Non-perturbative renormalization constants on the lattice from flavour non-singlet Ward identities

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Abstract

By imposing axial and vector Ward identities for flavour-non-singlet currents, we estimate in the quenched approximation the non-perturbative values of combinations of improvement coefficients, which appear in the expansion around the massless case of the renormalization constants of axial, pseudoscalar, vector, scalar non-singlet currents and of the renormalized mass. These coefficients are relevant for the completion of the improvement programme to $O(a)$ of such operators. The simulations are performed with a clover Wilson action non-perturbatively improved.

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1 Introduction

The programme of the improvement [1] of the Wilson action has been actively developed at the non-perturbative level over the last years [2, 3, 4, 5]. At first, in the framework of the Schrödinger functional it was possible to determine non perturbatively the dependence upon the bare coupling constant of the coefficient c_{SW} [3] of the clover term in the improved action [6].

Besides improving the action, the programme includes the improvement of the operators appearing in the correlation functions related to phenomenological interesting quantities such as pseudoscalar meson decay constants and the matrix element of the four-fermion operators of the weak effective Hamiltonian.

In general the operator improvement consists of two parts: the mixing with higher-dimensional operators with the same quantum numbers (in the literature the mixing coefficients are called c) and the multiplication by a suitable renormalization constant. The ultraviolet-finite renormalization constants can be expanded around the massless case:

$$Z_{\mathcal{O}}(m \neq 0) = Z_{\mathcal{O}} \cdot (1 + b_{\mathcal{O}} ma + ...), \tag{1}$$

where we have omitted corrections due to lattice artefacts of order a^2 and higher.

Some of these quantities have been calculated at the perturbative level for axial (A), vector (V), pseudoscalar (P) and scalar(S) currents as well as for the renormalized mass (m) [7, 8, 9]: non perturbative estimates are available for Z_A, Z_V, b_V [5] and for c_A [3] and c_V [10].

In this letter, we present a non-perturbative determination of the quantities:

$$b_A - b_P, \quad b_V - b_S, \quad b_m \tag{2}$$

and of the ratios:

$$Z_m Z_P / Z_A, \quad Z_m Z_S / Z_V \tag{3}$$

from a set of axial and vector Ward identities.

2 The method

The extraction of the b coefficients and of the Z ratios is based on the following Ward identities:

$$\begin{aligned} \partial_\mu \langle \mathbf{A}^I_\mu(x) \Omega^\dagger(0) \rangle &= 2m_{jk} \ \langle \mathbf{P}(x) \Omega^\dagger(0) \rangle + O(a^2) \\ \partial_\mu \langle \mathbf{V}^I_\mu(x) \Omega^\dagger(0) \rangle &= \Delta m_{jk} \ \langle \mathbf{S}(x) \Omega^\dagger(0) \rangle + O(a^2), \end{aligned} \tag{4}$$

which, after $\vec{x}$ integration, become:

$$\begin{aligned} \partial_t \langle A^I_0(t) \Omega^\dagger(0) \rangle &= 2m_{jk} \ \langle P(t) \Omega^\dagger(0) \rangle + O(a^2) \\ \partial_t \langle V^I_0(t) \Omega^\dagger(0) \rangle &= \Delta m_{jk} \ \langle S(t) \Omega^\dagger(0) \rangle + O(a^2). \end{aligned} \tag{5}$$

The suffix I for the axial current indicates that the current is improved by the appropriate mixing with the pseudoscalar density multiplied by the coefficient c_A. The value for c_A is taken from its non-perturbative determination in ref.[3]. For the vector current, the contribution of the mixing with the tensor current (see ref.[10] for non-perturbative determination of the mixing coefficient c_V) vanishes because of the antisymmetry of tensor indices.

The indices j, k refer to the flavour content of the bilinear operators, which can be written as

$$\mathcal{O}(t) = \sum_{\vec{x}} \mathbf{O}(x) = \sum_{\vec{x}} \bar{\psi}_j(x) \Gamma_{\mathcal{O}} \psi_k(x). \tag{6}$$

Equations (4) hold for any operator Ω at any time different from t, reflecting the fact that the W.I. are identities among operators. For the axial W.I. we use $\Omega = P(0)$ and $\Omega = A_0(0)$ while for the vector W.I. we use $\Omega = S(0)$ and $\Omega = V_0(0)$; in both cases the latter operator leads to much noisier results.

The quantities m_{jk} and Δm_{jk} are related respectively to the average and the difference of the renormalized mass, according to

$$\begin{aligned} m_{jk} &= m^R_{jk} \frac{Z_P}{Z_A} \frac{1 + b_P \overline{m_q a}}{1 + b_A \overline{m_q a}} \\ \Delta m_{jk} &= \Delta m^R_{jk} \frac{Z_S}{Z_V} \frac{1 + b_S \overline{m_q a}}{1 + b_V \overline{m_q a}} \end{aligned} \tag{7}$$

$$m^R_{jk} = \frac{1}{2}(m^R_j + m^R_k), \quad \Delta m^R_{jk} = m^R_j - m^R_k,$$

where $\overline{m_q a}$ is the average of the bare masses j and k:

$$\begin{aligned} \overline{m_q a} &= \frac{1}{2}(m_{q_j} a + m_{q_k} a) \\ a m_{q_j} &= \left(\frac{1}{2\kappa_j} - \frac{1}{2\kappa_c} \right) \end{aligned} \tag{8}$$

with κ_c the critical value of the Wilson hopping parameter. The parameter κ_c is determined from the chiral extrapolation of the mass defined through the axial Ward identities themselves.

The renormalization constants can be determined by replacing the "unrenormalized current masses" m_{jk} and Δm_{jk} with the renormalized ones through the above equation, and then the renormalized masses in terms of the bare masses:

$$m^R = Z_m m_q (1 + b_m m_q a). \tag{9}$$

Indeed, by including the lattice artefacts up to $O(a)$:

$$\begin{aligned} m_{jk} a &= \frac{Z_P Z_m}{Z_A} \left(\overline{m_q a} - (b_A - b_P)(\overline{m_q a})^2 + b_m \overline{(m_q a)^2} \right) \\ \Delta m_{jk} a &= \frac{Z_S Z_m}{Z_V} (m_{q_j} a - m_{q_k} a) \left(1 + (2b_m - (b_V - b_S)) \overline{m_q a}\right) \end{aligned} \tag{10}$$

From a fit of the bare mass dependence of these results we can determine non-perturbatively the combinations in eqs. (2,3).

Within the approximation used in the above formulae, we can derive an expression for $(b_A - b_P)$:

$$m_{jk}a - \frac{(m_{jj}a + m_{kk}a)}{2} = \frac{Z_P Z_m}{Z_A}\frac{1}{4}(b_A - b_P)(m_{q_i} - m_{q_j})^2 a^2 \tag{11}$$

which can be used to determine a value for the combination independently from the knowledge of the critical value of κ. We have compared such a determination of $b_A - b_P$ with the one coming from a global fit to expression eq.(10) and used it as a sensitive check of our estimate of κ_c.

The presence of flavour-non-diagonal currents is important for the fit of the axial W.I. and essential for the vector ones. We want to point out that $m_{jk}a$ depends either upon $(\overline{m_q a})^2$ or upon $\overline{(m_q a)^2}$, which are different variables if the flavours are not degenerate, making then possible to disentangle the coefficient b_m from the combination $b_A - b_P$.

Our fits are to the dependence upon the quark mass, and order a^2 corrections linear in the quark mass can in general fake the extraction of the coefficients. The simple lattice discretization of the time derivative $\frac{1}{2a}(f(t+a)-f(t-a))$ has an error of the order of $f^{(3)}a^2$, which becomes $f^{(1)}M^2a^2$ when a single state of mass M dominates the correlation function f. In the pseudoscalar channel, chiral symmetry makes this term proportional to the quark mass. In other channels, the mass M acquires a non-zero value when the quark masses vanish. The quantity $(Ma)^2$ still contains a term that is linear in the quark mass but not in the lattice spacing. In order to minimize such effects we have used the lattice discretization of the time derivative $a\partial_t$ correct up to term $f^{(5)}a^5$. While for the pseudoscalar case with spontaneous symmetry breaking the improvement to the fifth order of the derivative removes these fake linear terms in the quark mass, for the vector current a linear term survives at any finite order n, although suppressed by a coefficient of order $1/(n-1)!$. We have checked that improving the derivative to the next order does not change our results beyond their accuracy.

We cannot exclude the presence of other lattice artefacts in the ratio of matrix elements formally of order a^2 but linear in the quark mass. We have checked in the case of the axial current that the results are stable with respect to the choice of the operators and we interpret this as a sign for the absence of large extra terms of order $m\Lambda a^2$.

This method of computing the combinations in eqs. (2,3) is valid in the quenched approximation. The presence of dynamical quark loops would introduce an additional sea quark mass dependence, which would involve flavours different from those in the currents.

3 The numerical results

We have performed several simulations at different values of β, in order to derive the coupling constant dependence of the quantities in eqs. (2,3). The

$L^3\ T$	$16^3 48$	$16^3 32$	$16^3 32$	$16^3 32$
β	6.2	6.8	8.0	12.0
# confs	50	50	80	80
κ	0.124	0.124967	0.129382	0.126299
	0.1275	0.127517	0.130055	0.126941
	0.1295	0.128831	0.130736	0.127589
	0.132	0.131198	0.131078	0.127915
	0.13326	0.132589	0.131423	0.128243
	0.13362	0.132942	0.131700	0.128507
	0.134	0.133296	0.131908	0.128705
	0.1345	0.133796	0.132222	0.129004
	0.135	0.134371	0.132467	0.129237
	0.13535	0.134660	0.132749	0.129505
κ_c	0.13578(2)	0.13511(1)	0.13318(1)	0.129915(8)

Table 1: The values of κ used in our simulations at various β

values of β used in the simulations and the corresponding volumes are collected in Table 1, with a list of the values of κ and of our best estimate of the critical κ obtained from the W.I. themselves. The values of κ_c are well compatible within errors with those of ref. [3]. The variation of our results under a change of κ_c within the quoted error is smaller than the accuracy by which we can extract the non-perturbative quantities from our fit.

Simulations are performed with an updating sequence made by a standard heat-bath followed by 3 over-relaxation steps. The improved fermion propagator is calculated every 1000 gauge update using a stabilized biconjugate algorithm. For our runs we have used the 25 Gflops machine of the APE series, made of 512 nodes working in SIMD (Single Instruction Multiple Data) mode.

Each propagator was summed over the space volume distributed to the single node ($3 \times 3 \times 2$) and stored on disk. The use of these propagators leads to correlation functions that contain the correct local-gauge-invariant terms and other non-local, gauge-non-invariant terms that go to zero after summing over the gauge configurations. We have explicitly checked that with our statistics the residual noise due to imperfect cancellation of the gauge-non-invariant terms is much below the statistical fluctuations. Storing fermion propagators allows for an off-line calculation of all flavour-non-singlet correlations.

The W.I. are satisfied separately at each time: after some initial time, up to which higher-order lattice artefacts still dominate, m_{jk} and Δm_{jk} show a plateau. At $\beta = 6.2$ we run two temporal extensions (32 and 48) in order to monitor the stability of the plateau. We have used two methods of analysis: either we first average the result over the time interval of the plateau and

$L^3\ T$	$16^3 48$	$16^3 32$	$16^3 32$	$16^3 32$
β	6.2	6.8	8.0	12.0
# confs	50	50	80	80
$Z_m Z_P/Z_A$	1.09(1)	1.08(1)	1.08(1)	1.060(6)
$Z_m Z_S/Z_V$	1.24(2)	1.21(1)	1.142(4)	1.080(5)
$b_A - b_P$	0.15(2)	0.10(2)	0.06(2)	0.04(2)
b_m	−0.62(3)	−0.58(3)	−0.57(3)	−0.53(2)
$b_m - (b_V - b_S)/2$	−0.69(4)	−0.63(4)	−0.54(3)	−0.52(2)

Table 2: The results of our calculations

then perform a fit, or we first perform a fit at each time value inside the plateau and then average the results of the fit at different times. The two procedures give very similar results.

The choice of the spectator operator Ω affects the statistical error of the final results. We have found that the pseudoscalar and the scalar densities give the best results for the axial and the vector case respectively.

We perform a fit of $2m_{jk}$ with the function (here all masses are in lattice spacing units):

$$a_1(m_{q_i}+m_{q_j})+a_2(m_{q_i}^2+m_{q_j}^2)+a_3(m_{q_i}+m_{q_j})^2+a_4(m_{q_i}^3+m_{q_j}^3)+a_5 m_{q_i} m_{q_j}(m_{q_i}+m_{q_j})$$

The first three coefficients of the fit are related to the renormalization constants as follows: $a_1 = Z_m Z_P/Z_A$; $a_2/a_1 = b_m$; $a_3/a_1 = -(b_A - b_P)/2$. The last two coefficients in the fit can be introduced to parametrize order-a^2 corrections compatible with the flavour exchange symmetry of the Ward identity.

For the quantity Δm_{jk} we perform the fit with:

$$v_1(m_{q_i} - m_{q_j}) + v_2(m_{q_i}^2 - m_{q_j}^2) + v_3(m_{q_i}^3 - m_{q_j}^3) + v_4 m_{q_i} m_{q_j}(m_{q_i} - m_{q_j})$$

where $v_1 = Z_m Z_S/Z_V$; $v_2/v_1 = b_m - (b_V - b_P)/2$. As before, the extra coefficients v_3 and v_4 parametrize the possible order-a^2 corrections.

Our results normally refer to the fit with three parameters for the axial case and two for the vector case. Increasing the number of parameters in general does not improve the value of χ^2 much while it considerably increases the error. The results are compatible with the lower parameter fit, with the exception of the determination of b_V, which comes systematically higher with the four-parameter fit. We have included this effect in the corresponding error.

Table 2 contains the main results, the values for the various renormalization parameters at different β and volumes. The $\beta = 6.2$ results on the smaller temporal extension are fully compatible.
With the values of the fit we can check if the renormalized W.I. depend only upon the sum (for the axial) or the difference (for the vector) of the renormalized masses.

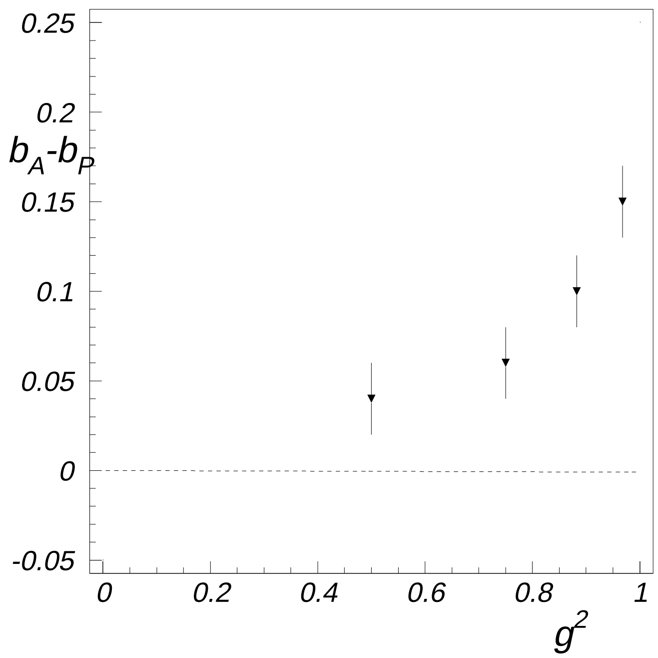

Figure 1: The non-perturbative result for $b_A - b_P$. The perturbative result of $O(g^2)$ is negligible on this scale.

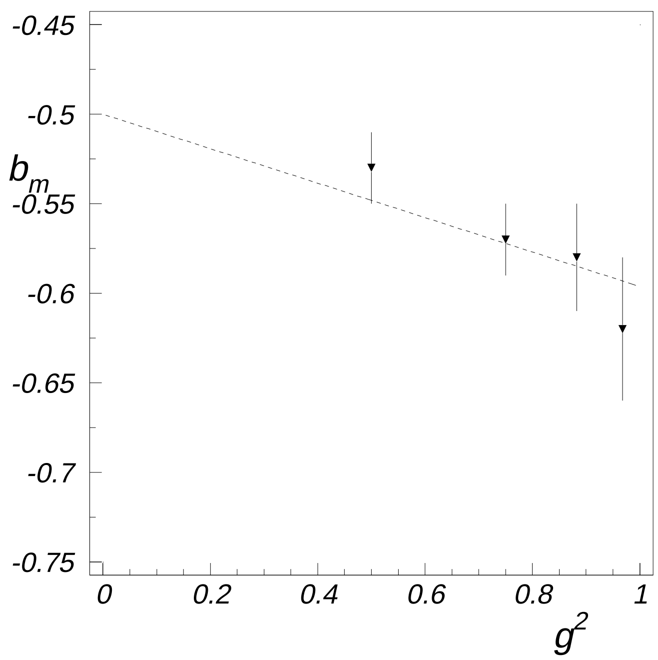

Figure 2: The non-perturbative estimate of b_m is compared with the perturbative result.

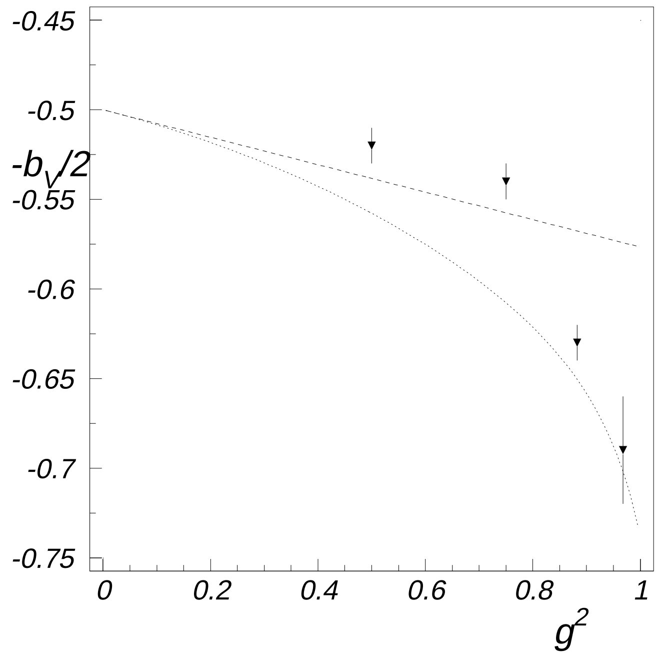

Figure 3: The non-perturbative result for b_V is compared, after using Lüscher's relation, with the one of ref. [6] (dotted curve) and with the perturbative result (dashed curve).

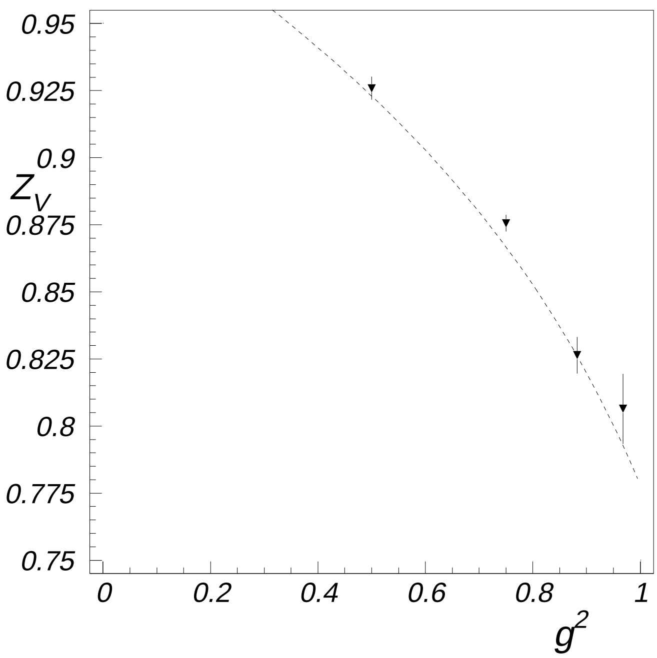

Figure 4: The non-perturbative result for Z_V is compared, after using Lüscher's relation, with the one of ref. [6].

The renormalized masses and currents manage to bring on the same straight line points that appear misaligned and on a curved line for the bare quantities. For large values of the masses, higher-order terms enter the game and produce again a misalignment of the corresponding points.

The fits in general do not include all κ values; we exclude the heavier masses until the stability of the results is reached.

Our results, when compared with those available from perturbation theory, show that higher-order terms seem to dominate for the differences $b_A - b_P$, which is very small at one-loop order (see fig. 1), while for b_m the presence of sizeable terms of order g^2 makes the effect of g^4 terms less prominent. Indeed, our results are not far from lowest-order perturbation theory for this case (see fig. 2).
For $b_V - b_S + 2b_m$, there is an argument due to Martin Lüscher [11] relating b_m with b_S and Z_S with Z_m in the quenched approximation: $2b_m - b_S = 0$ and $Z_S Z_m = 1$, which implies that from the W.I. for the vector current we actually obtain b_V and Z_V. The comparison with those of ref. [5] is shown in figs. 3 and 4: while for Z_V there is a perfect agreement, for b_V we are generally closer to the perturbative result. Our large errors are mainly systematic and reflect the instability of a four-parameter fit. Running at lower quark masses could reduce the discrepancy which might also be due to residual order-a^2 lattice artefacts.

The use of axial and vector Ward identities with flavour-non-singlet currents allows the determination in the quenched approximation of various non-perturbative renormalization constants. The calculation that we have presented could be refined by using the Schrödinger functional method which would allow a safe investigation of the very low quark mass region.

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