to the mass-perturbed Vafa-Witten theory remains invariant. In sec. 4 we reformulate the mass-perturbed theory as an equivariant extension associated to a $U(1)$ symmetry present in the Vafa-Witten theory on Kahler manifolds. Finally, in sect. 5 we present our conclusions.

2. Twisting of $N = 4$ supersymmetric gauge theory on Kahler manifolds

In this section we review some aspects of the twisting of four-dimensional $N = 4$ supersymmetric gauge theories, and we present the form of one of the twisted theories, the Vafa-Witten theory, for the case of Kahler manifolds.

2.1 $N = 4$ SUPERSYMMETRIC GAUGE THEORY

We begin by recalling several generalities about the $N = 4$ supersymmetric gauge theory on flat $\mathbb{R}^4$. From the point of view of $N = 1$ superspace, the theory contains one $N = 1$ vector multiplet and three $N = 1$ chiral multiplets. These supermultiplets are represented in $N = 1$ superspace by the superfields V and Φ_s ($s = 1, 2, 3$), which satisfy the constraints $V = V^\dagger$ and $\overline{D}_{\dot\alpha}\Phi_s = 0$, $\overline{D}_{\dot\alpha}$ being a superspace covariant derivative⋆. The physical component fields of these superfields will be denoted as follows:

$$
\begin{aligned}
V &\longrightarrow A_{\alpha\dot\alpha},\ \lambda_{4\alpha},\ \overline{\lambda}^{4}{}_{\dot\alpha},\\
\Phi_s, \Phi^{\dagger s} &\longrightarrow B_s,\ \lambda_{s\alpha},\ B^{\dagger s},\ \overline{\lambda}^{s}{}_{\dot\alpha}.
\end{aligned}
\tag{2.1}
$$

The $N = 4$ supersymmetry algebra has the automorphism group $SU(4)_I$. The field content of the corresponding field theory is conventionally arranged so that the gauge bosons are scalars under $SU(4)_I$, while the gauginos and the scalar fields transform respectively as $\mathbf{4}\oplus\overline{\mathbf{4}}$ and $\mathbf{6}$. All the above fields take values in the adjoint representation of some compact Lie group G. The action takes the following form in $N = 1$ superspace:

$$
\begin{aligned}
\mathcal{S} = &-\frac{i}{4\pi}\tau\int d^4x d^2\theta\,\mathrm{Tr}(W^2) + \frac{i}{4\pi}\overline{\tau}\int d^4x d^2\overline{\theta}\,\mathrm{Tr}(W^{\dagger 2})\\
&+\frac{1}{e^2}\sum_{s=1}^{3}\int d^4x d^2\theta d^2\overline{\theta}\,\mathrm{Tr}(\Phi^{\dagger s}\mathrm{e}^V\Phi_s)\\
&+\frac{i\sqrt{2}}{e^2}\int d^4x d^2\theta\,\mathrm{Tr}\big\{\Phi_1[\Phi_2,\Phi_3]\big\} + \frac{i\sqrt{2}}{e^2}\int d^4x d^2\overline{\theta}\,\mathrm{Tr}\big\{\Phi^{\dagger 1}[\Phi^{\dagger 2},\Phi^{\dagger 3}]\big\},
\end{aligned}
\tag{2.2}
$$

⋆ We follow the same conventions as in [11].

where $W_\alpha = -\frac{1}{16}\overline{D}^2 \mathrm{e}^{-V} D_\alpha \mathrm{e}^V$ and $\tau = \frac{\theta}{2\pi} + \frac{4\pi^2 i}{e^2}$.

The theory is invariant under the following four supersymmetries (in $SU(4)_I$ covariant notation):

$$
\begin{aligned}
\delta A_{\alpha\dot\alpha} &= -2i\overline{\xi}^u{}_{\dot\alpha}\lambda_{u\alpha} + 2i\overline{\lambda}^u{}_{\dot\alpha}\xi_{u\alpha},\\
\delta\lambda_{u\alpha} &= -iF^+{}_\alpha{}^\beta\xi_{u\beta} + i\sqrt{2}\overline{\xi}^{v\dot\alpha}\nabla_{\alpha\dot\alpha}\phi_{vu} - i\xi_{w\alpha}[\phi_{uv},\phi^{vw}],\\
\delta\phi_{uv} &= \sqrt{2}\{\xi_u{}^\alpha\lambda_{v\alpha} - \xi_v{}^\alpha\lambda_{u\alpha} + \epsilon_{uvwz}\overline{\xi}^w{}_{\dot\alpha}\overline{\lambda}^{z\dot\alpha}\},
\end{aligned}
\tag{2.3}
$$

where $F^+{}_\alpha{}^\beta = \sigma^{mn}{}_\alpha{}^\beta F_{mn}$ and $(u,v,w,z,\ldots)$ label the fundamental representation $\mathbf{4}$ of $SU(4)_I$. For future convenience we note that, according to our conventions, the supersymmetry transformations with parameters $\xi^\alpha_{v=4}$ and $\overline{\xi}^{w=4}_{\dot\alpha}$ are the ones which are manifest in the $N=1$ superspace formulation (2.2). In (2.3) $\lambda_u = \{\lambda_1,\lambda_2,\lambda_3,\lambda_4\}$, while

$$
\phi_{uv} = \begin{pmatrix} 0 & -B^{\dagger 3} & B^{\dagger 2} & -B_1 \\ B^{\dagger 3} & 0 & -B^{\dagger 1} & -B_2 \\ -B^{\dagger 2} & B^{\dagger 1} & 0 & -B_3 \\ B_1 & B_2 & B_3 & 0 \end{pmatrix}, \qquad \begin{cases} \phi_{uv} = -\phi_{vu}, \\ \phi^{uv} = (\phi_{uv})^\dagger = \phi^*_{vu} = -\frac{1}{2}\epsilon^{uvwz}\phi_{wz} \end{cases}
\tag{2.4}
$$

The global symmetry group of $N=4$ supersymmetric theories in $\mathbb{R}^4$ is $\mathcal{H} = SU(2)_L \otimes SU(2)_R \otimes SU(4)_I$, where $\mathcal{K} = SU(2)_L \otimes SU(2)_R$ is the rotation group $SO(4)$. The supersymmetry generators responsible for the transformations (2.3) are $Q^u{}_\alpha$ and $\overline{Q}_{u\dot\alpha}$. They transform as $(\mathbf{2},\mathbf{1},\overline{\mathbf{4}}) \oplus (\mathbf{1},\mathbf{2},\mathbf{4})$ under $\mathcal{H}$.

2.2 Twists of the $N=4$ theory

Since first introduced by Witten in [1], the twisting procedure has proved to be a very useful tool for intertwining between physical (supersymmetric) quantum field theories and the topology of low-dimensional manifolds. In four dimensions, the global symmetry group of the extended supersymmetric gauge theories is of the form $SU(2)_L \otimes SU(2)_R \otimes \mathcal{I}$, where $\mathcal{K} = SU(2)_L \otimes SU(2)_R$ is the rotation group, and $\mathcal{I}$ is the chiral $\mathcal{R}$-symmetry group. The twist can be thought of either as an exotic realization of the global symmetry group of the theory, or as the coupling to the spin connection of a certain subgroup of the global $\mathcal{R}$-current of the theory–see for example [12]. As this latter mechanism changes the energy-momentum tensor and hence the couplings (spins) of the different fields to gravity, both points of view are easily reconciled.

While in $N = 2$ supersymmetric gauge theories the $\mathcal{R}$-symmetry group is at most $U(2)$ and thus the twist is essentially unique, in the $N = 4$ supersymmetric gauge theory the $\mathcal{R}$-symmetry group is $SU(4)$ and there are three different possibilities, each of these corresponding to different non-equivalent homomorphisms of the rotation group into the $\mathcal{R}$-symmetry group [5,11,13].

Two of these possibilities give rise to topological field theories with two supercharges. One of these was considered by Vafa and Witten [5] in order to carry out an explicit test of S-duality on several four-manifolds, and is the object of the present paper. It has the unusual feature that the virtual dimension of its moduli space is exactly zero. This feature was analysed from the perspective of balanced topological field theories in [14], while the underlying structure had already been anticipated within the framework of supersymmetric quantum mechanics in [15].

The second possibility was first discussed in [16], where it was shown to correspond to a topological theory of complexified flat gauge connections. This idea was pursued further in [17], where a link to supersymmetric BF-theories in four dimensions was established. From a somewhat different viewpoint, it has been claimed in [11] that the theory is amphicheiral, this meaning that the twist with either $SU(2)_L$ or $SU(2)_R$ leads essentially to the same theory.

The remaining possibility leads to the "half-twisted theory", a topological theory with only one BRST supercharge [13]. This feature is reminiscent of the situation in twisted $N = 2$ supersymmetric gauge theories, and in fact [11], the theory is a close relative of the non-Abelian monopole theory [6,10,18], the non-abelian generalization of Witten's monopole theory [4], for the special case in which the matter fields are in the adjoint representation of the gauge group.

2.3 The Vafa-Witten theory

The twist of the $N=4$ supersymmetric gauge theory we are interested in arises as follows [13]. First break $SU(4)_I$ down to $SO(4) = SU(2)_F \otimes SU(2)_{F'}$, then replace $SU(2)_L$ by its diagonal sum $SU(2)'_L$ with $SU(2)_{F'}$. After the twisting, the symmetry group of the theory becomes $\mathcal{H}' = SU(2)'_L \otimes SU(2)_R \otimes SU(2)_F$. Under $\mathcal{H}'$, the supercharges split up as

$$Q^v{}_\alpha \to Q^i,\ Q^i{}_{\alpha\beta}, \qquad \overline{Q}_{v\dot\alpha} \to \overline{Q}^i_{\alpha\dot\alpha}, \tag{2.5}$$

where i is an $SU(2)_F$ index. The twist has produced two scalar supercharges, the $SU(2)_F$ doublet Q^i, which are defined in terms of the original supercharges as follows: $Q^{i=1} = Q^{v=1}{}_{\alpha=1} + Q^{v=2}{}_{\alpha=2}$, $Q^{i=2} = Q^{v=3}{}_{\alpha=1} + Q^{v=4}{}_{\alpha=2}$.

The fields of the $N=4$ supersymmetric multiplet decompose under $\mathcal{H}'$ in the following manner–in the notation of [13]:

$$\begin{aligned} &A_{\alpha\dot\alpha} \longrightarrow A_{\alpha\dot\alpha}, &\qquad &\overline{\lambda}^v{}_{\dot\alpha} \longrightarrow \psi^{i\alpha}{}_{\dot\alpha}, \\ &\lambda_{v\alpha} \longrightarrow \chi_{i\beta\alpha},\ \eta_i, &\qquad &\phi_{uv} \longrightarrow \varphi_{ij},\ G_{\alpha\beta}. \end{aligned} \tag{2.6}$$

Notice that the fields $\chi^i_{\alpha\beta}$ and $G_{\alpha\beta}$ are symmetric in their spinor indices and can therefore be regarded as components of self-dual two-forms. As argued in [11] (see also [19] for a related discussion), it is convenient to further break $SU(2)_F$ down to its T_3 subgroup, whose eigenvalues are then assumed to give the (non-anomalous) ghost numbers of the different fields in the theory. The resulting model has BRST charges $Q^+ = Q^2$ and $Q^- = iQ^1$ of opposite ghost number. The field content can now be organized as in [5], and consists of 3 scalar fields $\{\phi^{+2}, \overline{\phi}^{-2}, C^0\}$, 2 one-forms $\{A^0_{\alpha\dot\alpha}, \tilde{H}^0_{\alpha\dot\alpha}\}$ and 2 self-dual two-forms $\{(B^+_{\alpha\beta})^0, (H^+_{\alpha\beta})^0\}$ on the bosonic (commuting) side; and 2 scalar fields $\{\zeta^{+1}, \eta^{-1}\}$, 2 one-forms $\{\psi^1_{\alpha\dot\alpha}, \tilde{\chi}^{-1}_{\alpha\dot\alpha}\}$ and 2 self-dual two-forms $\{(\tilde{\psi}^+_{\alpha\beta})^{+1}, (\chi^+_{\alpha\beta})^{-1}\}$ on the fermionic (anticommuting) side. The superscript stands for the ghost number carried by each of the fields. These fields are related to the fields in the underlying $N=4$ supersymmetric gauge theory as follows:

$$
\begin{aligned}
\lambda_{\tilde{1}1} &= \tilde{\psi}^{+}_{11}, & \lambda_{\tilde{3}1} &= -i\chi^{+}_{11}, & \overline{\lambda}^{\tilde{1}}_{\dot\alpha} &= \tilde{\chi}_{2\dot\alpha}, \\
\lambda_{\tilde{1}2} &= \tilde{\psi}^{+}_{22}, & \lambda_{\tilde{3}2} &= -i\chi^{+}_{22}, & \overline{\lambda}^{\tilde{2}}_{\dot\alpha} &= -\tilde{\chi}_{1\dot\alpha}, \\
\lambda_{\tilde{2}1} &= \tilde{\psi}^{+}_{12} - \frac{i}{2}\zeta, & \lambda_{\tilde{4}1} &= \frac{1}{2}\eta - i\chi^{+}_{12}, & \overline{\lambda}^{\tilde{3}}_{\dot\alpha} &= i\psi_{2\dot\alpha}, \\
\lambda_{\tilde{2}2} &= \tilde{\psi}^{+}_{12} + \frac{i}{2}\zeta, & \lambda_{\tilde{4}2} &= -\frac{1}{2}\eta - i\chi^{+}_{12}, & \overline{\lambda}^{\tilde{4}}_{\dot\alpha} &= -i\psi_{1\dot\alpha}, \\
B_1 &= -B^{+}_{12} + iC, & B_2 &= -B^{+}_{22}, & B_3 &= -\overline{\phi}, \\
B_1^{\dagger} &= -B^{+}_{12} - iC, & B_2^{\dagger} &= B^{+}_{11}, & B_3^{\dagger} &= -\phi.
\end{aligned}
\tag{2.7}
$$

($\tilde{1}$, $\tilde{2}$, etc., denote $SU(4)_I$ indices).

In this paper we will make use of the transformations generated by Q^{+} only, which are readily obtained from (2.3) by simply declaring

$$
\xi_{(v=1,2)\alpha} = 0, \quad \xi_{(v=3,4)\alpha} \to \epsilon C_{(\beta=1,2)\alpha}, \quad \overline{\xi}^{v}{}_{\dot\alpha} = 0, \tag{2.8}
$$

and turn out to be (we give the off-shell version):

$$
\begin{aligned}
[Q^{+}, A_{\alpha\dot\alpha}] &= -2\psi_{\alpha\dot\alpha}, \\
\{Q^{+}, \psi_{\alpha\dot\alpha}\} &= -\sqrt{2}\mathcal{D}_{\alpha\dot\alpha}\phi, \\
[Q^{+}, \phi] &= 0, \\
[Q^{+}, B^{+}_{\alpha\beta}] &= \sqrt{2}\tilde{\psi}^{+}_{\alpha\beta}, \\
\{Q^{+}, \tilde{\psi}^{+}_{\alpha\beta}\} &= 2i\,[B^{+}_{\alpha\beta}, \phi], \\
[Q^{+}, C] &= \frac{1}{\sqrt{2}}\zeta, \\
\{Q^{+}, \zeta\} &= 4i\,[C, \phi],
\end{aligned}
\qquad
\begin{aligned}
[Q^{+}, \overline{\phi}] &= \sqrt{2}\,\eta, \\
\{Q^{+}, \eta\} &= 2i\,[\overline{\phi}, \phi], \\
\{Q^{+}, \tilde{\chi}_{\alpha\dot\alpha}\} &= \tilde{H}_{\alpha\dot\alpha} + \sqrt{2}s_{\alpha\dot\alpha}, \\
[Q^{+}, \tilde{H}_{\alpha\dot\alpha}] &= 2\sqrt{2}i\,[\tilde{\chi}_{\alpha\dot\alpha}, \phi] - \sqrt{2}[Q^{+}, s_{\alpha\dot\alpha}], \\
\{Q^{+}, \chi^{+}_{\alpha\beta}\} &= H^{+}_{\alpha\beta} + s_{\alpha\beta}, \\
[Q^{+}, H^{+}_{\alpha\beta}] &= 2\sqrt{2}i\,[\chi^{+}_{\alpha\beta}, \phi] - [Q^{+}, s_{\alpha\beta}],
\end{aligned}
\tag{2.9}
$$

where

$$
\begin{aligned}
s_{\alpha\dot{\alpha}} &= \mathcal{D}_{\alpha\dot{\alpha}}C + i\mathcal{D}_{\beta\dot{\alpha}}B^{+\beta}{}_{\alpha}, \\
s_{\alpha\beta} &= F^{+}_{\alpha\beta} + [B^{+}_{\gamma\alpha}, B^{+\gamma}_{\beta}] + 2i\,[B^{+}_{\alpha\beta}, C].
\end{aligned}
\tag{2.10}
$$

With our conventions, the on-shell formulation is simply obtained by setting $H^{+}_{\alpha\beta} = 0 = \tilde{H}_{\alpha\dot{\alpha}}$ in (2.9).

According to Witten's fixed-point theorem [20], the contributions to the partition function of the theory, which is the only non-trivial observable owing to the vanishing of the ghost number anomaly, come from the fixed points of the BRST symmetry. In view of (2.9) and (2.10), this means that the Vafa-Witten theory localizes on the moduli space defined by the equations

$$
\begin{cases}
\mathcal{D}_{\alpha\dot{\alpha}}C + i\mathcal{D}_{\beta\dot{\alpha}}B^{+\beta}{}_{\alpha} = 0, \\
F^{+}_{\alpha\beta} + [B^{+}_{\gamma\alpha}, B^{+\gamma}_{\beta}] + 2i\,[B^{+}_{\alpha\beta}, C] = 0,
\end{cases}
\tag{2.11}
$$

which are precisely the equations discussed in [5]. One of the main ingredients in the analysis in [5] is the existence, on certain four-manifolds (basically of the Kahler type), of a suitable vanishing theorem which guarantees that all the solutions to eqs. (2.11) are of the form:

$$
F^{+}_{\alpha\beta} = 0, \quad B^{+}_{\alpha\beta} = 0, \quad C = 0,
\tag{2.12}
$$

that is, that the moduli space reduces to the moduli space of ASD connections. In fact, under these circumstances, the partition function of the theory computes, for each value of the instanton number, the Euler characteristic of the corresponding instanton moduli space. Observe that the vanishing theorem allows only positive instanton numbers to contribute to the partition function; the presence of negative instanton number contributions will signal a failure of the vanishing theorem.

In [5,11,14,21] it was shown that the theory admits a nice geometric interpretation within the framework of the Mathai-Quillen formalism [22] (for a review of the Mathai-Quillen formalism in the context of topological field theories of cohomological type, see [7,23,24,25]). In this context, the equations (2.10) are interpreted as defining a section $s : \mathcal{M} \to \mathcal{V}$ in the trivial vector bundle $\mathcal{V} = \mathcal{M} \times \mathcal{F}$, where $\mathcal{M} = \mathcal{A} \times \Omega^0(X, \mathrm{ad}P) \times \Omega^{2,+}(X, \mathrm{ad}P)$ is the field space, and the fibre is $\mathcal{F} = \Omega^1(X, \mathrm{ad}P) \oplus \Omega^{2,+}(X, \mathrm{ad}P)$, whose zero locus –modded out by the gauge symmetry– is precisely the desired moduli space. $\mathcal{A}$ denotes the space of connections on a principal G-bundle $P \to X$, while $\Omega^0(X, \mathrm{ad}P)$ and $\Omega^{2,+}(X, \mathrm{ad}P)$ denote respectively the space of 0-forms and self-dual 2-forms on X taking values in the Lie algebra of G, while $\mathrm{ad}P$ denotes the adjoint bundle of P, $P \times_{\mathrm{ad}} \mathbf{g}$ ($\mathbf{g}$ stands for the Lie algebra of G). The space of sections of this bundle, $\Omega^0(X, \mathrm{ad}P)$, is the Lie algebra of the group $\mathcal{G}$ of gauge transformations (vertical automorphisms) of the bundle P.

In this setting, the fields of the theory play well-defined roles: A, B^+ and C belong to the field space; ψ and $\tilde{\psi}^+$ are ghosts living in the (co)tangent space $T^*\mathcal{M}$; $\tilde{\chi}$ and χ^+ are fibre antighosts associated to eqs. (2.10), while $\tilde{H}$ and H^+ are their corresponding auxiliary fields; finally, ϕ –or rather its vacuum expectation value $\langle\phi\rangle$– gives the curvature of the principal $\mathcal{G}$-bundle $\mathcal{M} \to \mathcal{M}/\mathcal{G}$, while $\overline{\phi}$ and η enforce the horizontal projection $\mathcal{M} \to \mathcal{M}/\mathcal{G}$. The BRST symmetry (2.9) is the Cartan model representative of the $\mathcal{G}$-equivariant differential on $\mathcal{V}$, while the ghost number is just a form degree. The exponential of the action of the theory gives, when integrated over the antighosts and their auxiliary fields, the Mathai-Quillen representative for the Thom form of the principal bundle $\mathcal{M} \times \mathcal{F} \to \mathcal{E} = \mathcal{M} \times_{\mathcal{G}} \mathcal{F}$.

The action itself (but for the theta-term) can be written as a Q^+ commutator. The appropriate gauge fermion is [11]:

$$\Psi = \frac{1}{e^2}\int_X d^4x\,\sqrt{g}\,\mathrm{Tr}\left\{ -\frac{1}{4}\tilde\chi^{\dot\alpha\alpha}\big(\tilde H_{\alpha\dot\alpha} - \sqrt{2}s_{\alpha\dot\alpha}\big) - \frac{1}{4}\chi^{\alpha\beta}\big(H_{\alpha\beta} - s_{\alpha\beta}\big)\right\}$$

$$+\frac{1}{e^2}\int_X d^4x\,\sqrt{g}\,\mathrm{Tr}\left\{ \frac{1}{2\sqrt{2}}\overline{\phi}\,\big(\mathcal{D}_{\alpha\dot\alpha}\psi^{\dot\alpha\alpha} + i\sqrt{2}\,[\tilde\psi_{\alpha\beta}, B^{\alpha\beta}] - i\sqrt{2}\,[\zeta, C]\big)\right\} \tag{2.13}$$

$$-\frac{1}{e^2}\int_X d^4x\,\sqrt{g}\,\mathrm{Tr}\left\{ \frac{i}{4}\eta[\phi,\overline{\phi}]\right\}.$$

We have not said a word about the role played by Q^-. In fact, the theory admits two Mathai-Quillen descriptions, related to each other by the Weyl group of $SU(2)_F$, in such a way that the roles of Q^+ and Q^- are interchanged, as are the roles of ψ and $\tilde\chi$, χ^+ and $\tilde\psi^+$, ζ and η, and ϕ and $\overline{\phi}$. The corresponding moduli space is defined by eqs. (2.11) with the substitution $C \to -C$, and the theory localizes –as was proved in [5]– actually on the intersection of both moduli spaces, which is defined by the equations

$$\begin{cases} \mathcal{D}_{\alpha\dot\alpha}C = 0, & \mathcal{D}_{\beta\dot\alpha}B^{+\beta}{}_{\alpha} = 0, \\ F^+_{\alpha\beta} + [B^+_{\gamma\alpha}, B^{+\gamma}_{\beta}] = 0, & [B^+_{\alpha\beta}, C] = 0. \end{cases} \tag{2.14}$$

2.4 The twist on Kahler manifolds

On a four-dimensional Kahler manifold the holonomy group is contained in $SU(2)_R \otimes U(1)_L$, where $U(1)_L$ is a certain subgroup of $SU(2)_L$. Under this reduction of the holonomy, left-handed spinors ψ_α decompose into pieces ψ_1 and ψ_2 of opposite $U(1)_L$ charges, in such a way that if the manifold is also spin, the spinor bundle S^+ has a decomposition $S^+ \simeq K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$, where $K^{\frac{1}{2}}$ is some square root of the canonical bundle of X, $K = \bigwedge^2_{\mathbf{C}} T^*X$. We can define a complex structure on X by taking the 1-forms $(\sigma_\mu)_{1\dot\alpha}dx^\mu$ to be of type $(1,0)$, and the 1-forms $(\sigma_\mu)_{2\dot\alpha}dx^\mu$ of type $(0,1)$. With this choice, the self-dual 2-form $(\sigma_{\mu\nu})_{\alpha\beta}dx^\mu \wedge dx^\nu$ can be regarded as a $(2,0)$-form for $\alpha=\beta=1$, as a $(0,2)$-form for $\alpha=\beta=2$, and as a $(1,1)$-form for $\alpha=1,\beta=2$. This decomposition corresponds to the splitting

$\Omega^{2,+}(X) = \Omega^{2,0}(X) \oplus \Omega^{0,2}(X) \oplus \varpi\Omega^0(X)$, valid on any Kahler surface (ϖ stands for the Kahler form).

With respect to the complex structure of the manifold, the fields of the theory naturally split into objects that can be thought of as components of forms of type (p,q). For example, the connection 1-form $A_{\alpha\dot{\alpha}}(\sigma_\mu)^{\dot{\alpha}\alpha}dx^\mu$ splits up into a $(1,0)$-form $A_{2\dot{\alpha}}(\sigma_\mu)^{\dot{\alpha}}_1 dx^\mu$ and a $(0,1)$-form $A_{1\dot{\alpha}}(\sigma_\mu)^{\dot{\alpha}}_2 dx^\mu$. Likewise, the self-dual 2-form $B^+_{\alpha\beta}(\sigma_{\mu\nu})^{\alpha\beta}dx^\mu \wedge dx^\nu$ gives rise to a $(2,0)$-form $B^+_{22}(\sigma_{\mu\nu})_{11}dx^\mu \wedge dx^\nu$ a $(0,2)$-form $B^+_{11}(\sigma_{\mu\nu})_{22}dx^\mu \wedge dx^\nu$ and a $(1,1)$-form for $B^+_{12}(\sigma_{\mu\nu})_{12}dx^\mu \wedge dx^\nu = B^+_{12}\varpi$. Notice that in our conventions the field B^+_{11} would correspond to the $(0,2)$-form $\overline{\beta}$, B^+_{22} to the $(2,0)$-form β and B^+_{12} to the 0-form b in [5]. Note that the field B^+_{12} can be thought of as a scalar field on X. In fact, we shall see in a moment that it naturally combines with the scalar field C into two complex scalars $B^+_{12} \pm iC$. Something similar happens with the other self-dual 2-forms χ^+ and $\tilde{\psi}^+$.

Let us recall that in our conventions the BRST operators $Q^\pm$ are obtained from the $N=4$ supercharges $Q^v{}_\alpha$, with the recipe

$$Q^+ = Q^{\tilde{3}}{}_1 + Q^{\tilde{4}}{}_2, \qquad Q^- = i(Q^{\tilde{1}}{}_1 + Q^{\tilde{2}}{}_2). \tag{2.15}$$

In the Kahler case, each of the individual components $Q^{\tilde{1}}{}_1$, $Q^{\tilde{2}}{}_2$, $Q^{\tilde{3}}{}_1$ and $Q^{\tilde{4}}{}_2$ is well-defined under the holonomy $SU(2)_R \otimes U(1)_L$. It is therefore possible to define four charges, of which only $Q^{\tilde{4}}{}_2$ is related to the underlying construction in $N=1$ superspace. Hence, it is the only topological symmetry that should be expected to survive after the mass terms are plugged in.

In what follows, we will be interested only in $Q^{\tilde{3}}{}_1$ and $Q^{\tilde{4}}{}_2$. The corresponding transformation laws (with parameters ρ_2 and ρ_1 respectively) can be extracted from the $N=4$ supersymmetry transformations (2.3) by setting:

$$\overline{\xi}^{v\dot{\alpha}} = 0, \quad \xi_{\tilde{1}\alpha} = \xi_{\tilde{2}\alpha} = 0, \quad \xi_{\tilde{3}\alpha} = \rho_2 C_{2\alpha}, \quad \xi_{\tilde{4}\alpha} = \rho_1 C_{1\alpha}, \tag{2.16}$$

The corresponding BRST charges will be denoted by $Q_1 = Q^{\tilde{4}}{}_2$ and $Q_2 = Q^{\tilde{3}}{}_1$. The on-shell transformations turn out to be:

$$\begin{aligned}
[Q_1, A_{1\dot{\alpha}}] &= -2\psi_{1\dot{\alpha}}, \\
[Q_1, A_{2\dot{\alpha}}] &= 0, \\
[Q_1, F^+_{11}] &= -2iD_{1\dot{\alpha}}\psi_1{}^{\dot{\alpha}}, \\
[Q_1, F^+_{22}] &= 0, \\
\{Q_1, \psi_{1\dot{\alpha}}\} &= 0, \\
\{Q_1, \psi_{2\dot{\alpha}}\} &= -\sqrt{2}D_{2\dot{\alpha}}\phi, \\
[Q_1, \phi] &= 0, \\
[Q_1, B^+_{11}] &= 0, \\
[Q_1, B^+_{12} + iC] &= 0, \\
\{Q_1, \tilde{\psi}^+_{11}\} &= 2i[B^+_{11}, \phi], \\
\left\{Q_1, \tilde{\psi}^+_{12} + \frac{i}{2}\zeta\right\} &= -2i[\phi, B^+_{12} + iC], \\
\{Q_1, \tilde{\chi}_{1\dot{\alpha}}\} &= -\sqrt{2}iD_{2\dot{\alpha}}B^+_{11},
\end{aligned}
\qquad
\begin{aligned}
[Q_1, F^+_{12}] &= -iD_{2\dot{\alpha}}\psi_1{}^{\dot{\alpha}}, \\
[Q_1, \overline{\phi}] &= \sqrt{2}\left(\frac{1}{2}\eta - i\chi^+_{12}\right), \\
\left\{Q_1, \frac{1}{2}\eta + i\chi^+_{12}\right\} &= -i[\phi, \overline{\phi}] + iF^+_{12} \\
&\quad + i[B^+_{12} - iC, B^+_{12} + iC] + i[B^+_{11}, B^+_{22}], \\
\left\{Q_1, \frac{1}{2}\eta - i\chi^+_{12}\right\} &= 0, \\
\{Q_1, \chi^+_{11}\} &= -2[B^+_{12} + iC, B^+_{11}], \\
\{Q_1, \chi^+_{22}\} &= F^+_{22}, \\
[Q_1, B^+_{22}] &= \sqrt{2}\tilde{\psi}^+_{22}, \\
[Q_1, B^+_{12} - iC] &= \sqrt{2}\left(\tilde{\psi}^+_{12} - \frac{i}{2}\zeta\right), \\
\{Q_1, \tilde{\psi}^+_{22}\} &= 0, \\
\left\{Q_1, \tilde{\psi}^+_{12} - \frac{i}{2}\zeta\right\} &= 0, \\
\{Q_1, \tilde{\chi}_{2\dot{\alpha}}\} &= -\sqrt{2}iD_{2\dot{\alpha}}(B^+_{12} + iC),
\end{aligned}
\tag{2.17}$$

for Q_1. The Q_2 transformations are easily computed from (2.9) and (2.17) after using $Q^+ = Q_1 + Q_2$. They read:

$$
\begin{aligned}
[Q_2, A_{1\dot\alpha}] &= 0, & [Q_2, F^+_{12}] &= -iD_{1\dot\alpha}\psi_2{}^{\dot\alpha},\\
[Q_2, A_{2\dot\alpha}] &= -2\psi_{2\dot\alpha}, & [Q_2, \overline{\phi}] &= \sqrt{2}\left(\frac{1}{2}\eta + i\chi^+_{12}\right),\\
[Q_2, F^+_{11}] &= 0, & \left\{Q_2, \frac{1}{2}\eta - i\chi^+_{12}\right\} &= -i[\phi,\overline{\phi}] - iF^+_{12}\\
[Q_2, F^+_{22}] &= -2iD_{2\dot\alpha}\psi_2{}^{\dot\alpha}, & &-i[B^+_{12} - iC, B^+_{12} + iC] - i[B^+_{11}, B^+_{22}],\\
\{Q_2, \psi_{1\dot\alpha}\} &= -\sqrt{2}D_{1\dot\alpha}\phi, & \left\{Q_2, \frac{1}{2}\eta + i\chi^+_{12}\right\} &= 0,\\
\{Q_2, \psi_{2\dot\alpha}\} &= 0, & \{Q_2, \chi^+_{11}\} &= F^+_{11},\\
[Q_2, \phi] &= 0, & \{Q_2, \chi^+_{22}\} &= 2[B^+_{12} - iC, B^+_{22}],\\
[Q_2, B^+_{11}] &= \sqrt{2}\tilde\psi^+_{11}, & [Q_2, B^+_{22}] &= 0,\\
[Q_2, B^+_{12} - iC] &= 0, & [Q_2, B^+_{12} + iC] &= \sqrt{2}\left(\tilde\psi^+_{12} + \frac{i}{2}\zeta\right),\\
\{Q_2, \tilde\psi^+_{11}\} &= 0, & \{Q_2, \tilde\psi^+_{22}\} &= 2i[B^+_{22}, \phi],\\
\left\{Q_2, \tilde\psi^+_{12} - \frac{i}{2}\zeta\right\} &= -2i[\phi, B^+_{12} - iC], & \{Q_2, \tilde\psi^+_{12} + \frac{i}{2}\zeta\} &= 0,\\
\{Q_2, \tilde\chi_{1\dot\alpha}\} &= \sqrt{2}iD_{1\dot\alpha}(B^+_{12} - iC), & \{Q_2, \tilde\chi_{2\dot\alpha}\} &= \sqrt{2}iD_{1\dot\alpha}B^+_{22},
\end{aligned}
\tag{2.18}
$$

It is straightforward to verify that $(Q_1)^2 = (Q_2)^2 = 0$ on-shell, while $\{Q_1, Q_2\}$ gives a gauge transformation generated by ϕ.

3. Mass perturbations

We now turn to the discussion of the possible ways of (softly) breaking $N = 4$ supersymmetry by suitably adding mass terms for the chiral multiplets. Let us consider first the situation that arises on a flat $\mathbb{R}^4$. By adding a bare mass term for just one of the chiral multiplets, say Φ_1,

$$\Delta L^{(1)} = m \int d^4x d^2\theta \mathrm{Tr}(\Phi_1)^2 + \mathrm{h.c.}, \tag{3.1}$$

$N = 4$ supersymmetry is broken down to $N = 1$. The corresponding low-energy effective theory, at scales below m, is $N = 1$ supersymmetric QCD, with $SU(2)$ as gauge group, coupled to two massless chiral superfields in the adjoint representation with a (tree-level) quartic superpotential induced by integrating out the massive superfield. As shown in [26], this theory has a moduli space of vacua where both a Coulomb and a Higgs phase coexist. On the other hand, equal bare mass terms for two of the chiral multiplets,

$$\Delta L^{(2)} = m \int d^4x d^2\theta \mathrm{Tr}(\Phi_1 \Phi_2) + \mathrm{h.c.}, \tag{3.2}$$

preserve $N = 2$ supersymmetry, whereas if the mass terms are different:

$$\Delta' L^{(2)} = m_1 \int d^4x d^2\theta \mathrm{Tr}(\Phi_1)^2 + m_2 \int d^4x d^2\theta \mathrm{Tr}(\Phi_2)^2 + \mathrm{h.c.}, \tag{3.3}$$

$N = 4$ supersymmetry is again broken down to $N = 1$. However, both theories flow in the infrared to a pure $N = 2$ supersymmetric gauge theory, which has a moduli space of vacua in the Coulomb phase. Finally, mass terms for the three chiral multiplets, no matter whether the mass parameters are equal or not, preserve only $N = 1$ supersymmetry. Of the three inequivalent ways of breaking $N = 4$ supersymmetry down to $N = 1$, we must choose the one in terms of which the analysis of the vacuum structure of the resultant $N = 1$ theory is simplest. The appropriate choice is [5]

$$\Delta L^{(3)} = m \int d^4x d^2\theta \mathrm{Tr}\big((\Phi_1)^2 + (\Phi_2)^2 + (\Phi_3)^2\big) + \text{h.c.}, \tag{3.4}$$

in terms of which the classical vacua of the resulting $N = 1$ theory can be classified by the complex conjugacy classes of homomorphisms of the $SU(2)$ Lie algebra to that of G. In the case that $G = SU(2)$, there are three discrete vacua, corresponding to the three singularities of the mass-deformed $N = 4$ supersymmetric gauge theory with gauge group $SU(2)$ [3].

On general curved manifolds the naıve construction sketched above simply doesn't work. As explained in [9,5], superpotentials of a twisted theory on Kahler manifolds must transform as $(2, 0)$-forms. According to our conventions, two of the chiral superfields, Φ_1 and Φ_3 (whose scalar components are $B^+_{12} \pm iC$ and ϕ, $\overline{\phi}$ resp.) are scalars in the twisted model, while the third one, Φ_2 (whose scalar components are B^+_{11} and B^+_{22}), is a $(2, 0)$-form. A suitable mass term for Φ_2 and one of the other scalar superfields, say Φ_1, can be readily written down and reads:

$$\Delta L(m) = m \int_X d^2\theta \mathrm{Tr}(\Phi_1\Phi_2) + \text{h.c.} \tag{3.5}$$

In (3.5) m is just a (constant) mass parameter. A mass term for the remaining superfield Φ_3 requires the introduction of the $(2, 0)$-form* ω [9]:

$$\Delta L(\omega) = \int_X \omega \wedge d^2\overline{z} d^2\theta \mathrm{Tr}(\Phi_3)^2 + \text{h.c.} \tag{3.6}$$

Therefore we now turn to studying the effect of the following mass terms for

$\star$ Of course, this sets on the manifold X the constraint $h^{(2,0)}(X) \neq 0$, which for Kahler manifolds is equivalent to demanding $b^+_2 > 1$. This excludes, for example, the case of $\mathbb{CP}^2$.

the chiral multiplets Φ_1, Φ_2 and Φ_3:

$$\begin{aligned}\Delta L(m,\omega) = m\int_X d^2\theta \mathrm{Tr}(\Phi_1\Phi_2) + m\int_X d^2\overline{\theta}\mathrm{Tr}(\Phi_1^\dagger\Phi_2^\dagger)\\ +\int_X d^2\theta\omega\mathrm{Tr}(\Phi_3)^2 + \int_X d^2\overline{\theta}\overline{\omega}\mathrm{Tr}(\Phi_3^\dagger)^2,\end{aligned} \tag{3.7}$$

where, for simplicity, $\omega = \omega_{11} = (\sigma_{\mu\nu})_{11}\omega_{\tau\lambda}\epsilon^{\mu\nu\tau\lambda}$ stands for the only non-vanishing component of the $(2,0)$-form ω, while $\overline{\omega} = \overline{\omega}_{22} = \omega_{11}^*$ stands for the only non-vanishing component of the $(0,2)$-form $\overline{\omega}$ conjugate to ω.

After expanding the fields and integrating out the auxiliary fields one gets the contributions

$$\begin{aligned}&-2\sqrt{2}i\omega B_3[B_1^\dagger,B_2^\dagger] - 2\sqrt{2}i\overline{\omega}B_3^\dagger[B_1,B_2] - 4|\omega|^2 B_3B_3^\dagger\\ &-\omega\lambda_3{}^\alpha\lambda_{3\alpha} - \overline{\omega}\overline{\lambda}^3{}_{\dot\alpha}\overline{\lambda}^{3\dot\alpha}\\ &-2\sqrt{2}imB_2[B_2^\dagger,B_3^\dagger] - 2\sqrt{2}imB_2^\dagger[B_2,B_3] - m^2B_2B_2^\dagger\\ &-m\lambda_1{}^\alpha\lambda_{2\alpha} - m\overline{\lambda}^1{}_{\dot\alpha}\overline{\lambda}^{2\dot\alpha}\\ &-2\sqrt{2}imB_1[B_3^\dagger,B_1^\dagger] - 2\sqrt{2}imB_1^\dagger[B_3,B_1] - m^2B_1B_1^\dagger.\end{aligned} \tag{3.8}$$

The $N=1$ transformations for the fermions get modified as follows:

$$\begin{aligned}\delta\lambda_{1\alpha} &= \ldots - \sqrt{2}\xi_{4\alpha}mB_2^\dagger,\\ \delta\lambda_{2\alpha} &= \ldots - \sqrt{2}\xi_{4\alpha}mB_1^\dagger,\\ \delta\lambda_{3\alpha} &= \ldots - 2\sqrt{2}\xi_{4\alpha}\overline{\omega}B_3^\dagger\end{aligned} \tag{3.9}$$

(and their corresponding complex conjugates). In terms of the twisted fields the mass contributions are –see (2.7):

$$\begin{aligned}
&\mathrm{Tr}\Big\{-2\sqrt{2}i\omega\overline{\phi}[B_{12}^{+}+iC,B_{11}^{+}]+2\sqrt{2}i\overline{\omega}\phi[B_{12}^{+}-iC,B_{22}^{+}]-4|\omega|^{2}\phi\overline{\phi}\\
&-2i\omega\chi_{11}^{+}\left(\frac{1}{2}\eta-i\chi_{12}^{+}\right)+\overline{\omega}\psi_{2\dot{\alpha}}\psi_{2}{}^{\dot{\alpha}}-2\sqrt{2}imB_{22}^{+}[B_{11}^{+},\phi]\\
&-2\sqrt{2}imB_{11}^{+}[B_{22}^{+},\overline{\phi}]+m^{2}B_{11}^{+}B_{22}^{+}+m\left(\tilde{\psi}_{12}^{+}+\frac{i}{2}\zeta\right)\left(\tilde{\psi}_{12}^{+}-\frac{i}{2}\zeta\right)\\
&+m\tilde{\psi}_{11}^{+}\tilde{\psi}_{22}^{+}+m\tilde{\chi}_{2\dot{\alpha}}\tilde{\chi}_{1}{}^{\dot{\alpha}}+\sqrt{2}im\phi[B_{12}^{+}+iC,B_{12}^{+}-iC]\\
&-\sqrt{2}im\overline{\phi}[B_{12}^{+}+iC,B_{12}^{+}-iC]-m^{2}|B_{12}^{+}+iC|^{2}\Big\}.
\end{aligned}\tag{3.10}$$

Notice that the mass terms (3.10) explicitly break the ghost number symmetry. In fact, as there are terms with ghost number $+2$, others with ghost number -2, and finally some with ghost number 0, the perturbation actually preserves a $\mathbb{Z}_2$ subgroup of the ghost number. The Q_1 transformations (2.17), which are the only ones to survive the perturbation a priori, also get modified in a way that is dictated by the underlying $N=1$ structure, so that in view of (3.9) they become:

$$\begin{aligned}
\{Q_1^{(m,\omega)},\tilde{\psi}_{11}^{+}\}&=2i[B_{11}^{+},\phi]-\sqrt{2}mB_{11}^{+},\\
\left\{Q_1^{(m,\omega)},\tilde{\psi}_{12}^{+}+\frac{i}{2}\zeta\right\}&=-2i[\phi,B_{12}^{+}+iC]+\sqrt{2}m(B_{12}^{+}+iC),\\
\{Q_1^{(m,\omega)},\chi_{11}^{+}\}&=-2[B_{12}^{+}+iC,B_{11}^{+}]+2\sqrt{2}i\overline{\omega}\phi.
\end{aligned}\tag{3.11}$$

(The rest of the transformations remain the same.) Notice that the fixed-point equations which stem from (3.11) are precisely the F-flatness conditions as derived from the superpotential

$$i\sqrt{2}\mathrm{Tr}\,(\Phi_1[\Phi_2,\Phi_3])+m\mathrm{Tr}\,(\Phi_1\Phi_2)+\omega\mathrm{Tr}(\Phi_3)^2.\tag{3.12}$$

We can analyse these equations following [5]. They admit a trivial solution $B_{11}^{+}=B_{12}^{+}=C=\phi=0$, which leaves at low energies the two vacua of the pure $N=1$ supersymmetric gauge theory with gauge group $SU(2)$. Unless the manifold

X is hyper-Kahler, this picture must be corrected near the zeroes of the mass parameter ω (which form a collection of complex one-dimensional submanifolds $\{C_i\}$) along the lines proposed in [9]. In addition to this trivial vacuum, eqs. (3.11) admit a non-trivial fixed-point in which ϕ, and therefore B^+_{11}, B^+_{12} and C, are not zero. On flat space-time this solution corresponds to a Higgs vacuum in which the gauge group is completely broken. From the viewpoint of the mass-perturbed $N=2$ supersymmetric gauge theory, it corresponds, at least for large m, to a singular point where an elementary quark hypermultiplet becomes massless [3]. This analysis is still valid on hyper-Kahler manifolds. However, on arbitrary Kahler manifolds, this vacuum bifurcates into a "Higgs" vacuum where the gauge group is completely broken, and an Abelianised vacuum with gauge bundle $E = K^{1/2} \oplus K^{-1/2}$ and instanton number $n = -\frac{2\chi+3\sigma}{4}$. This Abelianisation can be understood as follows [5]. On Kahler manifolds eqs. (2.10) can be decomposed in the following way (this can seen by looking at the $Q_{1,2}$-transformations (2.17), (2.18)):

$$
\begin{aligned}
&F^+_{11} = 0 = F^+_{22}, \qquad [B^+_{12} + iC, B^+_{11}] = 0 = [B^+_{12} - iC, B^+_{22}], \\
&F^+_{12} + [B^+_{12} - iC, B^+_{12} + iC] + [B^+_{11}, B^+_{22}] = 0.
\end{aligned}
\tag{3.13}
$$

These equations have a $U(1)$ symmetry (which will be further exploited below) $B^+_{11} \to \mathrm{e}^{i\alpha} B^+_{11}$, $B^+_{22} \to \mathrm{e}^{-i\alpha} B^+_{22}$, $B^+_{12} \pm iC \to \mathrm{e}^{\mp i\alpha}(B^+_{12} \pm iC)$. When the vanishing theorem fails, the contributions from the branch $B^+ \neq 0 \neq C$ come from the fixed points of the combined gauge-$U(1)$ action. If there is a non-trivial fixed point, the gauge connection has to be reducible there, and the gauge bundle is therefore Abelianised. The instanton number of such an Abelianised bundle is typically negative, which means that on a general Kahler manifold, the partition function of the theory will be computing not the Euler characteristic of the instanton moduli space (recall that the contribution of bundles with negative instanton number means that the vanishing theorem is failing), but the Euler characteristic of the $U(1)$-equivariant bundle defined by eqs. (3.13).

With the mass terms added, the action $S + \Delta L(m, \omega)$ is only invariant under $Q_1^{(m,\omega)}$. To get rid of the mass terms proportional to m, we shall proceed as follows. We will modify the Q_2 transformations by appropriately introducing mass terms (proportional to m), in such a way that $Q^+(m) = Q_1^{(m)} + Q_2^{(m)}$ (with mass m, and $\omega = 0$ at this stage) be a symmetry of the original action plus mass perturbations. We will show this by proving that $L + \Delta L(m, \omega = 0)$ is actually $Q^+(m)$-exact. To this end we make the replacements:

$$\begin{aligned}
\{Q_2, \tilde{\psi}_{22}^+\} = 2i[B_{22}^+, \phi] &\longrightarrow \{Q_2^{(m)}, \tilde{\psi}_{22}^+\} = 2i[B_{22}^+, \phi] + \sqrt{2}mB_{22}^+, \\
\left\{Q_2, \tilde{\psi}_{12}^+ - \frac{i}{2}\zeta\right\} = -2i[\phi, B_{12}^+ - iC] &\longrightarrow \left\{Q_2^{(m)}, \tilde{\psi}_{12}^+ - \frac{i}{2}\zeta\right\} \\
&\qquad = -2i[\phi, B_{12}^+ - iC] - 2\sqrt{2}m(B_{12}^+ - iC)
\end{aligned} \tag{3.14}$$

(the rest of the transformations remain the same). Notice that still $(Q_2^{(m)})^2 = 0$. Next we spell out the $Q^+(m) = Q_1^{(m)} + Q_2^{(m)}$-transformations:

$$\begin{aligned}
[Q^+(m), B_{11}^+] &= \sqrt{2}\tilde{\psi}_{11}^+, \\
[Q^+(m), B_{22}^+] &= \sqrt{2}\tilde{\psi}_{22}^+, \\
[Q^+(m), B_{12}^+ \pm iC] &= \sqrt{2}\left(\tilde{\psi}_{12}^+ \pm \frac{i}{2}\zeta\right), \qquad \{Q^+(m), \tilde{\psi}_{11}^+\} = 2i[B_{11}^+, \phi] - \sqrt{2}mB_{11}^+, \\
&\qquad\qquad\qquad\qquad\qquad\quad\; \{Q^+(m), \tilde{\psi}_{22}^+\} = 2i[B_{22}^+, \phi] + \sqrt{2}mB_{22}^+, \\
\left\{Q^+(m), \tilde{\psi}_{12}^+ \pm \frac{i}{2}\zeta\right\} &= 2i[B_{12}^+ \pm iC, \phi] \\
&\quad \pm \sqrt{2}m(B_{12}^+ \pm iC),
\end{aligned} \tag{3.15}$$

On any of these fields (which we denote generically by X) the charge $Q^+(m)$ satisfies the algebra:

$$(Q^+(m))^2 X = 2\sqrt{2}i[X, \phi] + 2mqX, \tag{3.16}$$

where $q = -1$ for B_{11}^+, $\tilde{\psi}_{11}^+$, $B_{12}^+ - iC$ and $\tilde{\psi}_{12}^+ - \frac{i}{2}\zeta$, and $q = +1$ for B_{22}^+, $\tilde{\psi}_{22}^+$, $B_{12}^+ + iC$ and $\tilde{\psi}_{12}^+ + \frac{i}{2}\zeta$. Notice that these charge assingments are compatible with the $U(1)$

symmetry that we discussed above, and in fact one can see the "central charge" $\delta_q X = 2mqX$ arising in the algebra (3.16) as an infinitesimal $U(1)$ transformation with parameter m.

We also extend the $Q^+(m)$ transformation off-shell by declaring its action on $\tilde{H}_{\alpha\dot{\alpha}}$ to be:

$$\begin{aligned} [Q^+(m), \tilde{H}_{1\dot{\alpha}}] &= \ldots - 2m\tilde{\chi}_{1\dot{\alpha}}, \\ [Q^+(m), \tilde{H}_{2\dot{\alpha}}] &= \ldots + 2m\tilde{\chi}_{2\dot{\alpha}}. \end{aligned} \tag{3.17}$$

In this way, $Q^+(m)$ closes on $\tilde{H}_{1\dot{\alpha}}, \tilde{\chi}_{1\dot{\alpha}}$ with $q = -1$, and on $\tilde{H}_{2\dot{\alpha}}, \tilde{\chi}_{2\dot{\alpha}}$ with $q = +1$.

Let us now prove that the above modifications suffice to render the m mass terms $Q^+(m)$-exact:

$$\frac{1}{2\sqrt{2}} m(\tilde{\psi}^+_{22} B^+_{11} - \tilde{\psi}^+_{11} B^+_{22}) \overset{Q^+(m)}{\longrightarrow} m^2 B^+_{11} B^+_{22} + m\tilde{\psi}^+_{11}\tilde{\psi}^+_{22} - \sqrt{2} im B^+_{22}[B^+_{11}, \phi], \tag{3.18}$$

and

$$\begin{aligned} -\frac{1}{2\sqrt{2}} m \left\{ (B^+_{12} - iC)\left(\tilde{\psi}^+_{12} + \frac{i}{2}\zeta\right) - (B^+_{12} + iC)\left(\tilde{\psi}^+_{12} - \frac{i}{2}\zeta\right) \right\} \overset{Q^+(m)}{\longrightarrow} \\ - m^2 |B^+_{12} + iC|^2 + \sqrt{2} im\phi[B^+_{12} + iC, B^+_{12} - iC] + m\left(\tilde{\psi}^+_{12} + \frac{i}{2}\zeta\right)\left(\tilde{\psi}^+_{12} - \frac{i}{2}\zeta\right). \end{aligned} \tag{3.19}$$

Notice, moreover, that these terms are likewise $Q_1^{(m,\omega)}$-exact:

$$-\frac{1}{\sqrt{2}} m\tilde{\psi}^+_{11} B^+_{22} \overset{Q_1^{(m,\omega)}}{\longrightarrow} m^2 B^+_{11} B^+_{22} + m\tilde{\psi}^+_{11}\tilde{\psi}^+_{22} - \sqrt{2} im B^+_{22}[B^+_{11}, \phi], \tag{3.20}$$

and

$$\begin{aligned} -\sqrt{2} m \left\{ (B^+_{12} - iC)\tilde{\psi}^+_{12} \right\} \overset{Q_1^{(m,\omega)}}{\longrightarrow} -m^2 |B^+_{12} + iC|^2 + \sqrt{2} im\phi[B^+_{12} + iC, B^+_{12} - iC] \\ + m\left(\tilde{\psi}^+_{12} + \frac{i}{2}\zeta\right)\left(\tilde{\psi}^+_{12} - \frac{i}{2}\zeta\right). \end{aligned} \tag{3.21}$$

But we have not yet reproduced the terms (see (3.10)): $-\sqrt{2} im\overline{\phi}[B^+_{12} + iC, B^+_{12} - iC]$, $-2\sqrt{2} im B^+_{11}[B^+_{22}, \overline{\phi}]$ and $m\tilde{\chi}_{2\dot{\alpha}}\tilde{\chi}_1{}^{\dot{\alpha}}$. These come from pieces already present in the gauge fermion. Explicitly,

$$\mathrm{Tr}\left\{-\frac{1}{4}\tilde{\chi}^{\dot{\alpha}\alpha}\tilde{H}_{\alpha\dot{\alpha}}\right\}\stackrel{Q^{+}(m)}{\longrightarrow} m\tilde{\chi}_{2\dot{\alpha}}\tilde{\chi}_{1}{}^{\dot{\alpha}}, \tag{3.22}$$

and

$$\begin{aligned}&\mathrm{Tr}\left\{\frac{i}{2}\overline{\phi}[\tilde{\psi}_{\alpha\beta},B^{\alpha\beta}]+\frac{i}{2}[\zeta,C]\right\}\stackrel{Q^{+}(m)}{\longrightarrow}\\&\quad-\sqrt{2}im\overline{\phi}[B^{+}_{12}+iC,B^{+}_{12}-iC]-2\sqrt{2}imB^{+}_{11}[B^{+}_{22},\overline{\phi}].\end{aligned} \tag{3.23}$$

The analysis of the terms containing the $(2,0)$-form ω can be carried out essentialy as in the Donaldson-Witten theory. The perturbation breaks up into a $Q_1^{(m,\omega)}$-exact piece:

$$\begin{aligned}&\{Q_1^{(m,\omega)},\mathrm{Tr}(\sqrt{2}i\omega\overline{\phi}\chi^{+}_{11})\}\\&\qquad=\mathrm{Tr}\left\{-2\sqrt{2}i\omega\overline{\phi}[B^{+}_{12}+iC,B^{+}_{11}]-4|\omega|^{2}\phi\overline{\phi}-2i\omega\chi^{+}_{11}\left(\frac{1}{2}\eta-i\chi^{+}_{12}\right)\right\},\end{aligned} \tag{3.24}$$

and an operator of ghost number $+2$:

$$J(\overline{\omega})=\int_{X}\mathrm{Tr}\left(2\sqrt{2}i\overline{\omega}\phi[B^{+}_{12}-iC,B^{+}_{22}]+\overline{\omega}\psi_{2\dot{\alpha}}\psi_{2}{}^{\dot{\alpha}}\right). \tag{3.25}$$

Equation (3.25) is not very useful as it stands. To rewrite it in a more convenient form we note that from (2.9) it follows that:

$$2\sqrt{2}i\overline{\omega}\mathrm{Tr}\left\{\phi[B^{+}_{12}-iC,B^{+}_{22}]\right\}=\sqrt{2}i\mathrm{Tr}\left(\{Q^{+},\overline{\omega}\phi\chi^{+}_{22}\}\right)-\sqrt{2}i\overline{\omega}\mathrm{Tr}\left(\phi F^{+}_{22}\right). \tag{3.26}$$

Hence,

$$J(\overline{\omega})=\{Q^{+},\cdots\}+\underbrace{\int_{X}\overline{\omega}\mathrm{Tr}\left(\psi_{2\dot{\alpha}}\psi_{2}{}^{\dot{\alpha}}-\sqrt{2}i\phi F^{+}_{22}\right)}_{I(\overline{\omega})},\qquad [Q^{+},I(\overline{\omega})]=0. \tag{3.27}$$

Moreover, as the m mass term does not enter in any of the above calculations, the results also hold for $Q^+(m)$.

The preceding analysis implies that if we denote vacuum expectation values in the twisted theory (which has topological symmetry $Q^+$and action L) by $\langle \ldots \rangle$, in the completely perturbed theory (with action $L+\Delta L(m,\omega)$ and symmetry $Q_1^{(m,\omega)}$) by $\langle \ldots \rangle_{m,\omega}$, and in the equivariantly extended theory (with action $L+\Delta L(m)$ and symmetry $Q^+(m)$) by $\langle \ldots \rangle_m$, the situation for the partition function is the following:

$$\langle 1 \rangle_{m,\omega} = \left\langle \mathrm{e}^{-J(\overline{\omega})} \mathrm{e}^{-\Delta L(m)} \right\rangle = \left\langle \mathrm{e}^{-J(\overline{\omega})} \right\rangle_m . \tag{3.28}$$

In the first equality we have discarded the $Q_1^{(m,\omega)}$-exact term (3.24). Notice that it is also possible, for the same reason, to discard the terms in (3.20) and (3.21). This leaves the $Q_1^{(m,\omega)}$-closed action $L+\Delta^{(1)}+J(\overline{\omega})$, where $\Delta^{(1)}$ are the mass terms (3.22) and (3.23), i.e.

$$\Delta^{(1)} = m \int_X \mathrm{Tr}\left(\tilde{\chi}_{2\dot{\alpha}} \tilde{\chi}_1^{\dot{\alpha}} - \sqrt{2} i \overline{\phi} [B_{12}^+ + iC, B_{12}^+ - iC] - 2\sqrt{2} i B_{11}^+ [B_{22}^+, \overline{\phi}] \right) . \tag{3.29}$$

Notice that $\Delta^{(1)}$ has ghost number -2, while $J(\overline{\omega})$ has ghost number $+2$. Also $L+\Delta^{(1)}$ is $Q^+(m)$-closed (in fact it is $Q^+(m)$-exact up to a θ-term). Hence, we can trade $J(\overline{\omega})$ for $\{Q^+(m), \cdots\} + I(\overline{\omega})$ and discard the $Q^+(m)$-exact piece in (3.26). We are left with the action

$$\underbrace{L+\Delta^{(1)}}_{Q^+(m)-\text{exact}} + \underbrace{I(\overline{\omega})}_{Q^+(m)-\text{closed}} . \tag{3.30}$$

Now, as noted in [9] in a closely related context, $I(\overline{\omega})$ (or rather $J(\overline{\omega})$) is the F-term of the chiral superfield Φ_3; therefore, it cannot develop a vev if supersymmetry is to remain unbroken. Strictly speaking, this applies to $\langle \psi_2 \psi_2 \rangle$. As for the remaining term $\phi[B_{12}^+ - iC, B_{22}^+]$, one can readily check that it vanishes on the moduli space.

Hence,

$$\left\langle \mathrm{e}^{I(\overline{\omega})} \right\rangle_m = \langle 1 \rangle_m = \left\langle \mathrm{e}^{-\Delta^{(1)}} \right\rangle . \tag{3.31}$$

Finally, since $\Delta^{(1)}$ has ghost number -2, its vev in the original theory must vanish as well, if the ghost number symmetry is to remain unbroken. Hence, under these assumptions, the partition function is invariant under the perturbation.

4. Equivariant extension of the Thom form

On Kahler manifolds there is a $U(1)$ symmetry acting on the moduli space. This symmetry was already noted in [5] within the discussion of the vanishing theorem, which guarantees localization on the moduli space of ASD connections, but not further use of it was made. We have discussed it in the previous section in connection with the mass perturbations. Its action on the different fields is the following:

$$\begin{cases} B^+_{11} \to \mathrm{e}^{-it} B^+_{11}, \\ B^+_{12} - iC \to \mathrm{e}^{-it}(B^+_{12} - iC), \\ \tilde{\psi}^+_{11} \to \mathrm{e}^{-it}\tilde{\psi}^+_{11}, \\ \tilde{\psi}^+_{12} - \frac{i}{2}\zeta \to \mathrm{e}^{-it}(\tilde{\psi}^+_{12} - \frac{i}{2}\zeta), \\ \tilde{\chi}_{1\dot{\alpha}} \to \mathrm{e}^{-it}\tilde{\chi}_{1\dot{\alpha}}, \\ \tilde{H}_{1\dot{\alpha}} \to \mathrm{e}^{-it}\tilde{H}_{1\dot{\alpha}}, \end{cases} \qquad \begin{cases} B^+_{22} \to \mathrm{e}^{it} B^+_{22}, \\ B^+_{12} + iC \to \mathrm{e}^{it}(B^+_{12} + iC), \\ \tilde{\psi}^+_{22} \to \mathrm{e}^{it}\tilde{\psi}^+_{22}, \\ \tilde{\psi}^+_{12} + \frac{i}{2}\zeta \to \mathrm{e}^{it}(\tilde{\psi}^+_{12} + \frac{i}{2}\zeta), \\ \tilde{\chi}_{2\dot{\alpha}} \to \mathrm{e}^{it}\tilde{\chi}_{2\dot{\alpha}}, \\ \tilde{H}_{2\dot{\alpha}} \to \mathrm{e}^{it}\tilde{H}_{2\dot{\alpha}}. \end{cases} \tag{4.1}$$

The gauge field A, the antighosts $\chi^+_{\alpha\beta}$ and η, and the scalar fields ϕ and $\overline{\phi}$, carry no charge under this $U(1)$. These transformations can be thought of as defining the one-parameter flow associated to the action on the field space $\mathcal{M}$ of the following vector field $X_{\mathcal{M}} \in T_{(A,B^+,C)}\mathcal{M}$:

$$X_{\mathcal{M}} = \big(0, -iB^+_{11}, iB^+_{22}, -i(B^+_{12} - iC), i(B^+_{12} + iC)\big)\,. \tag{4.2}$$

From the viewpoint of the Mathai-Quillen formalism, the unperturbed twisted theory provides a representation of the $\mathcal{G}$-equivariant de Rham cohomology (in the Cartan model) on the moduli space. However, the formulation is not equivariant with respect to the $U(1)$ action. In other words, the perturbed action is not invariant (i.e. it is not equivariantly closed) under the unperturbed twisted supercharge. On the other hand, it is invariant under the perturbed twisted supercharge. In fact, the twisted supercharge $Q^+(m)$ of the perturbed theory can be interpreted as the generator of the $U(1)$-equivariant extension of the $\mathcal{G}$-equivariant de Rham cohomology on the moduli space. This connection between massive extensions of twisted supersymmetric theories and equivariant cohomology was pointed out in [27], in the context of the non-Abelian monopole theory with massive hypermultiplets; it was subsequently exploited in [28], where the explicit construction leading to the idea of the equivariant extension was carried out in detail. In what follows, we will try to adapt the construction in [28] to our problem. We intend to be as sketchy as possible, and therefore refer the reader to the work cited above for the minute details of the construction.

The idea underlying the construction is the following. Prior to the perturbation, we have a topological field theory which admits a Mathai-Quillen description with BRST charge Q^+. This means, among other things, that the corresponding Lagrangian is a Q^+-commutator. After adding the mass terms proportional to m, it is possible to modify the Q^+ transformation laws so that the perturbed Lagrangian can be written as a $Q^+(m)$-commutator as well, where $Q^+(m)$ are the modified topological transformations. In view of this, it would be tempting to assume that there has to be a standard Mathai-Quillen construction associated to the new topological theory. However, the perturbation has not changed the geometrical setting of the problem, so there is a priori no reason why the Mathai-Quillen formulation should change at all. In fact, it does not, and it turns out that the perturbed theory admits no standard Mathai-Quillen formulation. However, as pointed out in [28], the formalism allows a natural generalization in those situations in which there is an additional symmetry group acting on the moduli space. The geometrical construction involved is an equivariant extension of the Thom form of $\mathcal{E}$ within the framework of the Mathai-Quillen formalism.

The Mathai-Quillen formalism provides an explicit representative of the Thom form of the oriented vector bundle $\mathcal{E} = \mathcal{M} \times_{\mathcal{G}} \mathcal{F}$. The bundle $\mathcal{E}$ is awkward to work with, and it is preferable to work equivariantly, i.e. to regard $\mathcal{E}$ explicitly as an associated vector bundle to the $\mathcal{G}$-principal vector bundle $\mathcal{M} \times \mathcal{F} \to \mathcal{E}$. The Mathai-Quillen representative of the Thom form of $\mathcal{E}$ is $\mathcal{G}$-equivariantly closed and basic on $\mathcal{M} \times \mathcal{F}$ (and hence descends naturally to $\mathcal{E}$). In the Weil model for the $\mathcal{G}$-equivariant cohomology of $\mathcal{E}$, the Mathai-Quillen form is an element in $\mathcal{W}(\mathbf{g}) \otimes \Omega^*(F)$ ($\mathcal{W}(\mathbf{g})$ is the Weil algebra of G) given by [24]:

$$U = \mathrm{e}^{-|x|^2} \int D\rho \exp\left(\frac{1}{4}\rho_i K_{ij} \rho_j + i\rho_i (dx_i + \theta_{ij} x_j)\right). \tag{4.3}$$

In (4.3) x_i are orthonormal coordinates on the fibre $\mathcal{F}$, and dx_i are their corresponding differentials. The ρ_i are Grassmann orthonormal coordinates for the fibre, while K and θ are the generators of $\mathcal{W}(\mathbf{g})$. The Chern-Weil homomorphism, which essentially substitutes the universal realizations K and θ by the actual curvature and connection in $\mathcal{M} \times \mathcal{F}$, gives the link between the Universal representative U and the Thom form $\Phi(\mathcal{E})$. The important point is that while U is $\mathcal{G}$-equivariantly closed by construction, it is not equivariantly closed with respect to the $U(1)$ action. It seems natural to look for a redefinition of the representative (4.3), which is $U(1)$-equivariantly closed. The equivariant extension of U with respect to the $U(1)$ action simply amounts to finding a suitable form p such that $U + p$ is $U(1)$-equivariantly closed. Within the framework of the Mathai-Quillen formalism this amounts to replacing the curvature K with a new equivariant curvature $K_{U(1)}$ [28], which is just the original curvature 2-form K plus an operator L_Λ involving the infinitesimal $U(1)$ action and the connection 1-form θ. In the Cartan model, which is the best suited to topological field theories, the connection form is set to zero, and hence the equivariant extension of the curvature is just the original one plus an operator implementing the infinitesimal $U(1)$ action. This may sound rather abstract, so we now proceed to the actual construction. The main ingredients are a $U(1)$ action defined on the moduli space and the fibre $\mathcal{F}$, under which the metrics on both the moduli space and the fibre must be invariant, while the section $s : \mathcal{M} \to \mathcal{V}$ has to transform equivariantly; that is, if $\phi_t^{\mathcal{M}}$ and $\phi_t^{\mathcal{F}}$ denote the action of $U(1)$ on $\mathcal{M}$ and $\mathcal{F}$ respectively, then

$$s \cdot \phi_t^{\mathcal{M}} = \phi_t^{\mathcal{F}} \cdot s \tag{4.4}$$

This can be easily verified in the present problem in view of the form of s (2.10) and the $U(1)$ actions (4.1). As for the metrics, it suffices to show that for two vector fields $(0, X^+, x)$ and $(0, Y^+, y)$, their scalar product is invariant under the $U(1)$ action (4.1). According to our conventions [11], $\mathrm{Tr} X^+_{\alpha\beta} Y^{+\alpha\beta} = -4 \mathrm{Tr} X \wedge * Y$, so a natural definition for the metric on the field space would be as follows ($\langle | \rangle$ denotes the scalar product on $T\mathcal{M}$):

$$\begin{aligned}
&\big\langle (0, X^+, x) | (0, Y^+, y) \big\rangle = -\int_X \mathrm{Tr}\left(X^+_{\alpha\beta} Y^{+\alpha\beta} \right) + 2 \int_X \mathrm{Tr} * (xy) \\
&= -\int_X \mathrm{Tr}\left(X^+_{11} Y^+_{22} + X^+_{22} Y^+_{11} \right) + \int_X \mathrm{Tr}\left[(X^+_{12} + ix)(Y^+_{12} - iy) + (X^+_{12} - ix)(Y^+_{12} + iy) \right],
\end{aligned} \tag{4.5}$$

which is indeed invariant under the $U(1)$ action.

To incorporate the $U(1)$ action to the Mathai-Quillen construction sketched in section 2.2, we modify the Q^+ transformations of the ghosts and the auxiliary fields charged under $U(1)$ by replacing the curvature ϕ with its equivariant extension $\phi(t) = \phi + \mathcal{L}_t$, where $\mathcal{L}_t$ generates on the fields an infinitesimal $U(1)$ transformation. According to (4.1), this affects only $\tilde{\psi}^+_{\alpha\beta}$, ζ and $\tilde{H}_{\alpha\dot{\alpha}}$. In view of (2.9), the new transformations read:

$$\begin{aligned}
\{Q^+(t), \tilde{\psi}^+_{11}\} &= 2i([B^+_{11}, \phi] - itB^+_{11}), \\
\{Q^+(t), \tilde{\psi}^+_{22}\} &= 2i([B^+_{22}, \phi] + itB^+_{22}), \\
\left\{ Q^+(t), \tilde{\psi}^+_{12} \pm \frac{i}{2}\zeta \right\} &= 2i\big([B^+_{12} \pm iC, \phi] \pm it(B^+_{12} \pm iC)\big), \\
[Q^+(t), \tilde{H}_{1\dot{\alpha}}] &= 2\sqrt{2}i([\tilde{\chi}_{1\dot{\alpha}}, \phi] - it\tilde{\chi}_{1\dot{\alpha}}) - \sqrt{2}[Q^+, s_{1\dot{\alpha}}], \\
[Q^+(t), \tilde{H}_{2\dot{\alpha}}] &= 2\sqrt{2}i([\tilde{\chi}_{2\dot{\alpha}}, \phi] + it\tilde{\chi}_{2\dot{\alpha}}) - \sqrt{2}[Q^+, s_{2\dot{\alpha}}],
\end{aligned} \tag{4.6}$$

If we set $t = -\frac{m}{\sqrt{2}}$ we see that eqs. (4.6) reduce precisely to the $Q^+(m)$ transformations (3.15) and (3.17). The transformations (4.6), when applied to the gauge fermion (2.13), reproduce the original unperturbed action plus the mass terms (3.22) and (3.23). To reproduce the remaining mass terms we note that, as is standard in topological (cohomological) field theories, there remains the possibility of adding to the action a $Q^+(t)$-exact piece without –hopefully– disturbing the theory. As discussed in [28], the requisite piece can be interpreted as the equivariantly-exact differential form which is conventionally added to prove localization in equivariant integration. It has the form $\{Q^+(t), \omega_{X_\mathcal{M}}\}$, where $\omega_{X_\mathcal{M}}$ is the differential form given by $\omega_{X_\mathcal{M}}(Y) = \langle X_\mathcal{M}|Y\rangle$, Y being a vector field on $\mathcal{M}$. In view of the form of the vector field $X_\mathcal{M}$ (4.2) and of the metric (4.5), and keeping in mind that the ghosts $(\psi, \tilde{\psi}^+, \zeta)$ provide a basis of differential forms on $\mathcal{M}$, this form gives a contribution

$$\Bigg\{Q^+(t), -\frac{it}{2}\int_X \mathrm{Tr}\Big(\tilde{\psi}^+_{22}(-iB^+_{11}) + \tilde{\psi}^+_{11}(iB^+_{22})\Big) + \frac{it}{2}\int_X \mathrm{Tr}\Bigg((-i)(B^+_{12} - iC)\left(\tilde{\psi}^+_{12} + \frac{i}{2}\zeta\right) + i(B^+_{12} + iC)\left(\tilde{\psi}^+_{12} - \frac{i}{2}\zeta\right)\Bigg)\Bigg\}. \tag{4.7}$$

But these are precisely the terms (3.18) and (3.19), which as we have seen give correctly the remaining mass terms.

5. Conclusions

The analysis presented in this paper supports the assumption made in [5] in the context of the abstract approach applied to one of the twisted $N=4$ supersymmetric gauge theories. Namely, we have shown that on Kahler four-manifolds the partition function of the Vafa-Witten theory, the only observable leading to topological invariants, remains unchanged under a mass perturbation. This result depends crucially upon the fact that the ghost number symmetry of the theory is non-anomalous. Likewise, we have shown that the mass-perturbed theory (with $\omega=0$) can be regarded as the equivariant extension of the original theory with respect to the $U(1)$ action on the moduli space described in [5].

As was stated in the introduction, the use of the abstract approach in the context of topological quantum field theory is very interesting, because it relies entirely on the properties of physical $N=1$ supersymmetric gauge theories. Thus, it constitutes an important (and truly independent) test of these properties. The predictions in this approach can be tested by confronting them to known mathematical results, or to alternative results obtained in the concrete approach. The results in the framework of the concrete approach recently presented in [8] constitute a very fruitful arena to test predictions based on the abstract approach. In this sense, a wide context is now available to test the properties that are usually attributed to physical $N=1$ supersymmetric gauge theories. To date, only three models have been studied within the abstract approach: the Donaldson-Witten theory with gauge group $SU(2)$, the non-Abelian monopole theory with gauge group $SU(2)$ and a matter multiplet in the fundamental representation, and the Vafa-Witten theory. Other models, as for example those involving higher-rank groups and/or an extended set of hypermultiplets, should also be considered. Our analysis shows that the validity of the abstract approach has to be analysed case by case.

The second twist of the $N=4$ supersymmetric $SU(2)$ gauge theory seems to be quite a promising example. This theory has an anomaly in the ghost number symmetry equal to $-\frac{3}{4}(2\chi + 3\sigma)$ for $SU(2)$ [11]. On Kahler manifolds, this is proportional to the square of the canonical class ($K \cdot K = 2\chi + 3\sigma$) and therefore vanishes on hyper-Kahler manifolds. This is as it should be, for the physical and the twisted theory coincide on hyper-Kahler manifolds [9], and therefore the second twist should be equivalent to the Vafa-Witten theory, which is anomaly-free. On more general four-manifolds, this is no longer the case, and in order to compute non-trivial topological correlators one has to insert operators whose overall ghost number matches the anomaly of the theory. Notice that since the anomaly does not depend on the instanton number, there is only a finite number (if any) of non-vanishing correlation functions. One could in principle try to compute these topological observables in the pure (i.e. massless) twisted theory. As there is no equivalent of the u-plane description for the low-energy dynamics of $N = 4$ theories, the concrete approach does not apply to this case. As for the abstract approach, the $N = 1$ low-energy theory which corresponds to $N = 4$ perturbed with a mass term for one of the chiral superfields, has a continuum of vacuum states in different phases, and therefore it is not very useful for making explicit computations.

For the mass-deformed twisted theory ($N = 4$ with masses for two of the chiral superfields) the situation is certainly different. The corresponding physical theory has $N = 2$ supersymmetry and its low-energy behaviour is known [3]. There is a definite picture of the structure of singularities and of the symmetries governing the dynamics on the u-plane, and it is therefore possible to make explicit computations within the concrete approach. As for the twisted theory, unlike the Vafa-Witten theory, the mass perturbation makes sense on any arbitrary spin four-manifold. The perturbation preserves the unique topological symmetry of the theory, and in fact it can be shown, by extending the construction presented in [28] for the theory of non-Abelian monopoles, that the structure of the perturbation is dictated by an equivariant extension with respect to a $U(1)$ action which is a symmetry of the non-Abelian adjoint monopole equations. However, as the ghost number symmetry is generally anomalous, one should expect the correlation functions to depend non-trivially on the mass parameter m. Of course, on hyper-Kahler manifolds one should recover the results of [5]. In particular, the generating function for the topological correlators should converge to the partition function presented there. As regards the abstract approach, the vacuum structure of the $N = 1$ effective theory is known (and we have discussed it above): there are three isolated vacua with a definite pattern of symmetries relating them. The space-time-dependent mass term, which breaks $N = 2$ down to $N = 1$, cannot simply be dropped as in the Vafa-Witten theory. Rather, as this term is essentially one of the observables of the theory, the effect of the perturbation can be absorbed, as in [9] or [10], in a redefinition of the parameters in the generating function. We expect to address these and other related issues in future work.

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