Symmetry analysis of the 1+1 dimensional relativistic imperfect fluid dynamics

C Alexa† and D Vrinceanu‡

† Department of Particle Physics, IFIN-HH, Bucharest 76900, RO

‡ Dept. of Theoretical Phys. & Mathematics, University of Bucharest, RO

Abstract. The flow of the relativistic imperfect fluid in two dimensions is discussed. We calculate the symmetry group of the energy-momentum tensor conservation equation in the ultrarelativistic limit. Group-invariant solutions for the incompressible fluid are obtained.

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1. Introduction

Many physical systems may be approximately regarded as perfect fluids. A perfect fluid is defined [1] as having at each point a velocity $\vec{v}$, such that an observer moving with this velocity sees the fluid around him as isotropic. This will be the case if the mean free path between collisions is small compared with the scale of lengths used by the observer. But one often has to deal with somewhat imperfect fluids, in which the pressure or velocity vary appreciable over distances of the order of a mean free path, or over times of the order of a mean free time, or both.

Numerical methods to solve the hydrodynamical equations have been discussed for instance in [2]. Any attempt to solve numerically the relativistic hydrodynamics equations is hardly discouraged by the puzzle of choosing the rest frame; this situation leads us to acausal and instable solutions [3].

It is therefore of interest to use different methods that are directly related to the solutions of the equations of the relativistic fluid dynamics. Important information may be achieved using the Lie symmetry group of the covariant relativistic hydrodynamics equations. Symmetry analysis is one of the systematic and accurate ways to obtain solutions of differential equations. The power of this technique consists in the possibility to explore the properties of physical systems like the symmetry structure and the invariants and then solving the corresponding reduced differential equations. Interesting systems were successfully studied using this approach for example [4, 5]. In [6] we have already investigate a particular simple form of the energy-momentum tensor conservation equation.

In the next section we briefly discuss the general relativistic fluid formalism, the ultrarelativistic approximation and the final form of the energy-momentum tensor conservation equations. In Sec. 3 we address the symmetry group of transformations and its Lie algebra. Sec. 4 is devoted to integrability conditions, invariants and Sec. 5 to group invariant-solutions analysis. In the last section we present miscellaneous comments and final conclusions.

2. Energy-momentum tensor

Relativistic fluid dynamics is well describe by the number of particles N and energy-momentum tensor $T_{\alpha\beta}$ conservation equations [1]. In the ideal case we have :

$$\begin{aligned} T_{\alpha\beta} &= p\eta_{\alpha\beta} + (p+\varepsilon)U_\alpha U_\beta \\ N_\alpha &= nU_\alpha \end{aligned} \tag{1}$$

where $\eta_{\alpha\beta}$ is the metric tensor, p is the pressure, ε is the energy density, n is the number of particles density and $U_\alpha : (\gamma\vec{\beta}, \gamma)$ is the 4-velocity field.

There are two ways of choosing the rest frame : in the Landau way, U_α is the energy transport velocity where $T_{i0} = 0$ in the rest frame, while in the Eckart way U_α is the particle transport velocity where $N_i = 0$ in the rest frame. The dissipation contribution is introduced by redefining the energy-momentum and number of particle tensor by adding correction terms :

$$\begin{aligned} T_{\alpha\beta} &= p\eta_{\alpha\beta} + (p+\varepsilon)U_\alpha U_\beta + \Delta T_{\alpha\beta} \\ N_\alpha &= nU_\alpha + \Delta N_\alpha \end{aligned} \tag{2}$$

In the Eckart frame $\Delta N_\alpha = 0$, so the dissipation contribution is present only in the energy-momentum terms. In the following we choose the Eckart approach. The construction of the most general dissipation term $\Delta T_{\alpha\beta}$ is based on the positivity of the entropy production [1]:

$$\Delta T^{\alpha\beta} = -\eta H^{\alpha\gamma} H^{\beta\delta} W_{\gamma\delta} - \chi \left(H^{\alpha\gamma} U^\beta + H^{\beta\gamma} U^\alpha \right) Q_\gamma - \zeta H^{\alpha\beta} \partial_\gamma U^\gamma \tag{3}$$

where we have shear tensor:

$$W_{\alpha\beta} = \partial_\beta U_\alpha + \partial_\alpha U_\beta - \frac{2}{3} \eta_{\alpha\beta} \, \partial_\gamma U^\gamma \tag{4}$$

heat-flow vector:

$$Q_\alpha = \partial_\alpha T + T \; U^\beta \, \partial_\beta \, U_\alpha \tag{5}$$

T is the temperature and projection tensor on the hyperplane normal to U_α

$$H_{\alpha\beta} = \eta_{\alpha\beta} + U_\alpha \, U_\beta \tag{6}$$

We identify χ, η, ζ as the coefficients of heat conduction, shear viscosity and bulk viscosity.

The conservation of the energy-momentum tensor $T_{\alpha\beta}$ gives us the main system of equations that controls the fluid dynamics:

$$\partial^\alpha T_{\alpha\beta} = 0 \tag{7}$$

A major obstacle in the application of symmetry analysis is the large number of tedious calculations usually involved. This is the reason to simplify the form of the equations in a reasonable way. Therefore, we are looking in the energy-momentum conservation equation for the powers of 4-velocity field U because in the ultrarelativistic limit $U_\mu^3 \gg U_\mu^2 \gg U_\mu$; $\mu = 1, 2, 3$ or 4 (no summation), more precisely $U_\mu \sim \gamma \Rightarrow U_\mu^2 \sim \gamma^2 \Rightarrow U_\mu^2 \gg U_\mu \equiv \gamma^2 \gg \gamma$ and $\beta \simeq 1$. Taking only the higher power term and terms without U_μ, we have [7]:

$$\partial^\alpha T_{\alpha\beta} = \partial_\beta \left[p + \left(\frac{2}{3}\eta - \zeta \right) \partial^\alpha U_\alpha \right] - 2\chi \cdot \partial^\alpha \left(U_\alpha U_\beta U^\gamma \partial_\gamma T \right) \tag{8}$$

Performing calculations and in the end, taking only the highest contribution from the velocity field, we obtained from the energy-momentum conservation equation the following set of equations in 1+1 dimensions:

$$\begin{aligned} p_x + \left(\tfrac{2}{3}\eta - \zeta \right) (u_{xx} - v_{xt}) - 2\chi \left(u^3 T_{xx} + u v^2 T_{tt} - 2u^2 v T_{xt} \right) = 0 \\ p_t + \left(\tfrac{2}{3}\eta - \zeta \right) (u_{xt} - v_{tt}) - 2\chi \left(u^2 v T_{xx} + v^3 T_{tt} - 2v^2 u T_{xt} \right) = 0 \end{aligned} \tag{9}$$

$U_\alpha : (\gamma\vec{\beta}, \gamma) = (u, v)$, $U^2 = U_\alpha U^\alpha = -1$, $u_{xx} = \partial^2 u / \partial x^2$, etc.. It is important to mention that from $\partial^\alpha \left(U_\alpha U_\beta U^\gamma \partial_\gamma T \right)$ we took only the terms with U_μ^3, i.e. $\partial^\alpha \left(U_\alpha U_\beta U^\gamma \partial_\gamma T \right) \to U_\alpha U_\beta U^\gamma \partial^\alpha \partial_\gamma T$ and in this expression we neglected again terms containing velocity field, u or v, at a power smaller than three. We also consider that the shear viscosity, the bulk viscosity and the heat conduction are constants; this is a major simplification for the symmetry group calculations. In fact they can be functions of temperature, for example in Weinberg's book [1], where for a particular kind of fluid we have

$$\chi = \frac{4}{3} k_B T^3 \tau \;; \eta = \frac{4}{15} k_B T^4 \tau \;; \zeta = 4 k_B T^4 \tau \left[\frac{1}{3} - \left(\frac{\partial p}{\partial \varepsilon} \right) \right]^2 \tag{10}$$

where τ is the mean free time and k_B is the Boltzmann constant.

3. Symmetry group of transformations and its Lie algebra

At the end of the last century, Lie considered the invariance of the differential equations under the transformation of dependent and independent variables [8]. Lie was able to classify and solve some types of ordinary as well as partial differential equations. In recent years, the symmetry methods have become more attractive, especially in the field of nonlinear dynamics [4, 9].

The symmetry group of a system of differential equations is the largest local group of transformation acting on the independent and dependent variables of the system with the property that it transform solutions of the system to other solutions. We restrict our attention to local Lie group of symmetries, leaving aside problems involving discrete symmetries such as reflections.

Let $\mathcal{S}$ be a system of differential equations. A symmetry-group of the system $\mathcal{S}$ is a local group of transformations $\mathcal{G}$ acting on an open subset $\mathcal{M}$ of the space of independent and dependent variables for the system with the property that whenever u=f(x) is a solution of $\mathcal{S}$, and whenever $g \cdot f$ is defined for $g \in \mathcal{G}$, then $u = g \cdot f(x)$ is also a solution of the system.

The symmetry group infinitesimal generator is defined by :

$$\vec{\mathcal{V}} = \xi\partial_x + \tau\partial_t + \Phi\partial_u + \Psi\partial_v + \Gamma\partial_T + \Omega\partial_p \tag{11}$$

and the first order prolongation of $\vec{\mathcal{V}}$ is:

$$\begin{aligned} pr^{(1)}\vec{\mathcal{V}} = \; & \xi\partial_x + \tau\partial_t + \Phi\partial_u + \Psi\partial_v + \Gamma\partial_T + \Omega\partial_p \\ & +\Phi^x\partial_{u_x} + \Phi^t\partial_{u_t} + \Psi^x\partial_{v_x} + \Psi^t\partial_{v_t} \\ & +\Gamma^x\partial_{T_x} + \Gamma^t\partial_{T_t} + \Omega^x\partial_{p_x} + \Omega^t\partial_{p_t} \end{aligned} \tag{12}$$

where, for example,

$$\Phi^x = D_x(\Phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{xt} \tag{13}$$

and $D_x\Phi = \Phi_x + \Phi_u u_x + \Phi_v v_x + \Phi_T T_x + \Phi_p p_x$ is the total derivative. The second order prolongation of $\vec{\mathcal{V}}$ is defined by the following relation:

$$\begin{aligned} pr^{(2)}\vec{\mathcal{V}} = \; & \xi\partial_x + \tau\partial_t + \Phi\partial_u + \Psi\partial_v + \Gamma\partial_T + \Omega\partial_p \\ & +\Phi^x\partial_{u_x} + \Phi^t\partial_{u_t} + \Psi^x\partial_{v_x} + \Psi^t\partial_{v_t} + \Gamma^x\partial_{T_x} + \Gamma^t\partial_{T_t} + \Omega^x\partial_{p_x} + \Omega^t\partial_{p_t} \\ & +\Phi^{xx}\partial_{u_{xx}} + \Phi^{xt}\partial_{u_{xt}} + \Phi^{tt}\partial_{u_{tt}} + \Psi^{xx}\partial_{v_{xx}} + \Psi^{xt}\partial_{v_{xt}} + \Psi^{tt}\partial_{v_{tt}} \\ & +\Gamma^{xx}\partial_{T_{xx}} + \Gamma^{xt}\partial_{T_{xt}} + \Gamma^{tt}\partial_{T_{tt}} + \Omega^{xx}\partial_{p_{xx}} + \Omega^{xt}\partial_{p_{xt}} + \Omega^{tt}\partial_{p_{tt}} \end{aligned} \tag{14}$$

where, for example,

$$\Phi^{xx} = D_x^2(\Phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt} \tag{15}$$

Suppose $\Delta_\nu(x, u^{(n)}) = 0, \nu = 1, ..., l$, is a system of differential equations of maximal rank (meaning that the Jacobian matrix $J_\triangle(x, u^{(n)}) = \left(\frac{\partial\triangle_\nu}{\partial x^i}, \frac{\partial\triangle_\nu}{\partial u_J^\alpha}\right)$ of with respect to all the variables $(x, u^{(n)})$ is of rank l whenever $\Delta(x, u^{(n)}) = 0$) defined over $\mathcal{M}$⊂X×U, where $u^{(n)} = (u, v, T, p, u_x, u_t, ..., p_{tt})$. If $\mathcal{G}$ is a local group of transformations acting on $\mathcal{M}$ and $pr^{(n)}\vec{\mathcal{V}}\left[\Delta_\nu(x, u^{(n)})\right] = 0,\ \nu = 1, ..., l,$ whenever $\Delta(x, u^{(n)}) = 0$, for every infinitesimal generator $\sqsubseteq$ of $\mathcal{G}$, then $\mathcal{G}$ is a symmetry group of the system.

The standard procedure † is based on finding the infinitesimal coefficient functions $\xi, \tau, \Phi, \Psi, \Gamma$ and Ω. Applying $pr^{(2)}\vec{\mathcal{V}}$ on the system (9) , we obtained the infinitesimal criterion

$$\begin{array}{l} \Omega^x + \left(\frac{2}{3}\eta - \zeta\right)(\Phi^{xx} - \Psi^{xt}) - 2\chi \cdot (3u^2\Phi\Gamma^{xx} + 2v\Phi\Psi\Gamma^{tt} - 4u\Phi\Psi\Gamma^{xt}) = 0 \\ \Omega^x + \left(\frac{2}{3}\eta - \zeta\right)(\Phi^{xx} - \Psi^{xt}) - 2\chi \cdot (2u\Phi\Psi\Gamma^{xx} + 3v^2\Psi\Gamma^{tt} - 4v\Phi\Psi\Gamma^{xt}) = 0 \end{array} \tag{16}$$

Substituting the general formulae for $\Phi^x, \Psi^x, etc.$ and equating the coefficients of various monomials in the first and second order partial derivatives of u, v, T and p, we find the defining equations. We wish to determine all possible coefficient functions $\xi, \tau, \Phi, \Psi, \Gamma$ and Ω by solving the defining equations system so that the corresponding one-parameter group $\exp(\varepsilon\vec{\mathcal{V}})$ is a symmetry group of the equations (9).

We will consider two cases: a) incompressible and b) compressible fluid. The basis of the corresponding Lie algebra is:

$$\begin{array}{ll} \textbf{Incompressible} & \textbf{Compressible} \\ V_1 = \partial_x & V_1 = \partial_x \\ V_2 = \partial_t & V_2 = \partial_t \\ V_3 = \partial_T & V_3 = \partial_T \\ V_4 = x\partial_T & V_4 = x\partial_T \\ V_5 = t\partial_T & V_5 = t\partial_T \\ V_6 = t\partial_x + x\partial_t - u\partial_u - v\partial_v & V_6 = t\partial_x + x\partial_t - u\partial_u - v\partial_v \\ V_7 = x\partial_x + t\partial_t & V_7 = x\partial_x + t\partial_t - p\partial_p \\ V_8 = u\partial_u + v\partial_v - 2T\partial_T & V_8 = u\partial_u + v\partial_v - 2T\partial_T + p\partial_p \\ & V_9 = \partial_p \end{array} \tag{17}$$

Using the following substitutions $x = \tau\cosh(\alpha)$ and $t = \tau\sinh(\alpha)$ we obtain that $t\partial_x + x\partial_t = \partial_\alpha$ - angle translation (rotation of the (x,t)-plane); if $u = \sinh(w)$ and $v = \cosh(w) \rightarrow w = \frac{1}{2}\log\frac{v+u}{v-u}$ (w is the rapidity) we have $u\partial_v + v\partial_u = \partial_w$ which is a rapidity translation; it is important to mention that V_6 is a Lorentz transformation.

4. Solvable group and invariants

Because we have the Lie algebra of the system (9) we want to know if the general solution of the system of differential equations can be found by quadratures. This thing is possible if the Lie group is solvable. The group $\mathcal{G}$ is solvable if there exists a chain of Lie subgroups

$$e = G^{(0)} \subset G^{(1)} \subset \ldots \subset G^{(r-1)} \subset G^{(r)} = G \tag{18}$$

such that for each k=1,. . .,r, $\mathcal{G}^{(k)}$ is a k-dimensional subgroup of $\mathcal{G}$ and $\mathcal{G}^{(k-1)}$ is a normal subgroup of $\mathcal{G}^{(k)}$. A subgroup H is normal subgroup if $ghg^{-1} \in H$ whenever $g \in \mathcal{G}$ and $h \in H$. Equivalently, there is a chain of subalgebras

$$e = g^{(0)} \subset g^{(1)} \subset \ldots \subset g^{(r-1)} \subset g^{(r)} = g \tag{19}$$

such that for k, $dim g^{(k)} = k$ and $g^{(k-1)}$ is a normal subalgebra of $g^{(k)}$:

$$[g^{(k-1)}, g^{(k)}] \subset g^{(k-1)} \tag{20}$$

† The method is well known and a good description can be found in [4]

The requirement for solvability is equivalent to the existence of a basis $\{V_1, \ldots, V_r\}$ of Lie algebra g such that

$$[V_i, V_j] = \sum_{k=1}^{j-1} c_{ij}^k V_k \tag{21}$$

whenever $i < j$.

Table 1. Commutator table for the incompressible fluid algebra.

[,]	V(1)	V(2)	V(3)	V(4)	V(5)	V(6)	V(7)	V(8)
V(1)	0	0	0	V(3)	0	V(2)	V(1)	0
V(2)	0	0	0	0	V(3)	V(1)	V(2)	0
V(3)	0	0	0	0	0	0	0	-2V(3)
V(4)	-V(3)	0	0	0	0	-V(5)	-V(4)	-2V(5)
V(5)	0	-V(3)	0	0	0	-V(4)	-V(5)	-2V(4)
V(6)	-V(2)	-V(1)	0	V(5)	V(4)	0	0	0
V(7)	-V(1)	-V(2)	0	V(4)	V(5)	0	0	0
V(8)	0	0	2V(3)	2V(4)	2V(5)	0	0	0

Table 2. Commutator table for the compressible fluid algebra.

[,]	V(1)	V(2)	V(3)	V(4)	V(5)	V(6)	V(7)	V(8)	V(9)
V(1)	0	0	0	V(3)	0	V(2)	V(1)	0	0
V(2)	0	0	0	0	V(3)	V(1)	V(2)	0	0
V(3)	0	0	0	0	0	0	0	-2V(3)	0
V(4)	-V(3)	0	0	0	0	-V(5)	-V(4)	-2V(5)	0
V(5)	0	-V(3)	0	0	0	-V(4)	-V(5)	-2V(4)	0
V(6)	-V(2)	-V(1)	0	V(5)	V(4)	0	0	0	0
V(7)	-V(1)	-V(2)	0	V(4)	V(5)	0	0	0	V(9)
V(8)	0	0	2V(3)	2V(4)	2V(5)	0	0	0	-V(9)
V(9)	0	0	0	0	0	0	-V(9)	V(9)	0

Looking at the commutator table of the Lie algebra we will see that the requirement of solvability is satisfy in both incompressible and compressible cases because we can construct the following chain of invariant sub-groups

$$\begin{aligned}\{e\} = G^{[0]} \subset G^{[1]} \subset G^{[1,2]} \subset G^{[1,2,3]} \subset G^{[1,2,3,4]} \\ \subset G^{[1,\ldots,5]} \subset G^{[1,\ldots,6]} \subset G^{[1,\ldots,7]} \subset G^{[1,\ldots,8]} = G\end{aligned} \tag{22}$$

where $G^{[i,\ldots,j]}$ is the subgroup generated by $V(i), ..., V(j)$ for the incompressible fluid and

$$\begin{aligned}\{e\} = G^{[0]} \subset G^{[1]} \subset G^{[1,2]} \subset G^{[1,2,3]} \subset G^{[1,\ldots,4]} \subset G^{[1,\ldots,5]} \\ \subset G^{[1,\ldots,6]} \subset G^{[1,\ldots,7]} \subset G^{[1,\ldots,8]} \subset G^{[1,\ldots,9]} = G\end{aligned} \tag{23}$$

for the compressible fluid.

We use the method of characteristics to compute the invariants of the Lie algebra hopping that the reduced system, which can be obtained using the invariants of the group, will help us to solve the system of equations (9). An n-th order differential invariant of a group G is a smooth function depending on the independent and dependent variables and their derivatives, invariant on the action of the corresponding n-th prolongation of G [4].

Suppose that we have the following generator:

$$V_i = \xi_i \partial_x + \tau_i \partial_t + \Phi_i \partial_u + \Psi_i \partial_v + \Gamma_i \partial_T + \Omega_i \partial_p \tag{24}$$

A local invariant ζ of V_i is a solution of the linear, homogeneous first order partial differential equation:

$$V_i(\zeta) = \xi_i \partial_x \zeta + \tau_i \partial_t \zeta + \Phi_i \partial_u \zeta + \Psi_i \partial_v \zeta + \Gamma_i \partial_T \zeta + \Omega_i \partial_p \zeta = 0 \tag{25}$$

The classical theory of such equations shows that the general solution of equation (25) can be found by integrating the corresponding characteristic system of differential equations, which is

$$\frac{dx}{\xi_i} = \frac{dt}{\tau_i} = \frac{du}{\Phi_i} = \frac{dv}{\Psi_i} = \frac{dT}{\Gamma_i} = \frac{dp}{\Omega_i} \tag{26}$$

Doing this integration we get, in this case, five invariants; we now re-express the next generator of Lie algebra in terms of these five invariants and then we perform another integration. We continue this calculation until we re-express and integrate the last generator; at this point we obtain a set of invariants that represent the system of independent invariants of this group. The system of invariants can be used to reduce the order of the original equations - constructing the reduced order system of equations. Doing this one can hope to find simple equations that can be integrated (for example [4]).

Unfortunately our system of independent invariants is not so friendly and we can't simplify the form of the equations. We do not present here the invariants because of their unpleasant form and specially because they are useless in this particular application; the only important thing is that one of these invariants is $u^2 - v^2$, which means that the unitarity of the velocity field is preserved.

This method will be very well applied on the next section where the invariants are much more simple and we will use them to find the group invariant-solutions. In the next section we will focus on the incompressible fluid because the absence of the pressure term in our equations will allow us to obtain analytical solutions by integrating the equations; this is due to the number of dependent variable which decrease from four (u, v, T and p) to three (u, v and T).

5. Group invariant-solutions

A solution of the system of partial differential equations is said to be $\mathcal{G}$-invariant if it is unchanged by all the group transformations in $\mathcal{G}$. In general, to each s-parameter subgroup $\mathcal{H}$ of the full symmetry group $\mathcal{G}$ of a system of differential equations, there will correspond a family of group-invariant solutions. Since there are almost always an infinite number of such subgroups, it is not usually feasible to list all possible group-invariant solutions to the system. We need an effective systematic means of classifying these solutions, leading to an optimal system of group-invariant solutions from which every other solution can be derived. Since elements $g \in \mathcal{G}$ not in the subgroup $\mathcal{H}$ will transform an $\mathcal{H}$-invariant solution to some other group-invariant solution, only those solutions not so related need to be listed in our optimal system.

An optimal system of s-parameter subgroups is a list of conjugancy inequivalent s-parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list (conjugacy map: h$\rightarrow$ghg^{-1}).

Let $\mathcal{G}$ be a Lie group with Lie algebra g. For each $v \in g$, the adjoint vector $ad\ v$ at $w \in g$ is

$$ad\ v|_w = [w, v] = -[v, w] \tag{27}$$

Now we can reconstruct the adjoint representation $Ad\ \mathcal{G}$ of the Lie group by summing the Lie series

$$Ad(\exp(\varepsilon v))w = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(ad\ v)^n(w) = w - \varepsilon[v,w] + \frac{\varepsilon^2}{2}[v,[v,w]] - ... \tag{28}$$

obtaining the adjoint table.

Table 3. Adjoint table

Ad	**V(1)**	**V(2)**	**V(3)**
V(1)	V(1)	V(2)	V(3)
V(2)	V(1)	V(2)	V(3)
V(3)	V(1)	V(2)	V(3)
V(4)	V(1)+εV(3)	V(2)	V(3)
V(5)	V(1)	V(2)+εV(3)	V(3)
V(6)	e^{ε}V(1)	e^{ε}V(2)	V(3)
V(7)	V(1)	V(2)	$e^{-2\varepsilon}$V(3)
V(8)	cosh(ε) V(1) + sinh(ε) V(2)	sinh(ε) V(1) + cosh(ε) V(2)	V(3)

Ad	**V(4)**	**V(5)**	**V(6)**	**V(7)**	**V(8)**
V(1)	V(4)-εV(3)	V(5)	V(6)-εV(2)	V(7)-εV(1)	V(8)
V(2)	V(4)	V(5)-εV(3)	V(6)-εV(1)	V(7)-εV(2)	V(8)
V(3)	V(4)	V(5)	V(6)	V(7)	V(8)+2εV(3)
V(4)	V(4)	V(5)	V(6)+εV(5)	V(7)+εV(4)	V(8)+2εV(4)
V(5)	V(4)	V(5)	V(6)+εV(4)	V(7)+εV(5)	V(8)+2εV(5)
V(6)	cosh(ε)V(4)-sinh(ε)V(5)	cosh(ε)V(5)-sinh(ε)V(4)	V(6)	V(7)	V(8)
V(7)	e^{ε}V(4)	e^{ε}V(5)	V(6)	V(7)	V(8)
V(8)	$e^{-2\varepsilon}$V(4)	$e^{-2\varepsilon}$V(5)	V(6)	V(7)	V(8)

The optimal system of our equations (9) is provided by those generated by

$$\begin{array}{rl}
1) & V(6)+aV(7)+bV(8) = (ax+t)\partial_x + (at+x)\partial_t + (bu-v)\partial_u + (bv-u)\partial_v - 2T\partial_T \\
2) & aV(7)+V(8) = ax\partial_x + at\partial_t + u\partial_u + v\partial_v - 2T\partial_T \\
3) & aV(7)+V(6) = (ax+t)\partial_x + (at+x)\partial_t - v\partial_u - u\partial_v \\
4) & aV(7)+V(6)\pm V(3) = (ax+t)\partial_x + (at+x)\partial_t - v\partial_u - u\partial_v \pm \partial_T \\
5) & aV(8)+V(6) = t\partial_x + x\partial_t + (au-v)\partial_u + (av-u)\partial_v - 2aT\partial_T \\
6) & V(7)\pm V(3) = x\partial_x + t\partial_t \pm \partial_T \\
7) & V(8)\pm V(1) = \pm\partial_x + u\partial_u + v\partial_v - 2T\partial_T\ ; V(8)\pm V(2) = \pm\partial_t + u\partial_u + v\partial_v - 2T\partial_T \\
8) & V(6)\pm V(3) = t\partial_x + x\partial_t - v\partial_u - u\partial_v \pm \partial_T \\
9) & V(1)+V(4)+aV(5) = \partial_x + (x+at)\partial_T\ ; V(1)+V(5)+aV(4) = \partial_x + (t+ax)\partial_T \\
10) & V(2)+V(4)+aV(5) = \partial_t + (x+at)\partial_T\ ; V(2)+V(5)+aV(4) = \partial_t + (t+ax)\partial_T \\
11) & V(4)+V(1)+aV(2) = \partial_x + a\partial_t + x\partial_T\ ; V(4)+V(2)+aV(1) = a\partial_x + \partial_t + x\partial_T \\
12) & V(5)+V(1)+aV(2) = \partial_x + a\partial_t + t\partial_T\ ; V(5)+V(2)+aV(1) = a\partial_x + \partial_t + t\partial_T \\
13) & V(1)+V(4) = \partial_x + x\partial_T\ ; V(1)+V(5) = \partial_x + t\partial_T \\
14) & V(2)+V(4) = \partial_t + x\partial_T\ ; V(2)+V(5) = \partial_t + t\partial_T \\
15) & V(i), i = 1, ..., 8
\end{array} \tag{29}$$

where a and b are arbitrary constants.

In this case, of incompressible fluid, the equations (9) are

$$\begin{array}{c} u_{xx}-v_{xt}-k\cdot\left(u^3T_{xx}+uv^2T_{tt}-2u^2vT_{xt}\right)=0 \\ u_{xt}-v_{tt}-k\cdot\left(u^2vT_{xx}+v^3T_{tt}-2uv^2T_{xt}\right)=0 \end{array} \tag{30}$$

where $k=2\chi/\left(\frac{2}{3}\eta-\zeta\right)$; we can re-write them in the following form:

$$\begin{array}{c} v\left(u_{xx}-v_{xt}\right)-u(u_{xt}-v_{tt})=0 \\ u_{xx}-v_{xt}-k\cdot\left(u^3T_{xx}+uv^2T_{tt}-2u^2vT_{xt}\right)=0 \end{array} \tag{31}$$

Finally, we will concentrate our attention on the classification of the group-invariant solutions, but we will focus on the equations that can be solved analytically.

1) In terms of the invariants $w=\rho+\frac{1}{a}\log(ax+t), \beta=T(at+x)^{-2b/a}$ and $y=(t+x)^{a+1}(t-x)^{1-a}$, where $\rho=0.5\log\frac{v+u}{v-u}$ the reduced system of equations are very complicated but if we choose $a=\pm1$ they are

$$\begin{array}{c} w_{yy}\pm w_y^2+1.5y^{-1}w_y=0 \\ 2y^2\beta_{yy}\pm\left(4b\pm1\right)y\beta_y+b\left(2b\mp1\right)\beta=0 \end{array} \tag{32}$$

and the solutions are [10]

$$T=\begin{cases} c_1+c_2\left(t\pm x\right) \\ c_1+2c_2\log\left(t\pm x\right) \\ \left(t\pm x\right)\left[c_1+2c_2\log\left(t\pm x\right)\right] \end{cases} \tag{33}$$

$$\mid\vec{v}\mid=\mp\tanh\left\{\log\left[\pm2c_3\mp c_4\left(t\pm x\right)\right]\right\} \tag{34}$$

where $c_1,...,c_4$ are constants.

2) using $\rho=0.5\log\frac{v+u}{v-u}$ we obtained in terms of the invariant $y=x/t$ the following reduced equation

$$\rho_{yy}+\rho_y^2\frac{y+\tanh(\rho)}{1+y\tanh(\rho)}+\rho_y\frac{2\tanh(\rho)}{1+y\tanh(\rho)}=0 \tag{35}$$

which is an equation that has to solved using numerical codes. The second reduced equation is much more complicated but it can be solved once we have the solution from the first equation.

3) the invariants are $w=\rho+\frac{1}{a}\log(ax+t)$ and $y=(t+x)^{a+1}(t-x)^{1-a}$ and for the same reason mentioned above we considered a=±1 obtaining

$$\begin{array}{c} w_{yy}\pm w_y^2-0.5y^{-1}w_y\mp0.5y^{-2}=0 \\ 2yT_{yy}+T_y=0 \end{array} \tag{36}$$

The temperature solution is $T=c_1+2c_2\sqrt{y}$; for the second equation we need numerical codes.

4) the invariants are $w=\rho+\frac{1}{a}\log(ax+t)$, $y=(t+x)^{a+1}(t-x)^{1-a}$ and $\beta=\pm T-\frac{1}{a}\log(at+x)$; for a=±1 we have

$$\begin{array}{c} w_{yy}\mp w_y^2+1.5y^{-1}w_y=0 \\ \beta_{yy}+0.5y^{-1}\beta_y\mp0.25y^{-2}=0 \end{array} \tag{37}$$

and the solutions are

$$\begin{array}{c} T=-\frac{5/4}{(t\pm x)^4}+\frac{0.5}{(t\pm x)^2}\pm2c_1\left(t\pm x\right)\pm c_2\pm\log\left(t\pm x\right) \\ \mid\vec{v}\mid=\mp\tanh\{\log[\pm2c_1\mp c_2(t\pm x)]\} \end{array} \tag{38}$$

5) the reduced system of equations is very unpleasant and we need numerical codes

6) here we have the same problem as in the case number 2) (the reduced system of equations is the same as in the second case)

7), 8), 9) and 10) because of the form of the invariants we can not construct the reduced system of equations

11) the invariants and the solutions are $\beta = T - \frac{xt}{a}$ and $y = ax - t$ and respectively, for a=±1:

$$\begin{array}{c} T = \pm xt + 0.25\left(\mp x - t\right)^2 - \frac{1/12}{c_1^2\left[c_1(\mp x - t) + c_2\right]^2} + c_3\left(\mp x - t\right) + c_4 \\ \mid \vec{v} \mid = \pm \tanh\left\{\log\left[c_1 + c_2\left(x \mp t\right)\right]\right\} \end{array} \tag{39}$$

For the second transformation we have the following invariants $\beta = T - xt$, $y = at - x$ and solutions, for a=±1:

$$\begin{array}{c} T = xt + 0.25\left(\mp t - x\right)^2 - \frac{1/12}{c_1^2\left[c_1(\mp t - x) + c_2\right]^2} + c_3\left(\mp t - x\right) + c_4 \\ \mid \vec{v} \mid = \pm \tanh\left\{\log\left[\mp c_1 + c_2\left(t \mp x\right)\right]\right\} \end{array} \tag{40}$$

12) the invariants are $y = ax - t$, $\beta = T - xt$ and the solutions are, for a=±1:

$$\begin{array}{c} T = xt + 0.25\left(\mp x - t\right)^2 - \frac{1/12}{c_1^2\left[c_1(\mp x - t) + c_2\right]^2} + c_3\left(\mp x - t\right) + c_4 \\ \mid \vec{v} \mid = \pm \tanh\left\{\log\left[c_1 + c_2\left(x \mp t\right)\right]\right\} \end{array} \tag{41}$$

For the second transformation we have the following invariants $\beta = T - \frac{xt}{a}$, $y = at - x$ and solutions, for a=±1:

$$\begin{array}{c} T = \pm xt + 0.25\left(\mp t - x\right)^2 - \frac{1/12}{c_1^2\left[c_1(\mp t - x) + c_2\right]^2} + c_3\left(\mp t - x\right) + c_4 \\ \mid \vec{v} \mid = \pm \tanh\left\{\log\left[\mp c_1 + c_2\left(t \mp x\right)\right]\right\} \end{array} \tag{42}$$

13) and 14) because of the form of the invariants we can't construct the reduced system of equations

15) consider the transformation $V(6) = x\partial_x + t\partial_t$, the reduced equations are the same as in the case 2); in all the other transformations we can not obtain reduced equations.

6. Summary and conclusions

The results of the symmetry group analysis of the energy-momentum tensor conservation equation for the imperfect fluid flow can be summarized by the following remarks:

- The ultrarelativistic limit was implemented in a simple analytical manageable way on the equations of motion.
- The local Lie symmetries of the equations were presented.
- The optimal system of transformation was calculated.
- We present all the analytical solutions of the reduced system of equations.
- The equation that has to be solved numerically was written in the reduced form using the invariants of the transformation.
- These analytical solutions can be very useful for the investigation of different physical systems where the dissipative processes are important. One of them is the relativistic heavy ion collisions where this kind of relativistic hydrodynamic equations are usually applied [11].

There are some questions that have not been addressed in this paper:

- we have not take into account the pressure and the energy density
- there are also other terms with smaller power of the velocity field that were neglected
- only longitudinal expansion was consider and the three-dimensional radial expansion of the fluid have not been discussed

We will give short answers to the questions mention above:

- for the first problem we need a relation between the pressure and the energy density which can be used for dissipative systems
- the second one will be the goal of our future analyses
- the last one needs numerical codes and a particular physical system with known initial conditions

We demonstrated the application of the Lie symmetry method on some particular equations proving that the differential invariants can help us to simplify very much the task of finding the solutions of some given differential equations.

The Lie group approach in its general form is particularly effective since it furnishes both general Lie symmetries and all their invariants in a constructive way.

We find that the application of this method will give us a straightforward way to decide the question of integrability. It appears that cases of exact solutions of differential equations are based on the use of symmetry of these equations with respect to certain transformations.

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