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Two-loop hybrid renormalization of local dimension-4 heavy–light operators

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Abstract

The renormalization of local dimension-4 operators containing a heavy and a light quark field at scales below the heavy-quark mass is discussed, using the formalism of the heavy-quark effective theory. The anomalous dimensions of these operators and their mixing are calculated to two-loop order. Some phenomenological applications are briefly discussed.

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1 Introduction

Local operators containing both heavy and light quark fields exhibit an interesting behaviour under renormalization at scales below the heavy-quark mass m_Q. Large logarithms of the type $\alpha_s \ln(m_Q/\mu)$ arise from the exchange of gluons that are "hard" with respect to the light quark but "soft" with respect to the heavy quark. Since such gluons see the heavy quark as a static colour source, the large logarithms can be summed to all orders in perturbation theory using an effective theory for static heavy quarks, the so-called heavy-quark effective theory (HQET) [1]–[3]. In the HQET, the 4-component heavy-quark field $Q(x)$ is replaced by a velocity-dependent 2-component field $h_v(x)$ satisfying $\not{v}\, h_v = h_v$, where v is the velocity of the hadron containing the heavy quark. Because of the particular hierarchy of the mass scales involved, the renormalization of heavy–light operators in the HQET is called "hybrid" renormalization. Operators in the effective theory have a different evolution than in usual QCD. For instance, whereas the vector current $\bar{q}\,\gamma^\mu Q$ is conserved in QCD (i.e. its anomalous dimension vanishes), the corresponding current $\bar{q}\,\gamma^\mu h_v$ in the HQET has a nontrivial anomalous dimension [4]–[6], which governs the evolution for scales below the heavy-quark mass. It has been calculated at the two-loop order in Refs. [7]–[9]. The renormalization of higher-dimensional heavy–light operators is known only at the leading logarithmic order [10]–[12]. Here we shall generalize these results and perform the calculation of the anomalous dimension matrix of local dimension-4 operators at the two-loop order. This calculation requires the evaluation of two-loop tensor integrals in the HQET that are infrared (IR) singular when one of the external lines is taken on-shell. A general algorithm to calculate these integrals has been developed recently [13].

The matrix elements of local dimension-4 heavy–light operators appear at order $1/m_Q$ in the heavy-quark expansion of heavy-meson decay constants [14] and weak transition form factors [15]. As such, they play an important role in many phenomenological applications of the HQET. The theoretical predictions for weak decay form factors involve operator matrix elements renormalized at the scale m_Q. Our results can then be used to rewrite these matrix elements in terms of ones renormalized at a scale $\mu \ll m_Q$, which may be identified with the scale at which a nonperturbative evaluation of these matrix elements is performed. Details of such applications will be discussed elsewhere.

2 Operator renormalization

We start by constructing, in the HQET, a basis of local dimension-4 operators containing a heavy and a light quark field. Since ultimately our interest is in the matrix elements of these operators between physical hadron states, it is sufficient to consider gauge-invariant operators that do not vanish by the equations of motions [16]. Then, in the limit where the light-quark mass is set to zero, any dimension-4 operator must contain a covariant derivative acting on one of the quark fields. A basis of such operators, which closes under renormalization, is provided by[1]

$$
\begin{aligned}
O_1 &= \bar q\,\Gamma\, iD_\alpha h_v\,, \\
O_2 &= \bar q\,(iD_\alpha)^\dagger\,\Gamma\, h_v\,, \\
O_3 &= \bar q\,(iv\!\cdot\! D)^\dagger v_\alpha \Gamma\, h_v\,, \\
O_4 &= \bar q\,(iv\!\cdot\! D)^\dagger \gamma_\alpha \not{v}\,\Gamma\, h_v\,,
\end{aligned} \tag{1}
$$

where Γ represents an arbitrary Dirac matrix, which may or may not contain the Lorentz index α. The Feynman rules of the HQET ensure that no Dirac matrices appear on the right-hand side of Γ. Moreover, for $m_q = 0$ only an even number of γ matrices can appear on the left-hand side of Γ.

The equations of motion, $iv\cdot D\,h_v = 0$ and $\bar q\,(i\not{D})^\dagger = 0$, imply that between physical states the operators $(O_1 - O_2)$, O_3 and O_4 can be replaced by total derivatives of some lower-dimensional operators, which are renormalized multiplicatively and irrespective of their Dirac structure. The additional power of the external momentum carried by the operators does not affect the ultraviolet (UV) behaviour of loop diagrams. Moreover, the reparametrization invariance of the HQET [17, 18] ensures that also the multiplicative renormalization of the operator O_1 is determined by the same Z factor that governs the renormalization of dimension-3 operators [11]. However, there is a nontrivial mixing between O_1 and the other three operators.

We define a matrix $\mathbf{Z}$ of renormalization constants, which absorb the UV divergences in the matrix elements of the bare operators, by the relation $O_i = \sum_j \mathbf{Z}_{ij}\,O_{j,\text{bare}}$. Then the matrix $\boldsymbol{\gamma}$ of the anomalous dimensions, which govern the scale dependence of the renormalized operators, is given by

$$
\boldsymbol{\gamma} = -\frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}\ln\mu}\,\mathbf{Z}^{-1}\,. \tag{2}
$$

From the arguments presented above, it follows that the matrix $\mathbf{Z}$ has the structure

$$
\mathbf{Z} = \left(\begin{array}{cccc}
Z_1 & Z_2 & Z_3 & Z_4 \\
0 & Z_1 + Z_2 & Z_3 & Z_4 \\
0 & 0 & Z_1 & 0 \\
0 & 0 & 0 & Z_1
\end{array} \right)\,, \tag{3}
$$

where Z_1 is the renormalization constant of local dimension-3 currents, which is known at the two-loop order [7]–[9]. Here we will calculate the remaining entries in the matrix $\mathbf{Z}$ with the same accuracy.

In a minimal subtraction scheme, the renormalization constants are defined to remove the $1/\epsilon$ poles arising in the calculation of the bare Green functions with insertions of the operators O_i in dimensional regularization, i.e. in $d = 4 - 2\epsilon$ space-time dimensions.

[1] We use the notation $iD = i\partial + g_s A$ and $(iD)^\dagger = -i\overleftarrow{\partial} + g_s A$.

Hence,

$$\mathbf{Z} = \mathbf{1} + \sum_{k=0}^{\infty} \frac{1}{\epsilon^k}\,\mathbf{Z}^{(k)} \,. \tag{4}$$

The requirement that the anomalous dimension matrix be finite in the limit $\epsilon \to 0$ implies the relations [19]

$$\begin{aligned} \boldsymbol{\gamma} &= 2\alpha_s \frac{\partial \mathbf{Z}^{(1)}}{\partial \alpha_s} = -2\alpha_s \frac{\partial (\mathbf{Z}^{-1})^{(1)}}{\partial \alpha_s} \,, \\ \alpha_s \frac{\partial \mathbf{Z}^{(2)}}{\partial \alpha_s} &= \alpha_s \frac{\partial \mathbf{Z}^{(1)}}{\partial \alpha_s} \left(\mathbf{Z}^{(1)} + \frac{\beta(\alpha_s)}{\alpha_s} \right) , \end{aligned} \tag{5}$$

where $\beta(\alpha_s) = \mathrm{d}\alpha_s/\mathrm{d}\ln\mu^2$ is the β function. The first equation shows that the anomalous dimension matrix can be obtained from the coefficient of the $1/\epsilon$ pole in $\mathbf{Z}$, whereas the second one implies a nontrivial constraint on the coefficient of the $1/\epsilon^2$ pole, arising at two-loop and higher order.

Previous authors have calculated the renormalization constants Z_i appearing in (3) at the one-loop order, finding that [10, 11]

$$Z_i = \delta_{1i} + \frac{C_F\alpha_s}{4\pi\epsilon} \left(-\frac{3}{2}, \frac{3}{2}, -1, -\frac{1}{2} \right) + O(\alpha_s^2) \,; \quad i = 1,2,3,4 \,. \tag{6}$$

This information is sufficient to reconstruct the matrix $\mathbf{Z}$ at order α_s. From the second relation in (5), it then follows that

$$Z_i^{(2)} = \left(\frac{\alpha_s}{4\pi} \right)^2 \left(-\frac{3}{4} C_F^2 - \frac{11}{6} C_F C_A + \frac{2}{3} C_F T_F\, n_f \right) \left(-\frac{3}{2}, \frac{3}{2}, -1, -\frac{1}{2} \right) + O(\alpha_s^3) \,. \tag{7}$$

Here $C_A = N$, $C_F = \frac{1}{2}(N^2-1)/N$ and $T_F = \frac{1}{2}$ are the colour factors for an $SU(N)$ gauge group, and n_f is the number of light-quark flavours. This relation will provide a check on our two-loop results.

The equations of motion impose two additional conditions on the renormalization constants. Substituting $\Gamma = v^\alpha\Gamma'$ or $\Gamma = \gamma^\alpha\Gamma'$ in the bare operator O_1 in (1), and using that between physical states $\bar q\,\Gamma'\, iv\cdot D h_v = 0$, whereas $\bar q\, i\not{D}\Gamma' h_v = i\partial_\alpha\, [\bar q\,\gamma^\alpha\Gamma' h_v]$ is renormalized multiplicatively, we find that

$$\begin{aligned} (\mathbf{Z}^{-1})_{12} + (\mathbf{Z}^{-1})_{13} + (\mathbf{Z}^{-1})_{14} &= 0 \,, \\ (\mathbf{Z}^{-1})_{13} - 2(1-\epsilon)(\mathbf{Z}^{-1})_{14}\Big|_{\text{poles}} &= 0 \,, \end{aligned} \tag{8}$$

where it is understood that in the second relation all pole terms proportional to $1/\epsilon^k$ must cancel. In terms of the constants Z_i, we obtain the conditions

$$\begin{aligned} & Z_2^{(k)} + Z_3^{(k)} + Z_4^{(k)} = 0 \,; \quad k \ge 1 \,, \\ & Z_3^{(1)} - 2Z_4^{(1)} \left[1 + 2Z_1^{(1)} + Z_2^{(1)} \right] + 2Z_4^{(2)} = 0 \,, \end{aligned} \tag{9}$$

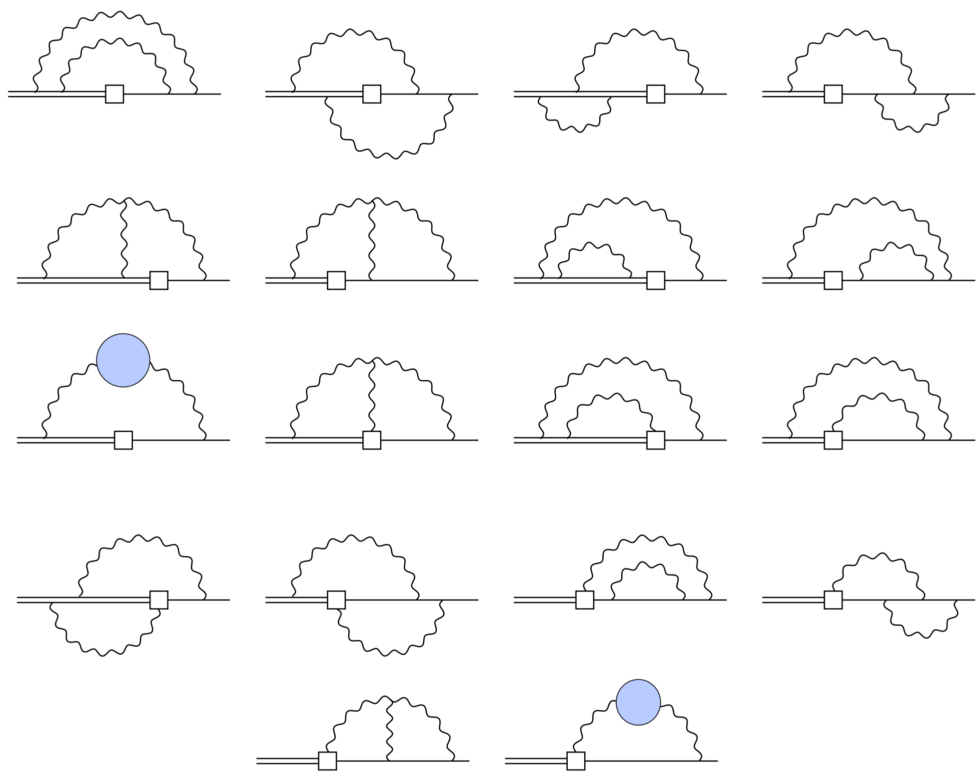

Figure 1: *Two-loop diagrams contributing to the calculation of the renormalization constants Z_i. The shaded circles represent one-loop insertions of the gluon self-energy.*

which will provide a highly nontrivial check on our two-loop results. Relations (8) allow to calculate two of the three constants Z_2, Z_3 and Z_4 in terms of the third one. The fact that the calculation of one of these constants suffices to reconstruct the full matrix $\mathbf{Z}$, using that Z_1 is fixed by reparametrization invariance, has been observed in Ref. [11]; however, the relations derived there are not fully correct beyond the one-loop order.

3 Two-loop calculation

To obtain the renormalization constants Z_i at order α_s^2, we calculate the insertions of the operator O_1 into the amputated Green function with a heavy and a light quark to two-loop order. The relevant diagrams are shown in Figure 1. We need to keep terms linear in the momentum p of the light quark, as only those contribute to the renormalization constants Z_2, Z_3 and Z_4. However, for completeness we will also keep terms linear in the heavy-quark momentum k_α, which contribute to Z_1. In that way we will reproduce the known two-loop result for Z_1 as a check of our calculation. Because the pole parts are polynomial in the external momenta, we can first take a derivative with respect to p and k and then set $p = 0$, so that all integrals are of propagator type and depend

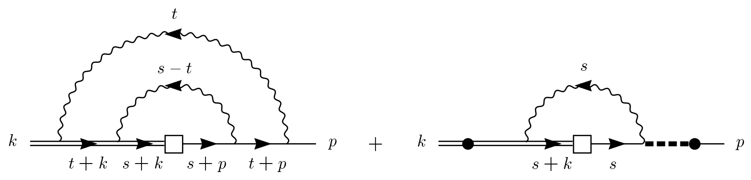

Figure 2: *Schematic representation of the R^* operation. The black dots represent the original vertices, the dashed line the IR counterterm.*

on the single variable $\omega = v \cdot k$. However, this method fails for some of the diagrams, for which setting $p = 0$ after differentiation leads to IR divergences. In these cases, we apply a variant of the so-called R^* operation [20], which compensates these IR poles by a construction of counterterms for the IR divergent subgraphs.

Consider, as an example, the first diagram in Figure 1. Its contribution is proportional to the integral

$$D_1 = \int \mathrm{d}^d s\, \mathrm{d}^d t\, \frac{(s+k)_\alpha\,(t+p)_\beta\,(s+p)_\gamma}{t^2(t+p)^2(s+p)^2(s-t)^2(v\cdot s+\omega)(v\cdot t+\omega)}\,. \tag{10}$$

When linearizing this expression in p, we encounter an IR divergence from the region $t \to 0$, which can be removed by adding and subtracting the IR subtraction term

$$D_1^{\mathrm{IR}} = \int \mathrm{d}^d s\, \mathrm{d}^d t\, \frac{(t+p)_\beta\, s_\alpha\, s_\gamma}{t^2(t+p)^2(s^2)^2(v\cdot s+\omega)\,\omega}\,. \tag{11}$$

The difference $D_1 - D_1^{\mathrm{IR}}$ can be evaluated using a naive linearization in p, since the behaviour of the IR subtraction term for $t \to 0$ is the same as that of the original integral. However, because the expansion of D_1^{IR} involves tadpole integrals that vanish in dimensional regularization, the difference $(D_1 - D_1^{\mathrm{IR}})_{\text{linearized}}$ coincides with the naive linearization of the original expression for D_1. The contribution D_1^{IR}, which is necessary to subtract the IR subdivergence of the original diagram, factorizes into an IR counterterm (the t integral) and the original diagram with the lines of the IR sensitive subgraph removed, and with t and p set to zero. The construction just described is schematically shown in Figure 2, where the dashed line represents the IR counterterm

$$\int \frac{\mathrm{d}^d t}{(2\pi)^d}\, \frac{\gamma^\beta (t+p)_\beta}{t^2(t+p)^2} = \frac{i\not{p}}{(4\pi)^{d/2}}\,(-p^2)^{-\epsilon}\, \frac{\Gamma(d/2)\,\Gamma(d/2-1)\,\Gamma(2-d/2)}{\Gamma(d-1)}\,. \tag{12}$$

The remaining two-loop tensor integrals are of the general form

$$\int \mathrm{d}^d s\, \mathrm{d}^d t\, \left(\frac{\omega}{v\cdot s+\omega}\right)^{\alpha_1} \left(\frac{\omega}{v\cdot t+\omega}\right)^{\alpha_2} \frac{s_{\mu_1}\dots s_{\mu_n} t^{\nu_1}\dots t^{\nu_m}}{(-s^2)^{\alpha_3}(-t^2)^{\alpha_4}[-(s-t)^2]^{\alpha_5}}$$
$$\equiv -\pi^d(-2\omega)^{2(d-\alpha_3-\alpha_4-\alpha_5)+n+m}\, I^{\nu_1\dots\nu_m}_{\mu_1\dots\mu_n}(v;\{\alpha_i\})\,,$$

$$\int \mathrm{d}^d s\, \mathrm{d}^d t\, \frac{s_{\mu_1}\dots s_{\mu_n} t^{\nu_1}\dots t^{\nu_m}}{(-s^2)^{\alpha_1}(-t^2)^{\alpha_2}[-(s-p)^2]^{\alpha_3}[-(t-p)^2]^{\alpha_4}[-(s-t)^2]^{\alpha_5}}$$
$$\equiv -\pi^d(-p^2)^{(d-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5)}\, R^{\nu_1\dots\nu_m}_{\mu_1\dots\mu_n}(p;\{\alpha_i\})\,. \tag{13}$$

Using the method of integration by parts [8, 21], we obtain the recurrence relations

$$\Big[(d-\alpha_1-\alpha_3-2\alpha_5+n)+\alpha_3\,\mathbf{3}^+(\mathbf{4}^--\mathbf{5}^-)+\alpha_1\,\mathbf{1}^+\mathbf{2}^-\Big]\;I^{\nu_1\ldots\nu_m}_{\mu_1\ldots\mu_n}(v;\{\alpha_i\})$$
$$=\sum_{j=1}^{n} I^{\mu_j\nu_1\ldots\nu_m}_{\mu_1\ldots[\mu_j]\ldots\mu_n}(v;\{\alpha_i\})\,,$$

$$\Big[(d-\alpha_1-\alpha_3-2\alpha_5+n)+\alpha_3\,\mathbf{3}^+(\mathbf{4}^--\mathbf{5}^-)+\alpha_1\,\mathbf{1}^+(\mathbf{2}^--\mathbf{5}^-)\Big]\;R^{\nu_1\ldots\nu_m}_{\mu_1\ldots\mu_n}(p;\{\alpha_i\})$$
$$=\sum_{j=1}^{n} R^{\mu_j\nu_1\ldots\nu_m}_{\mu_1\ldots[\mu_j]\ldots\mu_n}(p;\{\alpha_i\})\,, \tag{14}$$

which allow us to express any two-loop integral in terms of degenerate integrals, which have $\alpha_2=0$, $\alpha_4=0$ or $\alpha_5=0$. Here $\mathbf{1}^+$ is an operator raising the index α_1 by one unit etc., and $[\mu_j]$ means that this index is omitted. The degenerate integrals can be related in a straightforward way to products of one-loop tensor integrals [13]. Using this technique, we have calculated the pole parts of the two-loop diagrams in the 't Hooft–Feynman gauge. The results are summarized in the first three columns of Table 1. The renormalization scale μ is introduced by the replacement of the bare coupling constant with the renormalized one through the relation $g_s^{\rm bare}=\bar\mu^\epsilon Z_g\,g_s$, with $\bar\mu=\mu\,e^{\gamma_E/2}(4\pi)^{-1/2}$ in the $\overline{\rm MS}$ scheme.

The two-loop diagrams in Figure 1 contain subdivergences, which must be subtracted by UV counterterms. In addition to the one-loop counterterms for the quark and gluon propagators and vertices, local operator counterterms are required. To find these, we calculate at the one-loop order all insertions of the operator O_1 into the amputated Green functions with a non-negative degree of divergence. In our case, those are the two- and three-point functions with field content $\bar q h$ and $\bar q h A$. We find that in the 't Hooft–Feynman gauge the UV divergences of these functions are removed by the counterterms

$$\mathcal{L}_{\rm c.t.}=-C_F\,\frac{\alpha_s}{4\pi\epsilon}\left(O_1-\frac{3}{2}O_2+O_3+\frac{1}{2}O_4\right)-C_A\,\frac{\alpha_s}{4\pi\epsilon}\,\bar\mu^\epsilon g_s\,\bar q\,\Gamma A_\alpha h_v\,. \tag{15}$$

Since the two-loop calculation was performed off-shell, a gauge-dependent operator has to be included in addition to the operators O_i introduced in (1). However, there is no need to include operators that vanish by the equations of motion [16]. The results for the sum of all counterterm contributions are summarized in the last two columns of Table 1. When these contributions are added to the result for the sum of the two-loop diagrams, all nonlocal $1/\epsilon$ divergences proportional to $\ln(-2\omega/\mu)$ and $\ln(-p^2/\mu^2)$ cancel. This is a nontrivial check of our calculation.

The sum of the two-loop diagrams plus their counterterms determines the two-loop coefficients in the products $-Z_h^{1/2}Z_q^{1/2}Z_i$. To obtain the results for the renormalization constants at order α_s^2, we have to account for wave-function renormalization of the external quark fields. Using the one-loop expressions for the Z_i factors in (6), as well as the known one- and two-loop (for Z_1) wave-function renormalization constants of the

Table 1: *Sum of two-loop and counterterm contributions in units of* $(\alpha_s/4\pi)^2$

Structure		$\left(\frac{-2\omega}{\mu}\right)^{-4\epsilon}$	$\left(\frac{-2\omega}{\mu}\right)^{-2\epsilon}\left(\frac{-p^2}{\mu^2}\right)^{-\epsilon}$	$\left(\frac{-p^2}{\mu^2}\right)^{-2\epsilon}$	$\left(\frac{-2\omega}{\mu}\right)^{-2\epsilon}$	$\left(\frac{-p^2}{\mu^2}\right)^{-\epsilon}$
p_α	C_F^2	$-\frac{3}{8\epsilon^2}+\left(1-\frac{\pi^2}{3}\right)\frac{1}{\epsilon}$	$-\frac{1}{2\epsilon^2}-\frac{1}{\epsilon}$	$\frac{1}{2\epsilon^2}+\frac{9}{4\epsilon}$	$\frac{5}{4\epsilon^2}$	$-\frac{1}{2\epsilon^2}-\frac{1}{\epsilon}$
	$C_F C_A$	$-\frac{2}{3\epsilon^2}+\left(-\frac{49}{36}+\frac{\pi^2}{12}\right)\frac{1}{\epsilon}$	$-\frac{1}{2\epsilon^2}-\frac{1}{\epsilon}$	$-\frac{19}{12\epsilon^2}-\frac{151}{18\epsilon}$	$\frac{11}{6\epsilon^2}$	$\frac{11}{3\epsilon^2}+\frac{22}{3\epsilon}$
	$C_F T_F\, n_f$	$\frac{1}{3\epsilon^2}+\frac{1}{18\epsilon}$	0	$\frac{2}{3\epsilon^2}+\frac{31}{9\epsilon}$	$-\frac{2}{3\epsilon^2}$	$-\frac{4}{3\epsilon^2}-\frac{8}{3\epsilon}$
$v\cdot p\, v_\alpha$	C_F^2	$\frac{5}{4\epsilon^2}+\left(\frac{17}{12}+\frac{2\pi^2}{9}\right)\frac{1}{\epsilon}$	$-\frac{1}{\epsilon^2}-\frac{5}{\epsilon}$	0	$-\frac{3}{2\epsilon^2}+\frac{1}{2\epsilon}$	$\frac{1}{\epsilon^2}+\frac{2}{\epsilon}$
	$C_F C_A$	$\frac{11}{6\epsilon^2}+\left(-\frac{7}{9}-\frac{\pi^2}{18}\right)\frac{1}{\epsilon}$	0	0	$-\frac{11}{3\epsilon^2}+\frac{11}{3\epsilon}$	0
	$C_F T_F\, n_f$	$-\frac{2}{3\epsilon^2}+\frac{5}{9\epsilon}$	0	0	$\frac{4}{3\epsilon^2}-\frac{4}{3\epsilon}$	0
$v\cdot p\, \gamma_\alpha \not p$	C_F^2	$\frac{5}{8\epsilon^2}+\left(\frac{4}{3}+\frac{\pi^2}{9}\right)\frac{1}{\epsilon}$	$-\frac{1}{2\epsilon^2}-\frac{3}{\epsilon}$	0	$-\frac{3}{4\epsilon^2}+\frac{1}{2\epsilon}$	$\frac{1}{2\epsilon^2}+\frac{1}{\epsilon}$
	$C_F C_A$	$\frac{11}{12\epsilon^2}+\left(\frac{19}{36}-\frac{\pi^2}{36}\right)\frac{1}{\epsilon}$	0	0	$-\frac{11}{6\epsilon^2}$	0
	$C_F T_F\, n_f$	$-\frac{1}{3\epsilon^2}-\frac{1}{18\epsilon}$	0	0	$\frac{2}{3\epsilon^2}$	0
k_α	C_F^2	$\frac{1}{2\epsilon^2}+\left(-1+\frac{2\pi^2}{3}\right)\frac{1}{\epsilon}$	0	0	$-\frac{1}{\epsilon^2}$	0
	$C_F C_A$	$\frac{1}{\epsilon^2}+\left(1-\frac{\pi^2}{6}\right)\frac{1}{\epsilon}$	0	0	$-\frac{2}{\epsilon^2}$	0
	$C_F T_F\, n_f$	0	0	0	0	0

quark fields [22, 23], we obtain the final results:

$$
\begin{aligned}
Z_1 &= 1 - \frac{3C_F\alpha_s}{8\pi\epsilon}\Bigg\{1 + \frac{\alpha_s}{4\pi}\Bigg[C_F\left(-\frac{3}{4\epsilon} - \frac{5}{12} + \frac{4\pi^2}{9}\right) \\
&\qquad + C_A\left(-\frac{11}{6\epsilon} + \frac{49}{36} - \frac{\pi^2}{9}\right) + T_F\, n_f\left(\frac{2}{3\epsilon} - \frac{5}{9}\right)\Bigg]\Bigg\}, \\
Z_2 &= \frac{3C_F\alpha_s}{8\pi\epsilon}\Bigg\{1 + \frac{\alpha_s}{4\pi}\Bigg[C_F\left(-\frac{3}{4\epsilon} - \frac{5}{6} + \frac{2\pi^2}{9}\right) \\
&\qquad + C_A\left(-\frac{11}{6\epsilon} + \frac{41}{18} - \frac{\pi^2}{18}\right) + T_F\, n_f\left(\frac{2}{3\epsilon} - \frac{5}{9}\right)\Bigg]\Bigg\}, \\
Z_3 &= -\frac{C_F\alpha_s}{4\pi\epsilon}\Bigg\{1 + \frac{\alpha_s}{4\pi}\Bigg[C_F\left(-\frac{3}{4\epsilon} - \frac{13}{12} + \frac{2\pi^2}{9}\right) \\
&\qquad + C_A\left(-\frac{11}{6\epsilon} + \frac{26}{9} - \frac{\pi^2}{18}\right) + T_F\, n_f\left(\frac{2}{3\epsilon} - \frac{7}{9}\right)\Bigg]\Bigg\}, \\
Z_4 &= -\frac{C_F\alpha_s}{8\pi\epsilon}\Bigg\{1 + \frac{\alpha_s}{4\pi}\Bigg[C_F\left(-\frac{3}{4\epsilon} - \frac{1}{3} + \frac{2\pi^2}{9}\right)
\end{aligned}
$$

$$+ C_A \left(-\frac{11}{6\epsilon} + \frac{19}{18} - \frac{\pi^2}{18}\right) + T_F\, n_f \left(\frac{2}{3\epsilon} - \frac{1}{9}\right)\bigg]\bigg\} . \qquad (16)$$

Our result for Z_1 agrees with that derived in Refs. [7]–[9]. The two-loop results for the remaining constants Z_i are new. Note that these expressions do indeed obey the relations in (7) and (9). This a highly nontrivial check, which gives us confidence in the correctness of our results.

The anomalous dimension matrix $\boldsymbol{\gamma}$ governing the scale dependence and mixing of the renormalized operators can be obtained using the first relation in (5). Since the matrix coefficients in the perturbative expansion

$$\boldsymbol{\gamma} = \boldsymbol{\gamma_0}\, \frac{\alpha_s}{4\pi} + \boldsymbol{\gamma_1} \left(\frac{\alpha_s}{4\pi}\right)^2 + \dots \qquad (17)$$

have the same texture as the matrix $\mathbf{Z}$ in (3), it is sufficient to quote the entries of the first row, which we denote as γ_i with $i = 1, 2, 3, 4$. We obtain

$$\begin{aligned}
(\boldsymbol{\gamma_0})_{1i} \equiv \gamma_{i,0} &= C_F\, (-3\,,3\,,-2\,,-1) \;, \\
(\boldsymbol{\gamma_1})_{1i} \equiv \gamma_{i,1} &= C_F^2 \left(\frac{5}{2} - \frac{8\pi^2}{3}\,, -5 + \frac{4\pi^2}{3}\,, \frac{13}{3} - \frac{8\pi^2}{9}\,, \frac{2}{3} - \frac{4\pi^2}{9}\right) \\
&\quad + C_F C_A \left(-\frac{49}{6} + \frac{2\pi^2}{3}\,, \frac{41}{3} - \frac{\pi^2}{3}\,, -\frac{104}{9} + \frac{2\pi^2}{9}\,, -\frac{19}{9} + \frac{\pi^2}{9}\right) \\
&\quad + C_F T_F\, n_f \left(\frac{10}{3}\,, -\frac{10}{3}\,, \frac{28}{9}\,, \frac{2}{9}\right) .
\end{aligned} \qquad (18)$$

We note that $\gamma_2 + \gamma_3 + \gamma_4 = 0$ as a consequence of the first relation in (8), and that the one-loop matrix coefficient $\boldsymbol{\gamma_0}$ satisfies the simple relation ${\boldsymbol{\gamma_0}}^2 = -3C_F \boldsymbol{\gamma_0}$.

4 Nontrivial basis transformations

The results derived in this work are sufficient to calculate, at the two-loop order, the hybrid renormalization of any local dimension-4 operator containing a heavy and a light quark field. However, it some cases the choice of the operator basis in (1) may not be the most convenient one. Whereas the transformation between one operator basis and another is trivial at the one-loop order, it may become subtle at the next-to-leading order, because in dimensional regularization the relations between the operators of different bases may depend on ϵ.

Consider the general case where the operator basis $\{O_i\}$ is replaced by a new basis $\{Q_j\}$ with a linear relation of the form $O_i = \sum_j \mathbf{R}_{ij}(\epsilon)\, Q_j$. Depending on the choice of the Dirac matrix Γ, some of the operators O_i may not be independent, and thus it may happen that the new basis contains less than four operators. Therefore, in general the transformation matrix $\mathbf{R}(\epsilon)$ is a $4 \times n$ matrix with $n \leq 4$. We introduce an $n \times 4$

left-inverse of this matrix such that $\mathbf{L}(\epsilon)\,\mathbf{R}(\epsilon)=\mathbf{1}$. It then follows that the $n\times n$ matrix $\boldsymbol{\mathcal{Z}}$ of renormalization constants in the new operator basis satisfies the relation

$$\boldsymbol{\mathcal{Z}}^{-1}=\mathbf{1}+\mathbf{L}(\epsilon)\left(\mathbf{Z}^{-1}-\mathbf{1}\right)\mathbf{R}(\epsilon)\Big|_{\text{poles}}\,, \tag{19}$$

where only pole terms are kept on the right-hand side. The anomalous dimension matrix in the new basis is determined by the terms of order $1/\epsilon$ in $\boldsymbol{\mathcal{Z}}$. Expanding the transformation matrices as $\mathbf{R}(\epsilon)=\sum_n\mathbf{R_n}\epsilon^n$ and $\mathbf{L}(\epsilon)=\sum_n\mathbf{L_n}\epsilon^n$, we obtain

$$\boldsymbol{\mathcal{Z}}^{(1)}=\mathbf{L_0}\,\mathbf{Z}^{(1)}\,\mathbf{R_0}+\mathbf{L_1}\Big[\mathbf{Z}^{(2)}-(\mathbf{Z}^{(1)})^2\Big]\,\mathbf{R_0}+\mathbf{L_0}\Big[\mathbf{Z}^{(2)}-(\mathbf{Z}^{(1)})^2\Big]\,\mathbf{R_1}+\dots\,, \tag{20}$$

where the ellipses represent terms that do not contribute at the two-loop order. Denoting by $\bar{\boldsymbol{\gamma}}$ the anomalous dimension matrix in the new operator basis, we obtain, using (5),

$$\begin{aligned}\bar{\boldsymbol{\gamma}}_\mathbf{0}&=\mathbf{L_0}\,\boldsymbol{\gamma}_\mathbf{0}\,\mathbf{R_0}\,,\\ \bar{\boldsymbol{\gamma}}_\mathbf{1}&=\mathbf{L_0}\,\boldsymbol{\gamma}_\mathbf{1}\,\mathbf{R_0}-\mathbf{L_1}\left(\tfrac12\boldsymbol{\gamma}_\mathbf{0}{}^2+\beta_0\boldsymbol{\gamma}_\mathbf{0}\right)\mathbf{R_0}-\mathbf{L_0}\left(\tfrac12\boldsymbol{\gamma}_\mathbf{0}{}^2+\beta_0\boldsymbol{\gamma}_\mathbf{0}\right)\mathbf{R_1}\,.\end{aligned} \tag{21}$$

Alternatively, we may evaluate (19) directly to get the inverse matrix $\boldsymbol{\mathcal{Z}}^{-1}$, and then use the first relation in (5) to calculate the anomalous dimension matrix.

As an example, we discuss the mixing of the local dimension-4 operators appearing at order $1/m_Q$ in the heavy-quark expansion of the vector current[2] $V^\mu=\bar q\,\gamma^\mu Q$. There are six such operators, which can be chosen in the form [10, 11]

$$\begin{aligned}Q_1&=\bar q\,\gamma^\mu i\rlap{/}D h_v\,, & Q_4&=\bar q\,(iv\cdot D)^\dagger\gamma^\mu h_v\,,\\ Q_2&=\bar q\,v^\mu i\rlap{/}D h_v\,, & Q_5&=\bar q\,(iv\cdot D)^\dagger v^\mu h_v\,,\\ Q_3&=\bar q\,iD^\mu h_v\,, & Q_6&=\bar q\,(iD^\mu)^\dagger\,h_v\,.\end{aligned} \tag{22}$$

Each of the first three operators has the structure of O_1, with the substitutions $\Gamma=\gamma^\mu\gamma^\alpha$, $v^\mu\gamma^\alpha$ and $g^{\mu\alpha}$, respectively. In all three cases, the corresponding operators O_2, O_3 and O_4 can be written as linear combinations of Q_4, Q_5 and Q_6. For instance, the basis transformation for the case $\Gamma=\gamma^\mu\gamma^\alpha$ is

$$\begin{pmatrix}O_1\\O_2\\O_3\\O_4\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&0&0&2\\0&1&0&0\\0&2\epsilon&4(1-\epsilon)&0\end{pmatrix}\begin{pmatrix}Q_1\\Q_4\\Q_5\\Q_6\end{pmatrix}. \tag{23}$$

Using (19) and the constraints (8) imposed by the equations of motion, we find for the corresponding matrix of renormalization constants

$$\boldsymbol{\mathcal{Z}}_{(1,4,5,6)}=\begin{pmatrix}Z_1&2Z_4&2Z_3&2Z_2\\0&Z_1&0&0\\0&0&Z_1&0\\0&Z_4&Z_3&Z_1+Z_2\end{pmatrix}, \tag{24}$$

[2]The discussion for the axial vector current is identical, provided the basis operators are chosen in a convenient way [11].

where the subscript indicates the operators Q_j whose renormalization is accomplished by this matrix. Similarly, for the other two cases $\Gamma = v^\mu\gamma^\alpha$ and $g^{\mu\alpha}$ we find, respectively,

$$\boldsymbol{\mathcal{Z}}_{(2,5)} = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_1 \end{pmatrix}, \qquad \boldsymbol{\mathcal{Z}}_{(3,4,5,6)} = \begin{pmatrix} Z_1 & Z_4 & Z_3 & Z_2 \\ 0 & Z_1 & 0 & 0 \\ 0 & 0 & Z_1 & 0 \\ 0 & Z_4 & Z_3 & Z_1 + Z_2 \end{pmatrix}. \qquad (25)$$

Note that the 3×3 submatrix renormalizing the operators Q_4, Q_5 and Q_6 is the same in $\boldsymbol{\mathcal{Z}}_{(1,4,5,6)}$ and $\boldsymbol{\mathcal{Z}}_{(3,4,5,6)}$. This provides a nontrivial check of the formalism. The above results can thus be combined into a 6×6 matrix of renormalization constants for the six vector-current operators in (22). The corresponding anomalous dimension matrix $\boldsymbol{\gamma_V}$ has the block form [11]

$$\boldsymbol{\gamma_V} = \gamma_1 \, \mathbf{1} + \begin{pmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}, \qquad (26)$$

where

$$\mathbf{A} = \begin{pmatrix} 2\gamma_4 & 2\gamma_3 & 2\gamma_2 \\ 0 & 0 & 0 \\ \gamma_4 & \gamma_3 & \gamma_2 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma_4 & \gamma_3 & \gamma_2 \end{pmatrix}. \qquad (27)$$

The exact two-loop expressions for these matrices, which are obtained from (18), are an important result of our work.

5 Conclusion

We have calculated, at the two-loop order in the HQET, the renormalization and mixing of local dimension-4 operators containing a heavy and a light quark field. Our formalism allows for an arbitrary Dirac and Lorentz structure of these operators. The relevant two-loop diagrams have been evaluated with the help of tensor recurrence relations, once the IR divergent subgraphs have been regulated using IR counterterms. We find that a set of (up to) four operators closes under renormalization. The equations of motion imply nontrivial relations between the anomalous dimensions of these operators, which we have derived. We have also shown how general basis transformations, which may depend on the dimensional regulator ϵ, can be implemented in our approach.

The matrix elements of local dimension-4 heavy–light operators play an important role in many phenomenological applications of the heavy-quark expansion. In particular, they appear at order $1/m_Q$ in the heavy-quark expansion of heavy-meson decay constants, and of the semileptonic form factors describing, e.g., $\bar{B} \to \pi\,\ell\,\bar{\nu}$ and $\bar{B} \to \rho\,\ell\,\bar{\nu}$ transitions. Our results are an important step towards a full next-to-leading order analysis of these quantities. However, still missing is the two-loop mixing of some nonlocal operators with the local operators considered here. This will be discussed elsewhere [24].

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