

CERN-TH/97-277

hep-th/9710126

Global Anomalies in M -theory

Måns Henningson

*Theory Division, CERN**CH-1211 Geneva 23, Switzerland*

henning@nxth04.cern.ch

Abstract

We first consider M -theory formulated on an open eleven-dimensional spin-manifold. There is then a potential anomaly under gauge transformations on the E_8 bundle that is defined over the boundary and also under diffeomorphisms of the boundary. We then consider M -theory configurations that include a five-brane. In this case, diffeomorphisms of the eleven-manifold induce diffeomorphisms of the five-brane world-volume and gauge transformations on its normal bundle. These transformations are also potentially anomalous. In both of these cases, it has previously been shown that the perturbative anomalies, i.e. the anomalies under transformations that can be continuously connected to the identity, cancel. We extend this analysis to global anomalies, i.e. anomalies under transformations in other components of the group of gauge transformations and diffeomorphisms. These anomalies are given by certain topological invariants, that we explicitly construct.

CERN-TH/97-277

October 1997

1. Introduction

The consistency of a theory with gauge-fields or dynamical gravity requires that the effective action is invariant under gauge transformations and space-time diffeomorphisms, usually referred to as cancelation of gauge and gravitational anomalies. The first step towards establishing that a given theory is anomaly free is to consider transformations that are continuously connected to the identity. The cancelation of the corresponding anomalies, often called perturbative anomalies, imposes some constraints on the chiral field content of the theory. Given that the perturbative anomalies cancel, it makes sense to investigate transformations in other components of the group of gauge transformations and diffeomorphisms. An anomaly under such a transformation is usually referred to as a global anomaly. A general formula for the quantum contribution of chiral fields to global anomalies was given in [1]. Provided that the perturbative anomalies cancel, the global anomaly is a topological invariant, i.e. it is invariant under smooth deformations of the data, and thus only depends on the topological classes of for example the space-time manifold and the gauge-bundle. The requirement that the anomaly vanishes for an arbitrary transformation imposes some restrictions on these objects.

In string theory, the requirements of supersymmetry and vanishing anomalies are particularly constraining, because of the high dimensionality of space-time. For the purpose of computing anomalies, it is enough to know the low-energy effective supergravity theory. The non-chiral type IIA supergravity theory obviously has no anomalies. The chiral type IIB supergravity theory has a potential perturbative gravitational anomaly, but the contributions from the various chiral fields ‘miraculously’ cancel against each other [2]. Type I supergravity coupled to some super-Yang-Mills theory always has a non-vanishing perturbative quantum anomaly. It can however be cancelled by a Green-Schwarz mechanism [3] involving an anomalous transformation law at tree-level for the two-form field, provided that the gauge group is $SO(32)$ or $E_8 \times E_8$. This discovery actually preceded the construction of the $SO(32)$ and $E_8 \times E_8$ heterotic string theories. Global anomalies in string theory was first considered in [1]. The result is that the known string theories are free from global anomalies when formulated in ten-dimensional Minkowski space. However, there are interesting non-trivial restrictions on consistent compactifications to lower dimensions.

There is by now mounting evidence that the different string theories should be seen as particular limits of a conjectured theory called M -theory. In the long wave-length limit,

this theory should reduce to eleven-dimensional supergravity. It should also admit certain types of topological defects, in particular space-time boundaries and five-brane solitons. Much has been learned about M -theory by studying the mechanism for cancellation of perturbative anomalies for various configurations including such defects. In this way, it was discovered that when M -theory is defined on an open eleven-manifold, there is an $E_8 \times E_8$ super-Yang-Mills multiplet propagating on the boundary. The cancellation of perturbative gauge and gravitational anomalies involves a subtle interplay between contributions from this multiplet, the bulk degrees of freedom and a generalized Green-Schwarz mechanism [4][5]. The situation is even more interesting for M -theory configurations involving a five-brane. The perturbative gauge and gravitational anomalies from the five-brane world-volume theory can be partially cancelled by an anomaly inflow from the surrounding eleven-dimensional space [6][7]. There is however a remaining part, whose cancellation seems to require additional world-volume interactions and also imposes a certain topological restriction on the five-brane configuration [7][8].

Given our present incomplete understanding of M -theory, it seems that any further information about this theory would be valuable. The purpose of the present paper is to carry the analysis described above one step further by investigating also global anomalies. In section two, we consider the case of M -theory defined on an open eleven-manifold. In section three, we instead consider M -theory configurations including a five-brane. In both of these cases, we derive an explicit formula for the topological invariants describing the global anomalies.

2. Anomalies on open eleven-manifolds

We consider M -theory on an eleven-dimensional open spin-manifold Y . The massless degrees of freedom propagating in the bulk of Y are those of the eleven-dimensional supergravity multiplet, i.e. a metric G_{MN} , a three-form potential C_{MNP} and a fermionic Rarita-Schwinger field ψ_M . At low energies, the dynamics of these fields is governed by the eleven-dimensional supergravity action [9]. The bulk action possesses a classical invariance under diffeomorphisms of Y , and since we are in an odd number of space-time dimensions, this symmetry is obviously not spoiled by any chiral anomaly.

However, there is potentially an anomaly, often called the parity anomaly, in the bulk of Y , which is associated with the Rarita-Schwinger field ψ_M [10]. The operator in the kinetic term of this field is Hermitian in eleven dimensions, but has infinitely many

positive and negative eigenvalues, leading to a potential sign problem in the definition of the fermionic path-integral measure [2]. This will show up as a change $\Delta\Gamma_{bulk}$ of the effective bulk action Γ_{bulk} under a diffeomorphism π of Y . To describe this anomaly, it is convenient to introduce a twelve-dimensional manifold $(Y \times S^1)_\pi$, called the mapping torus, as follows: We start with the cylinder $Y \times I$, where I is an interval, and equip it with a metric that smoothly interpolates between the original metric on Y at one of the boundaries of I and the metric obtained from it by the transformation π at the other boundary. Finally, we glue together the two boundaries of I to form $(Y \times S^1)_\pi$. The bulk anomaly is then given by [11]

$$\Delta\Gamma_{bulk} = \pi i \left(\frac{1}{2} \text{Index}_{(Y \times S^1)_\pi}(RS) - \frac{3}{2} \text{Index}_{(Y \times S^1)_\pi}(D_0) \right), \quad (2.1)$$

where $\text{Index}_{(Y \times S^1)_\pi}(RS)$ and $\text{Index}_{(Y \times S^1)_\pi}(D_0)$ denote the indices of the Rarita-Schwinger and Dirac operators on $(Y \times S^1)_\pi$. It follows from charge conjugation symmetry that these indices are even in twelve dimensions, so $\Delta\Gamma_{bulk}$ is a multiple of πi , corresponding to the sign ambiguity in the fermionic path integral measure. Precisely in twelve dimensions, the combination of indices that appears in (2.1) is related to the signature $\sigma_{(Y \times S^1)_\pi}$ of $(Y \times S^1)_\pi$ as

$$\frac{1}{8} \sigma_{(Y \times S^1)_\pi} = \text{Index}_{(Y \times S^1)_\pi}(RS) - 3 \text{Index}_{(Y \times S^1)_\pi}(D_0). \quad (2.2)$$

It follows that $\sigma_{(Y \times S^1)_\pi}$ is a multiple of 16 and that the bulk anomaly can be written as

$$\Delta\Gamma_{bulk} = \frac{\pi i}{16} \sigma_{(Y \times S^1)_\pi}. \quad (2.3)$$

The massless degrees of freedom on the boundary M of Y include a left-handed Rarita-Schwinger field ψ_μ and a right-handed spinor field λ originating from the Rarita-Schwinger field ψ_M of the bulk theory. There is also a set of left-handed spinor fields χ in the adjoint representation of E_8 that together with a set of gauge fields A_μ make up a super Yang-Mills multiplet propagating on the boundary. These fields give rise to an anomaly under diffeomorphisms of Y that induce diffeomorphisms of M , and also under gauge transformations of the E_8 bundle V over M . To describe such a transformation π , we consider the mapping torus $(M \times S^1)_\pi$ and the E_8 bundle V_π over it. These objects are constructed in analogy with $(Y \times S^1)_\pi$ by identifying the boundaries of the cylinder $M \times I$ after a twist by π . We note that $(Y \times S^1)_\pi$ is bounded by $(M \times S^1)_\pi$. The anomalous change

$\Delta\Gamma_{eff}$ under the transformation π of the effective action Γ_{eff} obtained by integrating out the fermionic fields is then given by a general formula derived in [1] as

$$\Delta\Gamma_{eff} = \frac{\pi i}{2}\eta, \quad (2.4)$$

where η denotes a certain η -invariant on $(M \times S^1)_\pi$.

The η -invariant on a closed manifold C is defined as

$$\eta = \lim_{\epsilon \rightarrow 0} \sum_i' \text{sign}(\lambda_i) \exp(-\epsilon|\lambda_i|), \quad (2.5)$$

where i indexes the eigenvalues λ_i of a certain operator on C , and the sum runs over all i such that $\lambda_i \neq 0$. In general, the expression (2.5) is prohibitively difficult to evaluate. The situation is better if C bounds some twelve-manifold B , and the gauge bundle can be extended to a bundle over B . Whether this is actually possible or not is a problem in cobordism theory. In the situation at hand, where $C = (M \times S^1)_\pi$, it is indeed possible since we can for example choose B to be $(V \times S^1)_\pi$. In any case, if C is the boundary of B , the η -invariant on C can be expressed in terms of a certain operator D on B . The operator D is in fact the one that arises in a calculation of the perturbative anomaly, i.e. the anomaly under a transformation π that can be continuously connected to the identity, as we will now describe. The Atiyah-Singer index theorem (see for example [12]) gives the index of D on a *closed* twelve-manifold as the integral of some characteristic class I_{12} . The perturbative anomaly is then obtained through a descent procedure [13]: Since I_{12} is closed, it can be written locally as $I_{12} = d\omega_{11}$, where ω_{11} is the associated Chern-Simons form. The latter form is not invariant under infinitesimal diffeomorphisms and gauge transformations, but its variation is a total derivative, i.e. $\delta\omega_{11} = d\alpha_{10}^1$, where the superscript 1 indicates that the form α_{10}^1 is linear in the parameter of the transformation. The perturbative anomaly is now given by the integral of α_{10}^1 over the space-time manifold M . Returning to the case of an open manifold B with boundary C , the Atiyah-Patodi-Singer index theorem (see for example [12]) now states that the η -invariant on C is given by

$$\frac{1}{2}\eta = \text{Index}_B(D) - \int_B I_{12} + \int_C \omega_{11}. \quad (2.6)$$

In our case, $C = (M \times S^1)_\pi$. Recalling the definition of this manifold as the cylinder $M \times I$ with the two boundaries identified after a twist by π , we see that the last term in the expression for $\frac{1}{2}\eta$ does not really make sense, since the integrand ω_{11} is in general

not invariant under such a transformation and therefore is not well-defined on $(M \times S^1)_\pi$. This term should therefore more properly be written as an integral over $M \times I$, so that the anomalous change of the effective action is

$$\Delta\Gamma_{eff} = \pi i \left(\text{Index}_B(D) - \int_B I_{12} + \int_{M \times I} \omega_{11} \right). \quad (2.7)$$

As explained in [4], the anomaly from the ψ_μ and λ is given by half the standard anomaly $I_{Sugra}(R)$ from these fields in type I supergravity, whereas the anomaly from χ is the standard anomaly $I_{SYM}(R, F)$ from this field in super Yang-Mills theory with E_8 gauge group. The standard anomaly formulas [14] give

$$\begin{aligned} I_{Sugra}(R) &= \frac{1}{(2\pi)^6} \left(-\frac{1}{1296} (\text{tr}R^2)^3 + \frac{7}{1080} \text{tr}R^2 \text{tr}R^4 - \frac{31}{2835} \text{tr}R^6 \right) \\ I_{SYM}(R, F) &= \frac{1}{(2\pi)^6} \left(-\frac{1}{24} (\text{tr}F^2)^3 + \frac{1}{16} (\text{tr}F^2)^2 \text{tr}R^2 - \frac{5}{192} \text{tr}F^2 (\text{tr}R^2)^2 - \frac{1}{48} \text{tr}F^2 \text{tr}R^4 \right. \\ &\quad \left. + \frac{31}{10368} (\text{tr}R^2)^3 + \frac{31}{4320} \text{tr}R^2 \text{tr}R^4 + \frac{31}{5670} \text{tr}R^6 \right), \end{aligned} \quad (2.8)$$

where R and F are the Riemann curvature and field-strength two-forms respectively and tr for a power of F denotes $1/30$ of the trace in the adjoint representation of E_8 . Here we have used the E_8 identities $\text{tr}F^4 = \frac{3}{10}(\text{tr}F^2)^3$ and $\text{tr}F^6 = \frac{1}{8}(\text{tr}F^2)^3$. The integrals of $I_{Sugra}(R)$ and $I_{SYM}(R, F)$ on a closed twelve-manifold equal $\text{Index}(RS) - 3 \text{Index}(D_0)$ and $\text{Index}(D_V)$ respectively, where RS , D_0 and D_V are the Rarita-Schwinger operator, the Dirac-operator and the Dirac operator for fermions in the adjoint of E_8 respectively. Again, $\text{Index}(RS) - 3 \text{Index}(D_0) = \frac{1}{8}\sigma$, where σ denotes the signature in twelve dimensions. Indeed, $I_{Sugra}(R)$ equals the Hirzebruch L -polynomial in this dimension.

The characteristic class $I_{12} = \frac{1}{2}I_{Sugra}(R) + I_{SYM}(R, F)$ describing the anomaly from the fields ψ_μ , λ and χ does not vanish in general, so the perturbative anomalies from the fermionic fields do not cancel. However, it factorizes as $I_{12} = I_4 \wedge I_8$, where

$$\begin{aligned} I_4 &= \frac{1}{(2\pi)^2} \left(\frac{1}{4} \text{tr}F^2 - \frac{1}{8} \text{tr}R^2 \right) \\ I_8 &= \frac{1}{(2\pi)^4} \left(-\frac{1}{6} (\text{tr}F^2)^2 + \frac{1}{6} \text{tr}F^2 \text{tr}R^2 - \frac{1}{48} (\text{tr}R^2)^2 - \frac{1}{12} \text{tr}R^4 \right). \end{aligned} \quad (2.9)$$

We can thus write the anomalous change of the effective action as

$$\Delta\Gamma_{eff} = \pi i \left(\frac{1}{16} \sigma_B + \text{Index}_B(D_V) - \int_B I_4 \wedge I_8 + \int_{M \times I} \omega_3 \wedge I_8 \right), \quad (2.10)$$

where ω_3 is the Chern-Simons form of I_4 , i.e. $d\omega_3 = I_4$.

The quantum anomaly (2.10) can be cancelled by a generalized Green-Schwarz mechanism involving the three-form potential C of eleven-dimensional supergravity [4][5]. With some changes, the following discussion can also be adapted to the case of M -theory on a \mathbf{Z}_2 orbifold, where the role of the boundary is taken over by the orbifold fixed points. We begin by decomposing I_8 as

$$I_8 = -\frac{8}{3}I_4 \wedge I_4 + I'_8, \quad (2.11)$$

where

$$I'_8 = \frac{1}{(2\pi)^4} \left(\frac{1}{48}(\text{tr}R^2)^2 - \frac{1}{12}\text{tr}R^4 \right). \quad (2.12)$$

The Green-Schwarz counterterms are now

$$\Gamma_{GS} = \pi i \left(-\frac{8}{3} \int_Y C \wedge G \wedge G + \int_Y C \wedge I'_8 \right), \quad (2.13)$$

where $G = dC$ is the invariant four-form field strength. Note that these terms are bulk interactions, although the anomaly to be cancelled is supported on the boundary. The first of these terms is the familiar ‘Chern-Simons’ interaction of eleven-dimensional supergravity. The existence of the second term can be inferred from a one-loop calculation for type IIA strings [15] lifted to eleven dimensions, or from the requirement of perturbative anomaly cancelation on the M -theory five-brane world-volume [6]. To be able to write this term, it is crucial that I'_8 depends only on R and not on F , since the latter field only propagates on the boundary M and not in the bulk of Y .

The change in the Green-Schwarz terms under the transformation π is given by

$$\Delta\Gamma_{GS} = \pi i \int_{Y \times \partial I} C \wedge \left(-\frac{8}{3}G \wedge G + I'_8 \right), \quad (2.14)$$

where the integral over $Y \times \partial I$ means the difference of the integrals over Y at the two boundary points of the interval I . By using Stokes’ theorem and the fact that $\partial(Y \times I) = M \times I + Y \times \partial I$, we can rewrite this as

$$\Delta\Gamma_{GS} = -\pi i \int_{M \times I} C \wedge I_8 + \pi i \int_{Y \times I} G \wedge \left(-\frac{8}{3}G \wedge G + I'_8 \right), \quad (2.15)$$

where in the first term we have used (2.11) and the condition that

$$G = I_4 \quad (2.16)$$

on $M \times I$. This condition follows from the requirement that the boundary interactions preserve half of the supersymmetry of the bulk theory [4]. In particular, the pullback of I_4 to $M \times I$ is trivial in cohomology. We now assign an anomalous transformation law to C such that the quantity

$$H = \omega_3 - C \quad (2.17)$$

is invariant on $M \times I$. This is consistent with the boundary condition (2.16), which amounts to

$$dH = 0 \quad (2.18)$$

on $M \times I$. The anomalous change of the total action $\Gamma = \Gamma_{bulk} + \Gamma_{eff} + \Gamma_{GS}$ can now be written as

$$\begin{aligned} \Delta\Gamma = \pi i \left(\frac{1}{16} \sigma_{(Y \times S^1)_\pi} + \frac{1}{16} \sigma_B + \text{Index}_B(D_V) \right. \\ \left. - \int_B I_4 \wedge I_8 + \int_{(M \times S^1)_\pi} H \wedge I_8 + \int_{(Y \times S^1)_\pi} G \wedge \left(-\frac{8}{3} G \wedge G + I'_8 \right) \right). \end{aligned} \quad (2.19)$$

Note that since H and G are invariant, the integrands in the last two terms are indeed well-defined on $(M \times S^1)_\pi$ and $(Y \times S^1)_\pi$ respectively.

Before we continue, we will first verify that the expression (2.19) does not depend on the choice of B . We can replace B by some other twelve-manifold \tilde{B} with the same boundary $(M \times S^1)_\pi$. The expression for $\Delta\Gamma$ then changes by

$$\begin{aligned} \pi i \left(\frac{1}{16} \sigma_{\tilde{B}} + \text{Index}_{\tilde{B}}(D_V) - \frac{1}{16} \sigma_B - \text{Index}_B(D_V) \right. \\ \left. - \frac{1}{16} \sigma_{\tilde{B} \oplus (-B)} - \text{Index}_{\tilde{B} \oplus (-B)}(D_V) \right). \end{aligned} \quad (2.20)$$

Here $\tilde{B} \oplus (-B)$ denotes the closed twelve-manifold constructed by gluing together \tilde{B} and B with opposite orientation along their boundaries, and the last two terms originate from the integrals $-\int_{\tilde{B}} I_4 \wedge I_8 + \int_B I_4 \wedge I_8$. We can now use the Novikov formula (see for example [16])

$$\sigma_{\tilde{B} \oplus (-B)} = \sigma_{\tilde{B}} - \sigma_B \quad (2.21)$$

to cancel the signature terms. Furthermore, it follows from charge conjugation symmetry and the reality of the adjoint representation of E_8 that $\text{Index}(D_V)$ is always even in twelve dimensions, so the expression (2.20) vanishes modulo $2\pi i$. Since the action Γ always appears as $\exp \Gamma$, such an ambiguity in $\Delta\Gamma$ is harmless. To simplify the expression for the

anomaly, we can now choose $B = (Y \times S^1)_\pi$ and use the properties that the signature σ is a multiple of 16 and $\text{Index}(D_V)$ is even. It then follows from (2.19) that, modulo $2\pi i$,

$$\Delta\Gamma = \pi i \left(- \int_B I_4 \wedge I_8 + \int_{(M \times S^1)_\pi} H \wedge I_8 + \int_{(Y \times S^1)_\pi} G \wedge \left(-\frac{8}{3} G \wedge G + I'_8 \right) \right). \quad (2.22)$$

We should also check that $\Delta\Gamma$ does not depend on the precise form of H . For this to be true, we must actually require I_8 to be trivial in cohomology on $(M \times S^1)_\pi$. (This is of course automatic if the eight-dimensional cohomology group of this space is trivial, as would be the case for example in compactifications to $d \geq 4$ space-time dimensions.) It then follows from (2.18) that the term in (2.22) involving H actually vanishes so that

$$\Delta\Gamma = \pi i \left(- \int_B I_4 \wedge I_8 + \int_{(Y \times S^1)_\pi} G \wedge \left(-\frac{8}{3} G \wedge G + I'_8 \right) \right), \quad (2.23)$$

again modulo $2\pi i$.

The total anomaly (2.23) is in fact a topological invariant. In particular, $\Delta\Gamma$ vanishes for a transformation π that can be continuously connected to the identity, i.e. the perturbative anomalies cancel. To see this, we consider a smooth deformation of the data. The variation of the characteristic classes I_4 , I_8 and I'_8 are total derivatives, and the same is true for the field strength G , i.e. $\delta I_4 = d\Lambda_3$, $\delta I_8 = d\Lambda_7$, $\delta I'_8 = d\Lambda'_7$ and $\delta G = d\Lambda'_3$. To preserve the conditions (2.11) and (2.16), the relations

$$\begin{aligned} \Lambda_7 &= -\frac{16}{3} I_4 \wedge \Lambda'_3 + \Lambda'_7 \\ \Lambda_3 &= \Lambda'_3 \end{aligned} \quad (2.24)$$

must hold modulo closed forms. It is then easy to see that the expression (2.23) is invariant, again provided that I_8 is trivial in cohomology on the boundary $(M \times S^1)_\pi$ of B and $(Y \times S^1)_\pi$.

The total anomaly $\Delta\Gamma$ is thus determined by the topological classes of the various objects. We will not address the difficult problem of determining the conditions for it to vanish. A particularly interesting case is of course M -theory on an eleven-manifold of the form $Y = X \times J$ for some closed ten-manifold X and an interval J , which is believed to describe the strong-coupling limit of the $E_8 \times E_8$ heterotic string on X [4]. In this case, the considerations in [17][18] concerning global anomalies for the $E_8 \times E_8$ heterotic string can be carried over to the M -theory setting.

3. Anomalies on the five-brane world-volume

We consider an M -theory configuration including a five-brane. The world-volume of the five-brane defines a six-manifold W embedded in the eleven-dimensional spin-manifold Y on which the theory is defined. (For simplicity, in this section we will only consider the case when Y is closed and orientable.) The normal bundle N of W in Y is then an $SO(5)$ bundle over W . The massless fields of the world-volume theory are those of a six-dimensional $N = 4$ tensor multiplet [19], i.e. five scalars ϕ^i , $i = 1, \dots, 5$, a two-form β with anti-selfdual field strength $T = d\beta$ and fermionic spinors ψ that take their values in the bundle S constructed from the normal bundle N by using the spinor representation of the $SO(5)$ structure group.

The classical theory is invariant under diffeomorphisms of Y that map the five-brane world-volume W to itself. (The invariance under other diffeomorphisms is explicitly broken by the five-brane.) As in the previous section, such a transformation π is described by the mapping torus $(Y \times S^1)_\pi$. The transformation π induces a diffeomorphism of W and a gauge transformation on the bundle S , which we describe by the mapping torus $(W \times S^1)_\pi$ and an $SO(5)$ bundle S_π over it. Obviously, $(W \times S^1)_\pi$ is a seven-dimensional submanifold of the twelve-manifold $(Y \times S^1)_\pi$. For the purpose of computing anomalies, we can regard the world-volume theory as an $SO(5)$ gauge theory with fermions in the spinor representation. This essentially amounts to replacing the eleven-manifold Y by the total space of the normal bundle N . As stated above, the theory also contains a chiral two-form β and is coupled to non-dynamical gravity induced from the embedding of W in Y . Because of the anti-selfduality constraint on the field strength $T = d\beta$, there is no description in terms of a covariant action, but this can be remedied by adding further anomaly-free fields [2]. The anomalous change $\Delta\Gamma_{eff}$ of the effective action Γ_{eff} under the transformation π again follows from the general formula in [1]. We thus get $\Delta\Gamma_{eff} = \frac{\pi i}{2}\eta$, where η now is an η -invariant on $(W \times S^1)_\pi$. Assuming that $(W \times S^1)_\pi$ bounds some eight-manifold E , this can be expressed as

$$\Delta\Gamma_{eff} = \pi i \left(\text{Index}_E(D) - \int_E J_8 + \int_{W \times I} \omega_7 \right). \quad (3.1)$$

The anomaly polynomial J_8 is here given by

$$J_8 = \frac{1}{(2\pi)^4} \left(\frac{1}{256} (\text{tr} F^2)^2 - \frac{1}{192} \text{tr} F^4 - \frac{1}{384} \text{tr} F^2 \text{tr} R^2 - \frac{1}{768} (\text{tr} R^2)^2 + \frac{1}{192} \text{tr} R^4 \right), \quad (3.2)$$

where tr for a power of F denotes the trace in the fundamental representation of $SO(5)$, which is related to the trace Tr for the spinor representation as $\text{Tr}F^2 = \frac{1}{2}\text{tr}F^2$ and $\text{Tr}F^4 = \frac{3}{16}(\text{tr}F^2)^2 - \frac{1}{4}\text{tr}F^4$. The integral of J_8 over a closed eight-manifold yields $\text{Index}(D) = \frac{1}{2}\text{Index}(D_S) - \frac{1}{8}\sigma$, where D_S is the Dirac operator for chiral fermions with values in the bundle S and σ is the signature. Finally, ω_7 is the Chern-Simons form of J_8 , i.e. $d\omega_7 = J_8$.

The total anomaly on W also receives a contribution from the bulk theory on Y . The anomalous interaction is in fact the second term in the Green-Schwarz interaction (2.13). In the present context, this term is better written as

$$\Gamma_{bulk} = \pi i \int_Y G \wedge \omega'_7, \quad (3.3)$$

where the Chern-Simons form ω'_7 obeys $d\omega'_7 = I'_8$ and I'_8 was defined in (2.12). The reason is that the three-form C is not globally well-defined in the presence of the magnetically charged five-brane. The field strength G makes sense, though, and obeys

$$dG = \frac{1}{16}\delta_W, \quad (3.4)$$

where δ_W is a representative of the Poincaré dual of W supported in an infinitesimal neighborhood of W . (The factor of $\frac{1}{16}$ is due to our normalization of G .) The anomalous change of Γ_{bulk} under the transformation π is thus

$$\Delta\Gamma_{bulk} = \pi i \int_{Y \times \partial I} G \wedge \omega'_7 = \pi i \left(\int_{Y \times I} G \wedge I'_8 + \frac{1}{16} \int_{W \times I} \omega'_7 \right), \quad (3.5)$$

where we have used Stokes' theorem and (3.4). The last term involves the Chern-Simons form ω'_7 of I'_8 restricted to W . To evaluate this term in the present context, one should note that there is an important change in notation between this section and the previous one: In the formula (2.12) for I'_8 , R denotes the curvature on Y , whereas in this section we take R to denote the induced curvature on W . We should therefore rewrite (2.12) using the decomposition of the tangent bundle of Y restricted to W as the direct sum of the tangent bundle of W and the normal bundle N . The latter bundle is regarded as an $SO(5)$ bundle with field strength F . In this way we get

$$I'_8 = \frac{1}{(2\pi)^4} \left(\frac{1}{48}(\text{tr}F^2)^2 - \frac{1}{12}\text{tr}F^4 + \frac{1}{24}\text{tr}F^2\text{tr}R^2 + \frac{1}{48}(\text{tr}R^2)^2 - \frac{1}{12}\text{tr}R^4 \right) \quad (3.6)$$

on W . The combined anomaly of $\Gamma_{eff} + \Gamma_{bulk}$ is thus

$$\Delta\Gamma_{eff} + \Delta\Gamma_{bulk} = \pi i \left(\frac{1}{2}\text{Index}_E(D_S) - \frac{1}{8}\sigma_E - \int_E J_8 + \int_{Y \times I} G \wedge I'_8 + \int_{W \times I} \omega''_7 \right), \quad (3.7)$$

where ω_7'' is the Chern-Simons form of $I_8'' = J_8 + \frac{1}{16}I_8'$, i.e. $d\omega_7'' = I_8''$. We see that

$$I_8'' = \frac{1}{(2\pi)^4} \left(\frac{1}{192}(\text{tr}F^2)^2 - \frac{1}{96}\text{tr}F^4 \right), \quad (3.8)$$

which in fact equals $1/24$ times the second Pontrjagin class $p_2(N)$ of the normal bundle N .

We will now describe a mechanism, outlined in [7] and further elaborated in [8], to cancel the remaining perturbative anomaly in (3.7). In the following, we will use a vector sign over a differential form to denote that it takes its values in the normal bundle N . Bilinears in such forms are understood to be multiplied via the fiber-metric on N . In this way, we can write the characteristic class I_8'' as

$$I_8'' = \frac{1}{24}\vec{\chi} \wedge \vec{\chi}, \quad (3.9)$$

where the N -valued four-form $\vec{\chi}$ is a bilinear in the field strength F contracted with the invariant rank five tensor of $SO(5)$. (The field strength F takes its values in the adjoint representation of $SO(5)$, i.e. in the antisymmetric product of two copies of N .) We also introduce the N -valued Chern-Simons three-form $\vec{\omega}$ corresponding to $\vec{\chi}$ so that $D\vec{\omega} = \vec{\chi}$, where D denotes the $SO(5)$ covariant exterior derivative. Although the square of D does not vanish, it follows from the Bianchi identity for F that $D\vec{\chi} = 0$. Furthermore, we introduce an N -valued three-form \vec{H} as follows: When restricted to W , the tangent bundle of Y decomposes as a direct sum of the tangent bundle of W and the normal bundle N . The field strength G , which is a section of the fourth exterior power of the tangent bundle of Y , can be decomposed accordingly. The form \vec{H} is then proportional to the component which is a three-form on W with values in N . We must also require that

$$D\vec{H} = \vec{\chi} \quad (3.10)$$

when restricted to W , i.e. $\vec{\chi}$ must be covariantly exact. This is a topological restriction for the anomaly cancelation mechanism to work. It also fixes the normalization of \vec{H} . The requisite counterterm is now

$$\Gamma_{ct} = \frac{1}{24} \int_W \vec{\omega} \wedge \vec{H}. \quad (3.11)$$

Its change under the transformation π is

$$\Delta\Gamma_{ct} = \frac{1}{24} \int_{W \times \partial I} \vec{\omega} \wedge \vec{H} = \frac{1}{24} \int_{W \times I} \left(\vec{\chi} \wedge \vec{H} - \vec{\omega} \wedge \vec{\chi} \right), \quad (3.12)$$

where we have used Stokes' theorem, the relationship of $\vec{\omega}$ to $\vec{\chi}$, and the condition (3.10). The anomalous change of the total action $\Gamma_{total} = \Gamma_{eff} + \Gamma_{bulk} + \Gamma_{ct}$ is thus

$$\Delta\Gamma_{total} = \pi i \left(\frac{1}{2} \text{Index}_E(D_S) - \frac{1}{8} \sigma_E - \int_E J_8 + \int_{(Y \times S^1)_\pi} G \wedge I'_8 + \frac{1}{24} \int_{(W \times S^1)_\pi} \vec{\chi} \wedge \vec{H} \right), \quad (3.13)$$

where we have used the relationship between the Chern-Simons forms ω''_7 and $\vec{\omega}$ that follows from (3.9). Note that the integrands of the last two terms are well-defined on $(Y \times S^1)_\pi$ and $(W \times S^1)_\pi$ respectively, since the field strength G (and thus also \vec{H}) transforms covariantly under π .

We will now discuss some properties of the expression (3.13) for the total anomaly. First of all, $\Delta\Gamma_{total}$ should be independent modulo $2\pi i$ of the choice of the eight-manifold E as long as it is bounded by $(W \times S^1)_\pi$. For this to be true, we must, in addition to the eight-dimensional analogue of the Novikov formula (2.21), assume that

$$\text{Index}_{\tilde{E} \oplus (-E)}(D_S) = \text{Index}_{\tilde{E}}(D_S) - \text{Index}_E(D_S) \quad (3.14)$$

modulo 4 for any eight-manifolds E and \tilde{E} with common boundary. Furthermore, $\Delta\Gamma_{total}$ is independent of the exact form of G and \vec{H} as long as (3.4) and (3.10) are fulfilled and I'_8 and $\vec{\chi}$ are (covariantly) exact. (The exactness of I'_8 is again automatic in compactifications to $d \geq 4$ dimensions, whereas the covariant exactness of $\vec{\chi}$ is assured by (3.10).) Finally, $\Delta\Gamma_{total}$ is a topological invariant. Indeed, the variations of the characteristic classes $\vec{\chi}$, J_8 and I'_8 under a smooth deformation of the data must be (covariantly) exact, i.e. $\delta\vec{\chi} = D\vec{\lambda}$, $\delta J_8 = d\Lambda_7$ and $\delta I'_8 = d\Lambda'_7$. To preserve the relationships (3.9) and (3.10), we must have $\delta\vec{H} = \vec{\lambda}$ and

$$\frac{1}{12} \vec{\lambda} \wedge \vec{\chi} = \Lambda_7 + \frac{1}{16} \Lambda'_7 \quad (3.15)$$

modulo closed forms. It is then easy to see that the expression (3.13) is invariant.

The requirement that the anomaly vanish for any diffeomorphism π of the eleven-manifold Y that leaves the world-volume W invariant is a necessary restriction on a consistent M -theory configuration. Obviously, a first question to settle is the correct interpretation of the condition (3.10), which entered already at the perturbative level. Provided that this equation is fulfilled, it makes sense to consider the expression (3.13) for the global anomaly. It is not clear to what extent its vanishing follows from already known restrictions on M -theory configurations. Here we just remark that there is no anomaly for pure gauge transformations, i.e. transformations induced by diffeomorphisms of Y that act

trivially on W . The reason is that since the homotopy group $\pi_6(SO(5))$ is trivial, all pure gauge transformations can be continuously connected to the identity. One should therefore consider diffeomorphisms of W that are not continuously connected to the identity, possibly combined with gauge transformations. A basic case is when W is topologically a six-sphere S^6 , in which case $(W \times S^1)_\pi$ is actually the connected sum of one of the 27 exotic seven-spheres and $S^6 \times S^1$ [1]. In fact, if the global anomaly does not vanish in this situation, it will not vanish for any W . This follows from the fact that a diffeomorphism of S^6 always has an analogue in the diffeomorphism group of an arbitrary six-manifold.

References

- [1] E. Witten, ‘Global gravitational anomalies’, *Commun. Math. Phys.* **100** (1985) 197.
- [2] L. Alvarez-Gaumé and E. Witten, ‘Gravitational anomalies’, *Nucl. Phys.* **B234** (1983) 269.
- [3] M. B. Green and J. H. Schwarz, ‘Anomaly cancellations in $D = 10$ gauge theory and superstring theory’, *Phys. Lett.* **149B** (1984) 117.
- [4] P. Horava and E. Witten, ‘Heterotic and type I string dynamics from eleven dimensions’, *Nucl. Phys.* **B460** (1996) 506, [hep-th/9510209](#) ;
P. Horava and E. Witten, ‘Eleven dimensional supergravity on a manifold with boundary’, *Nucl. Phys.* **B475** (1996) 94, [hep-th/9603142](#).
- [5] S. P. de Alwis, ‘Anomaly cancellation in M -theory’, *Phys. Lett.* **B392** (1997) 332, [hep-th/9609211](#).
- [6] M. Duff, J. Liu and R. Minasian, ‘Eleven-dimensional origin of string/string duality: a one loop test’, *Nucl. Phys.* **B452** (1995) 261, [hep-th/9506126](#).
- [7] E. Witten, ‘Five-brane effective action in M -theory’, [hep-th/9610234](#).
- [8] S. P. de Alwis, ‘Couplings of branes and normalization of effective actions in string/ M -theory’, [hep-th/9705139](#).
- [9] E. Cremmer, B. Julia and J. Scherk, ‘Supergravity theory in 11 dimensions’, *Phys. Lett.* **B76** (1978) 409.
- [10] E. Witten, ‘On flux quantization in M -theory and the effective action’, [hep-th/9609122](#).
- [11] L. Alvarez-Gaumé, S. Della Pietra and G. Moore, ‘Anomalies and odd dimensions’, *Ann. Phys.* **163** (1985) 288.
- [12] T. Eguchi, P. B. Gilkey and A. J. Hanson, ‘Gravitation, gauge theories and differential geometry’, *Phys. Rep.* **66** (1980) 213.
- [13] B. Zumino, ‘Chiral anomalies and differential geometry’, in B. S. DeWitt and R. Stora eds., *Relativity, groups and topology II*, North Holland, Amsterdam, 1984.

- [14] L. Alvarez-Gaumé and P. Ginsparg, ‘The structure of gauge and gravitational anomalies’, *Ann. Phys.* **161** (1985) 423.
- [15] C. Vafa and E. Witten, ‘A one loop test of string duality’, *Nucl. Phys.* **B447** (1995) 261, [hep-th/9505053](#).
- [16] M. F. Atiyah and I. M. Singer, ‘Index of elliptic operators III’, *Ann. Math.* **87** (1968) 546.
- [17] E. Witten, ‘Global anomalies’, in G. W. Gibbons, S. W. Hawking and P. K. Townsend eds., *Supersymmetry and its applications*, Cambridge University Press, Cambridge 1985.
- [18] T. P. Killingback, ‘Global anomalies, string theory and spacetime topology’, *Class. Quant. Grav.* **5** (1988) 1169.
- [19] E. Witten, ‘Five-branes and M -theory on an orbifold’, *Nucl. Phys.* **B463** (1996) 383, [hep-th/9512219](#).