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U-duality and D-brane Combinatorics

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Abstract

We investigate D-brane instanton contributions to R^4 couplings in any toroidal compactification of type II theories. Starting from the 11D supergravity one-loop four-graviton amplitude computed by Green, Gutperle and Vanhove, we derive the non-perturbative $O(e^{-1/\lambda})$ corrections to R^4 couplings by a sequence of T-dualities, and interpret them as precise configurations of bound states of D-branes wrapping cycles of the compactification torus. Dp-branes explicitely appear as fluxes on D$(p+2)$-branes, and as gauge instantons on D$(p+4)$-branes. Specific rules for weighting these contributions are obtained, which should carry over to more general situations. Furthermore, it is shown that U-duality in $D \leq 6$ relates these D-brane configurations to $O(e^{-1/\lambda^2})$ instantons for which a geometric interpretation is still lacking.

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Although a non-perturbative definition of superstring theory is still elusive, the discovery of string dualities has made it clear that this theory should include various p-brane objects in its BPS spectrum. Under compactification, these objects can wrap on r-cycles of the compactification manifold to yield $(p-r)$-branes in lower dimensions, or instanton configurations if $r=p+1$ [1]. An exact calculation of physical couplings should take these instanton effects into account, but the rules for weighting them are still largely unknown. In some cases, the constraints of duality are strong enough to determine the exact non-perturbative completion of these couplings [2, 3, 4, 5, 6, 7], thereby opening a window on the string theory instanton rules. This way, D-instantons [3], D-particles [8, 9] and (p,q) strings [10] contributions to R^4 couplings in toroidal compactifications of type II string have been brought under control, as well as general D-brane contributions to four derivative couplings in $K3$ compactifications of the same [5]. In this letter, we want to take a more systematic approach to this problem, and derive the contributions of general Dp-brane instantons to R^4 couplings in toroidally compactified type II theory, by applying a sequence of perturbative T-dualities on the well understood D-particle contribution. Imposing S-duality will force us to include additional contributions, the origin of which remains to be elucidated.

The general contribution of D-particles to R^4 couplings in toroidally compactified type IIA theory has been obtained in Ref. [9] from the one-loop scattering amplitude of four supergravitons in 11-dimensional supergravity compactified on a $(N+1)$-torus:

$$A_4 = 2\pi\mathcal{V}_{11}\int_0^\infty \frac{dt}{t^{5/2}}\hat{\sum_{n^I}} e^{-\frac{\pi}{t}n^I g_{IJ} n^J} \ , \tag{1}$$

where g_{IJ} is the volume $\mathcal{V}_{11}$ metric of the torus in eleven-dimensional Planck units. The sum runs over $(N+1)$-uplets of non-zero integers (as denoted by the hat over the sum) dual to the momentum of the supergraviton running in the loop. In terms of type IIA variables, the metric g_{IJ} decomposes as

$$ds_{11}^2 = R_{11}^2(dx^{11}+\mathcal{A}_i dx^i)^2 + \frac{1}{R^{11}} dx^i g_{ij} dx^j \ , \tag{2}$$

where the eleven-dimensional radius R_{11} is related to the type IIA coupling [11] through

$R_{11} = e^{2\phi/3} = \lambda^{2/3}$, and g_{ij} denotes the metric in the string frame. The Kaluza–Klein connection $\mathcal{A}$ coincides with the Ramond–Ramond (RR) one-form gauge potential of type IIA superstring (RR potentials will be denoted by curl letters in the following). Going to the string frame and Poisson-resumming on $n^{11} \to m$ (the details of the procedure are explained at length in Ref. [10]), the amplitude (1) can be expanded at weak coupling as

$$A_4 = 2\zeta(3)\mathcal{V}e^{-2\phi} + 2\mathcal{V}\hat{\sum_{n^i}}\frac{1}{n^i g_{ij} n^j} + 4\pi\mathcal{V}e^{-\phi}\hat{\sum_{m}}\hat{\sum_{n^i}}\frac{|m|}{\sqrt{n^i g_{ij} n^j}}K_1\left(2\pi e^{-\phi}|m|\sqrt{n^i g_{ij} n^j}\right)e^{2\pi i m n^i \mathcal{A}_i} \,. \tag{3}$$

In the above expression, $\mathcal{V}$ is the volume of the N-torus in string units, $\zeta(3)$ is Apery's transcendental number and $K_s(z)$ is the Bessel K function, which for a large argument approximates to

$$\frac{|m|}{\sqrt{n^i g_{ij} n^j}}K_1\left(2\pi e^{-\phi}|m|\sqrt{n^i g_{ij} n^j}\right) \simeq \frac{(\pi|m|)^{1/2}}{(n^i g_{ij} n^j)^{3/2}}e^{-2\pi e^{-\phi}|m|\sqrt{n^i g_{ij} n^j}}\left(1 + O(e^{\phi})\right) \,. \tag{4}$$

The expansion in Eq. (3) precisely displays the tree-level and one-loop field-theoretical perturbative contributions to R^4 couplings, together with a sum of non-perturbative instantons that can be interpreted as D0-branes (or D-particles) whose Euclidean world-line winds on minimal cycles of the N-dimensional compactification torus. This is hardly surprising, given the fact that the type IIA field theory is the dimensional reduction of 11D supergravity under which the D0-branes are the Kaluza-Klein modes [11]. Indeed, the action of the instantons

$$S_{cl} = e^{-\phi}\sqrt{n^i g_{ij} n^j} + i\, n^i \mathcal{A}_i \tag{5}$$

is precisely the Born-Infeld action of a D0-brane wrapped on a cycle $\sum n^i \gamma_i$ of the N-torus. The integer charge m can be interpreted as the number of D0-brane bound together, and emerges as the momentum of the supergraviton along the eleven-dimensional circle. Moreover, Eq. (4) shows that each D0-instanton background receives an infinite number of perturbative corrections, whose coefficients are easily obtained from the asymptotics of the K_1 Bessel function.

Equation (3) therefore gives a precise prescription for including the effect of D0-brane instantons, at least for the case of the above R^4 couplings. Even if one could have guessed the form of the instanton sum, it is by no means clear how one could have obtained the precise summation prescription including the "zero-mode" factors in Eq. (4) from first principles, not to mention the perturbative corrections around the instanton background. Note that Eq. (3) also leaves room for interpretation, since we could rewrite it as a sum over the winding numbers $m^i = m\, n^i$ and thereby obscure the role of bound states of D0-branes. Note however that this would introduce a Jacobian $\mu(\{m^i\}) = \sum_{D|m^i} 1$, whereas we would not expect arithmetic functions to enter instanton calculus when formulated in terms of the natural objects. This "naturality" argument is a useful guideline in understanding instanton calculus rules.

Here we want to generalize this result and investigate the form of higher-dimensional D-brane corrections, which we will obtain by T-duality from the above result. It will be sufficient for our purposes to use a sequence of T-dualities on one cycle (say the first direction) of the compactification torus. To do this, it is convenient to decompose the N-dimensional torus as a $U(1)$ fibration:

$$ds^2 = R^2(dx^1 + A_a dx^a)^2 + dx^a g_{ab} dx^b \ , \quad B_a = B_{1a} \ , \tag{6}$$

where B_{ij} denotes the Neveu-Schwarz (NS) two-form. T-duality in the NS sector takes the well-known form:

$$R \leftrightarrow 1/R \ , \quad A_a \leftrightarrow B_b \ , \quad B_{ab} \leftrightarrow B_{ab} - A_a B_b + B_a A_b \ , \quad e^{-2\phi} R = \text{const.} \ , \tag{7}$$

mapping IIA to IIB. Note that in contrast to usual practice we do *not* canonically reduce the NS two-form B_{ab} on the first circle, so that our B_{ab} is not inert under T-duality. In order to write down the action on the RR gauge potentials, it is convenient to group them into an inhomogeneous differential form of even or odd degree:

$$\mathcal{R} = \sum \mathcal{R}_\alpha = \left[\begin{array}{ll} \mathcal{A}_i dx^i + \mathcal{C}_{ijk} dx^i \wedge dx^j \wedge dx^k + \dots \ , & \text{type IIA} \\ a + \mathcal{B}_{ij} dx^i \wedge dx^j + \mathcal{D}_{ijkl} dx^i \wedge dx^j \wedge dx^k \wedge dx^l + \dots \ , & \text{type IIB} \end{array} \right. \tag{8}$$

The action of T-duality can now be written as:

$$\mathcal{R} \leftrightarrow 1 \cdot \mathcal{R} + 1 \wedge \mathcal{R} \tag{9}$$

where the operators $1\cdot$ and $1\wedge$ are the interior and exterior products with the first direction, for instance

$$1 \cdot \mathcal{C} = \mathcal{C}_{1ij} dx^i \wedge dx^j \ , \quad 1 \wedge \mathcal{C} = \mathcal{C}_{ijk} dx^1 \wedge dx^i \wedge dx^j \wedge dx^k \tag{10}$$

In particular, Dp-brane states charged under the RR $(p+1)$-form potential are mapped to states charged under both the p- and $(p+2)$- forms of the dual theory, therefore to a superposition of D$(p-1)$- and D$(p+1)$-branes. Note that Eq. (10) holds only at zeroth order in the NS 2-form. Indeed, the lower components of the RR fields $\mathfrak{a}$ and $\mathcal{A}_i$ have (at the perturbative level) Peccei–Quinn symmetries, and should therefore be mapped to fields with a Peccei–Quinn symmetry as well. However, a constant $(SL(2,\mathbb{R}))$ shift $\mathfrak{a} \to \mathfrak{a} + c$ of the type IIB RR scalar has to be accompanied by a transformation of the RR two-form $\mathcal{B} \to \mathcal{B} - cB$, so that only $\tilde{\mathcal{B}} = \mathcal{B} + \mathfrak{a} B$ has a Peccei–Quinn symmetry. The correct mapping is therefore $\mathcal{A}_a \to \tilde{\mathcal{B}}_{1a}$, and $\mathcal{B}_{ab} \to \mathcal{C}_{1ab}$. A similar correction occurs in the $\mathcal{C} \to \tilde{\mathcal{D}} = \mathcal{D} + B \wedge \mathcal{B} + \mathfrak{a} B \wedge B$ transformation.

Our first aim is to study the action of T-duality on the classical action of the D0-brane of Eq. (5). Upon dualizing the first direction, we obtain

$$S_{cl} \to e^{-\phi} \sqrt{(n^1 + B_{1a} n^a)^2 + n^a \ g_{11} g_{ab} \ n^b} + i \ (n^1 \mathfrak{a} + n^a \tilde{\mathcal{B}}_{1a}) \ . \tag{11}$$

As it stands, this result is definitely not invariant under $SL(N,\mathbb{Z})$ reparametrizations of the N-torus. It can however be made invariant by reinterpreting n^1 as a scalar charge n, and introducing a two-form integer charge $n^{ij} = -n^{ji}$ of which n^a is simply the component n^{1a}. The action (11) then takes the form

$$S_{cl} = e^{-\phi} \sqrt{\left(n + \frac{1}{2} n^{ij} B_{ij}\right)^2 + \frac{1}{2} n^{ij} \ g_{ik} g_{jl} \ n^{kl}} + i \ (n \mathfrak{a} + \frac{1}{2} n^{ij} \tilde{\mathcal{B}}_{ij}) \ , \tag{12}$$

and states corresponding to images of D0-branes under T-duality on the first circle correspond to $n^{ij} = 0$ except for $i = 1$ or $j = 1$. This condition can be cast in a more intrinsic form by noting that states obtained from the D0-brane by T-duality on *any* circle are such that Rank $n^{ij}=2$. Therefore T-duality a priori only requires a sum over n, n^{ij} such that n^{ij} has rank two (at most). Note that this restriction is immaterial for $N \leq 3$.

We now show that S_{cl} has a natural interpretation as the Born–Infeld action of the type IIB D-string wrapping on two-cycles of the internal torus. The D-string is described by N embedding coordinates X^i together with a $U(1)$ gauge field A_α living on the two-dimensional world-volume. Supersymmetric mappings of a two-torus to a N-torus are described by a set of $2N$ integers N^i_α:

$$X^i = N^i_\alpha \sigma^\alpha \ , \tag{13}$$

where σ^α are the coordinates on the D-string worldsheet torus. The gauge field in two dimensions consists only of its zero-mode part, and its curvature, being the first Chern class of a U(1) bundle, has to have integral flux:

$$F_{\alpha\beta} = n \ \epsilon_{\alpha\beta} \ . \tag{14}$$

We can therefore evaluate the Born–Infeld action on this configuration and find (hatted quantities are pulled back from target space to the world-volume):

$$\int e^{-\phi}\sqrt{\det(\hat{G}+\hat{B}+F)} = e^{-\phi}\sqrt{\left(n+\frac{1}{2}n^{ij}B_{ij}\right)^2 + \frac{1}{2}n^{ij}\ g_{ik}g_{jl}\ n^{kl}} \ , \tag{15}$$

in precise agreement with Eq. (12), upon identifying

$$n^{ij} = \epsilon^{\alpha\beta} N^i_\alpha N^j_\beta \ . \tag{16}$$

The integer two-form n^{ij} is independent from the parametrization of the D-string world-sheet, and describes the homology class of the two-cycle inside the N-torus. Note in particular that Rank $n^{ij}=2$. Moreover, the phase in Eq. (12) is easily seen to be reproduced by the topological coupling on the D-string world-sheet [12]:

$$\int e^{\hat{B}+F} \wedge \hat{\mathcal{R}} = \int \left(\hat{\mathcal{B}} + a(\hat{B}+F)\right) = n a + \frac{1}{2} n^{ij} \tilde{\mathcal{B}}_{ij} \ . \tag{17}$$

This coupling here appears as a simple consequence of T-duality. Setting $n^{ij}=0$, the action (12) reduces to n times the action of a D-instanton $S_{cl}=\mathcal{a}+i\ e^{\phi}$. The integer flux n can therefore be identified with the D-instanton charge, and the D-string with $n\neq 0$ as a "bound state" [1] of a D-string with n D-instantons. This is closely related to proposals for bound states of Dp- and D$(p+4)$-brane [12, 13], but here occurs between Dp- and D$(p+2)$-branes due to the existence of nontrivial fluxes on a torus. The integer m again corresponds to the number of D-strings bound together, and there is no sign of the non-abelian nature of the interaction [13] in our result.

The sum over D0-branes winding around the cycles of the compactification manifold therefore implies by T-duality a sum over D-strings wrapping the two-cycles of the same:

$$A_4^{D1}=4\pi\mathcal{V}e^{-\phi}\hat{\sum_m}\hat{\sum_{n,n^{ij}}}\frac{|m|}{\sqrt{\det(\hat{G}+\hat{B}+F)}}K_1\left(2\pi|m|e^{-\phi}\int\sqrt{\det(\hat{G}+\hat{B}+F)}\right)e^{2\pi im\int\tilde{\mathcal{B}}+\mathcal{a}F}\tag{18}$$

As in the case of the D0-brane, by virtue of the asymptotic expansion of the K_1 Bessel function, this sum exhibits an infinite series of perturbative corrections around each D-string background. The interpretation of the "zero-mode" part in front of $e^{-2\pi S_{cl}}$ is by no means clear at this point. However, it becomes transparent by going to an alternative description, namely a sum over (p,q) string world-sheet instantons, which was the object of Ref. [10]. This description is obtained by performing a Poisson resummation over the D-instanton flux n (this is similar to the transformation from an instanton vacuum to a θ vacuum, but for the fact that the D-instanton is really a flux and not a gauge instanton). Under this operation, the Bessel function $K_1(z)$ turns into $K_{1/2}(z)=e^{-z}\sqrt{\frac{\pi}{2z}}$, and we find

$$A_4^{D1}=4\pi\mathcal{V}\hat{\sum_l}\sum_{p\wedge q=1}\sum_{\text{Rank }n^{ij}=2}\frac{e^{-2\pi l|p+q\tau|\sqrt{n^{ij}\ g_{ik}g_{jl}\ n^{kl}}+2\pi iln^{ij}(q\mathcal{B}_{ij}-pB_{ij})}}{\sqrt{n^{ij}\ g_{ik}g_{jl}\ n^{kl}}}\tag{19}$$

where $\tau=\mathcal{a}+ie^{\phi}$ is the type IIB $SL(2,\mathbb{Z})$ modulus. The term with $(p,q)=(1,0)$

[1] We shall make a rather loose use of the term "state", keeping with the philosophy that instanton effects in dimension D can be seen as loops of physical states in dimension $D+1$. This seems to fail for the case of the D-instantons.

corresponds to the one-loop world-sheet instantons on the fundamental string:

$$A_4^{(1,0)} = 4\pi\mathcal{V}\bar{\sum_{N_\alpha^i}} \frac{e^{-2\pi\sqrt{d^{ij}\ g_{ik}g_{jl}\ n^{kl}}-2\pi i d^{ij}B_{ij}}}{\sqrt{d^{ij}\ g_{ik}g_{jl}\ d^{kl}}}\ , \tag{20}$$

where $d^{ij}=\epsilon^{\alpha\beta}N_\alpha^i N_\beta^j$ and the sum runs over the different $SL(2,\mathbb{Z})$ orbits of N_α^i such that the d_{ij} are not all zero. Indeed, it can be checked that the number of orbits corresponding to a given rank-2 set of d_{ij} with greatest common divisor D is $\sum_{l|D} l$ (this generalizes the result obtained in Ref. [10] for the particular cases of $N=2,3$); the sum over the integers N_α^i modulo $SL(2,\mathbb{Z})$ can then be traded for a sum over l and the rank 2 $n_{ij}=d_{ij}/l$ integer matrix, with Jacobian l. This indeed reproduces Eq. (19), and justifies the rather mysterious "zero-modes" coefficients. The (p,q) string therefore appears as a *coherent superposition* of p D-strings with an arbitrary number of D-instantons, and not as a superposition of p D-string and q D-instantons as one might have naively guessed. Moreover, the (p,q) string background appears to generate *no* perturbative corrections, in contrast to what occurs around a D-string background in Eq. (18). These perturbative corrections are effectively summed up by going to the "θ vacuum".

Having obtained the type IIB D-string instanton effects from the knowledge of the type IIA D0-brane contribution, we now want to investigate higher-brane effects in type IIA by applying one further T-duality. A generic D-string–D-instanton configuration will now be mapped to a superposition of D0-branes that we started with and D2-branes, which we shall again discover by covariantizing the result under the reparametrization group of the N-torus. Upon T-dualization of the first circle, the action (15) turns into

$$\begin{aligned} S_{cl} &= e^{-\phi}\Bigg(R^2\left[n+A_a n^{1a}+\frac{1}{2}(B_{ab}-A_aB_{1b}+A_bB_{1a})n^{ab}\right]^2+\frac{1}{2}R^2 n^{ab}g_{ac}g_{bd}n^{cd} \\ &\quad +(n^{1a}+n^{ab}B_{1b})g_{ac}(n^{1c}+n^{cd}B_{1d})\Bigg)^{1/2}+i\ \left(n\mathcal{A}_1+n^{1a}\mathcal{A}_a+\frac{1}{2}n^{ab}\mathcal{C}_{1ab}\right) \end{aligned} \tag{21}$$

Defining $n^1=n, n^i=n^{1i}$ and introducing the three-form integer charge n^{ijk} such that $n^{1jk}=n^{ij}$, we can rewrite the above action as

$$S_{cl} = e^{-\phi}\sqrt{\left(n^i+\frac{1}{2}n^{ijk}B_{jk}\right)g_{il}\left(n^l+\frac{1}{2}n^{lmn}B_{mn}\right)+\frac{1}{6}n^{ijk}\ g_{il}g_{jm}g_{kn}\ n^{lmn}}$$

$$+i\ \left(n^i \mathcal{A}_i + \frac{1}{6} n^{ijk} \mathcal{C}_{ijk}\right) \tag{22}$$

When $n^{ijk} = 0$, we recover the action for the D0-brane we started with. $n_{ijk} \neq 0$ on the other hand corresponds to states charged under the type IIA RR three-form, therefore to D2-brane states. The transformation of the integer charges can be conveniently summarized by defining an integer inhomogeneous antisymmetric form

$$\mathcal{N} = \sum \mathcal{N}_\alpha = \begin{bmatrix} n^i dp_i + n^{ijk} dp_i \wedge dp_j \wedge dp_k + \dots\ , & \text{type IIA} \\ n + n^{ij} dp_i \wedge dp_j + n^{ijkl} dp_i \wedge dp_j \wedge dp_k \wedge dp_l + \dots\ , & \text{type IIB} \end{bmatrix} \tag{23}$$

which transforms under T-duality in the same way as $\mathcal{R}$, namely $\mathcal{N} \leftrightarrow 1 \cdot \mathcal{N} + 1 \wedge \mathcal{N}$. We note that the set of charges obtained by T-duality from Eq. (15) satisfies the conditions Rank $\mathcal{N}_3 = 3$ (*i.e.* $\mathcal{N}_3 \wedge \mathcal{N}_3 = 0$) and $\mathcal{N}_1 \wedge \mathcal{N}_3 = 0$. The imaginary coupling to RR gauge potentials can now be written as $\mathcal{N} \cdot \mathcal{R}$ and is now obviously invariant under T-duality.

We now would like, in the same spirit as before, to identify the actual D2-brane configuration corresponding to Eq. (22). The wrappings of the D2-brane world-volume on the compactification N-torus are now described by a set of $3N$ integers N^i_α (where now $\alpha = 1 \dots 3$), transforming as N triplets under the reparametrization group $SL(3, \mathbb{Z})$ of the three-torus. Evaluating the Born–Infeld action for this configuration leads precisely to the action (22), upon identifying

$$n^{ijk} = \epsilon^{\alpha\beta\gamma} N^i_\alpha N^j_\beta N^k_\gamma\ ,\ \ n^i = \frac{1}{2} \epsilon^{\alpha\beta\gamma} N^i_\alpha F_{\beta\gamma}\ . \tag{24}$$

This identification is in perfect agreement with the conditions Rank $\mathcal{N}_3 = 3, \mathcal{N}_1 \wedge \mathcal{N}_3 = 0$. Again, the D0-brane appears as a non-trivial flux of the $U(1)$ gauge field on the D2-brane world-volume. As a side remark, we note that the world-sheet coupling yielding the imaginary part of Eq. (22) is $\int \hat{\mathcal{C}} + F \wedge \hat{\mathcal{A}}$, and *not* $\int \hat{\mathcal{C}} + (F + \hat{B}) \wedge \hat{\mathcal{A}}$ as claimed in Ref. [12]. The latter would conflict not only with T-duality but also with gauge invariance, since a gauge transformation of the NS two-form $B \to B + d\lambda$ has to be accompanied with a transformation of the RR three-form $\mathcal{C} \to \mathcal{C} + \mathcal{A} \wedge d\lambda$, as required by the common eleven-dimensional origin of $B = \mathcal{C}^{(11)}_{11ij}$ and $\mathcal{C} = \mathcal{C}^{(11)}_{ijk} + \mathcal{A}_i B_{jk} + \mathcal{A}_j B_{ki} + \mathcal{A}_k B_{ij}$.

We furthermore obtain the precise summation prescription:

$$A_4^{D2} = 4\pi\mathcal{V}e^{-\phi}\hat{\sum_m}\hat{\sum_{n^i,n^{ijk}}}\frac{|m|}{\sqrt{\det(\hat{G}+\hat{B}+F)}}K_1\left(2\pi|m|e^{-\phi}\int\sqrt{\det(\hat{G}+\hat{B}+F)}\right)e^{2\pi i m\int\hat{\mathcal{C}}+\hat{\mathcal{A}}\wedge F} \tag{25}$$

At this point, we have to ask whether this result is consistent with U-duality. In particular, seen as a M-theory coupling, it should be invariant under $SL(N+1,\mathbb{Z})$ reparametrizations of the $(N+1)$-torus, as was the case for the D0-brane contribution of Eq. (1), obtained by Poisson resummation on $m \to n^{11}$ from the loop amplitude Eq. (3). We therefore carry out the same operation on Eq. (25), and obtain:

$$A_4^{D2} = 2\pi\mathcal{V}_{11}\int_0^\infty \frac{dt}{t^{5/2}}\hat{\sum} e^{-\frac{\pi}{t}\mathcal{M}^2} \ , \tag{26}$$

with

$$\begin{aligned}\mathcal{M}^2 \ &= \ R_{11}^2\left(n^{11}+\mathcal{A}_i n^i+\frac{1}{6}n^{ijk}\tilde{\mathcal{C}}_{ijk}\right)^2+\left(n^i+\frac{1}{2}n^{ijk}B_{jk}\right)\frac{g_{il}}{R_{11}}\left(n^l+\frac{1}{2}n^{lmn}B_{mn}\right)\\ &\quad +\frac{R_{11}^2}{6}n^{ijk}\ \frac{g_{il}g_{jm}g_{kn}}{R_{11}^3}\ n^{lmn} \ .\end{aligned} \tag{27}$$

By a now familiar line of reasoning, we discover that eleven-dimensional covariance forces us to introduce the integer four-form $n^{11ijk}=n^{ijk}$ in terms of which

$$\mathcal{M}^2 = \left(n^I+\frac{1}{6}n^{IJKL}\mathcal{C}^{(11)}_{JKL}\right)g_{IM}\left(n^M+\frac{1}{6}n^{MNPQ}\mathcal{C}^{(11)}_{NPQ}\right)+\frac{1}{24}n^{IJKL}g_{IM}g_{JN}g_{KP}g_{LQ}n^{MNPQ} \ . \tag{28}$$

For $N=3$, that is for seven-dimensional type IIA string theory, the above expression is simply a rewriting of Eq. (27) which makes $SL(3+1,\mathbb{Z})$ invariance manifest. Quite satisfyingly, it is also invariant under the *full* $SL(5,\mathbb{Z})$ U-duality group. Indeed, it can be checked (for instance by applying a T-duality on the type IIB $SL(5,\mathbb{R})$ symmetric matrix in Eq. (5.12) of Ref. [10]) that $\mathcal{M}^2$ in Eq. (28) is the norm of the integer vector $(n^1,n^2,n^3,n^4,n^{1234})$ under the quadratic form parametrizing the scalar manifold $SL(5,\mathbb{R})/SO(5)^2$. The R^4 coupling

[2] up to a factor $\mathcal{V}_{11}^{-4/5}$, that would appear by translating Eq. (26) to the Einstein frame.

in Eq. (26) can therefore be expressed as a weight-3/2 Eisenstein series for $SL(5,\mathbb{Z})$, as already found in Ref. [10] from the type IIB point of view.

For $N \geq 3$ however, we find *more* integer charges in Eq. (28) than expected from D0- and D2-brane states in Eq. (27). For six-dimensional type IIA string theory ($N=4$), the D-brane charges n^{11}, n^i, n^{ijk} together with the extra integer n^{1234} fill in a **10** representation $(n^I, m_I = \epsilon_{IJKLM} n^{JKLM})$ of the $SO(5,5,\mathbb{Z})$ U-duality group, in such a way that $\mathcal{M}^2$ is duality invariant. The R^4 amplitude is therefore again given by a weight-3/2 Eisenstein series for $SO(5,5,\mathbb{Z})$, proving the conjecture in Ref. [10]. The correct interpretation of states with non-zero n^{ijkl} is still missing at this stage. It can easily be checked that they give rise to non-perturbative effects of order e^{-1/λ^2}, much smaller that the effects of D-brane instantons. They however appear on the same footing as D2-branes from the eleven-dimensional point of view, and presumably also originate from the M-theory membrane, although their quantum numbers n^{IJKL}, n^I would be more suggestive of a 3+1 extended object with a three-form on the world-volume.

Leaving these exotic states aside for now, we can pursue our line of reasoning one step further, and obtain the D3-brane contribution by T-duality from the D2-brane result in Eq. (22). Again, one is led to introduce an integer four-form $n^{1jkl} = n^{jkl}$, and finds

$$\begin{aligned} S_{cl} \;=\; & e^{-\phi} \left[\left(n + \frac{1}{2} n^{ij} B_{ij} + \frac{1}{8} n^{ijkl} B_{ij} B_{kl} \right)^2 + \frac{1}{2} \left(n^{ij} + \frac{1}{2} n^{ijkl} B_{kl} \right) g_{im} g_{jn} \left(n^{mn} + \frac{1}{2} n^{mnpq} B_{pq} \right) \right. \\ & \left. + \frac{1}{24} n^{ijkl} g_{im} g_{jn} g_{kp} g_{lq} n^{mnpq} \right]^{1/2} + i \left(n a + \frac{1}{2} n^{ij} \tilde{\mathcal{B}}_{ij} + \frac{1}{24} n^{ijkl} \tilde{\mathcal{D}}_{ijkl} \right) , \end{aligned} \tag{29}$$

together with the constraints $\mathcal{N}_2 \wedge \mathcal{N}_2 = \mathcal{N}_0 \mathcal{N}_4$ and $\mathcal{N}_2 \wedge \mathcal{N}_4 = 0$. This turns out to be the effective action of a D3-brane with the identifications

$$n^{ijkl} = \epsilon^{\alpha\beta\gamma\delta} N^i_\alpha N^j_\beta N^k_\gamma N^l_\delta \ , \quad n^{ij} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} N^i_\alpha N^j_\beta F_{\gamma\delta} \ , \quad n = \frac{1}{8} \epsilon^{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta} \ , \tag{30}$$

The above constraints generalize the condition Rank $\mathcal{N}_2 = 0$, *i.e.* $\mathcal{N}_2 \wedge \mathcal{N}_2 = 0$, that we previously found in the absence of D3-branes. The D-string therefore appears as a flux on the D3-brane, while the D-instanton is nothing but a gauge instanton in the world-volume gauge theory[3]. This equivalence between topological invariants of a gauge bundle and wrapping of D-branes is simply a reflection of the isomorphism between the integer cohomology lattice, in which the characteristic classes take their values, and the integer homology lattice, which describes the possible wrappings of extended objects on the manifold.

We could reiterate this reasoning a few times in order to obtain the contributions of D-4 and higher-branes, but the pattern is by now clear, and the generalization of Eqs. (22) and (29) obvious. Note moreover that for any finite value of N, the process terminates and yields a result invariant under T-duality. However, it is by no means guaranteed to be invariant under U-duality. In fact, just as in the type IIA case, the $SL(2,\mathbb{Z})_\tau$ symmetry of type IIB string theory forces us to introduce extra states, that we can simply obtain by applying a T-duality on the type IIA states in Eq. (28):

$$\begin{aligned}\mathcal{M}^2 \;=\; & \frac{e^{2\phi}}{\mathcal{V}}\Big(m + a n + \frac{1}{2}n^{ij}\left(\mathcal{B}_{ij} + a B_{ij}\right) + \frac{1}{24}\left(n^{ijkl} + a m^{ijkl}\right)\mathcal{D}_{ijkl} \\ & \qquad + \frac{1}{8}\left(n^{ijkl}B_{ij} - m^{ijkl}\mathcal{B}_{ij}\right)\left(\mathcal{B}_{kl} + a B_{kl}\right)\Big)^2 \\ & + \frac{1}{\mathcal{V}}\left(n + \frac{1}{2}n^{ij}B_{ij} + \frac{1}{24}m^{ijkl}\mathcal{D}_{ijkl} + \frac{1}{8}\left(n^{ijkl}B_{ij} - m^{ijkl}\mathcal{B}_{ij}\right)B_{kl}\right)^2 \\ & + \frac{1}{\mathcal{V}}\left(n^{ij} + \frac{1}{2}\left(n^{ijkl}B_{kl} - m^{ijkl}\mathcal{B}_{kl}\right)\right) g_{im}g_{jn}\left(n^{mn} + \frac{1}{2}\left(n^{mnpq}B_{pq} - m^{mnpq}\mathcal{B}_{pq}\right)\right) \\ & + \frac{1}{\mathcal{V}}\left(n^{ijkl} + a m^{ijkl}\right) g_{im}g_{jn}g_{kp}g_{lq}\left(n^{mnpq} + a m^{mnpq}\right) + \frac{e^{-2\phi}}{\mathcal{V}} m^{ijkl} g_{im}g_{jn}g_{kp}g_{lq} m^{mnpq}\end{aligned} \tag{31}$$

In the above expression, n, n^{ij}, n^{ijkl} are the D-brane charges in Eq. (29), m is the Poisson dual of the integer m in Eq. (18), *i.e.* the analog of n^{11}, and m^{ijkl} is the image under T-duality of the type IIA exotic four-form n^{ijkl}. Note that in contrast to D-brane charges, the rank of this form is not changed under T-duality. Under $SL(2,\mathbb{Z})_\tau$ duality, the following

[3] Actually, anti-self-dual $U(1)$ connections do not exist on a torus with a generic flat metric. What we really mean here is that the D-instanton number appears as the first Pontryagin number of the $U(1)$ bundle on the 3-brane world-volume. I am grateful to P. van Baal for correspondence on this subject.

quantities transform as a doublet:

$$\begin{pmatrix} \mathcal{B}_{ij} \\ B_{ij} \end{pmatrix} , \quad \begin{pmatrix} m \\ n \end{pmatrix} , \quad \begin{pmatrix} n^{ijkl} \\ m^{ijkl} \end{pmatrix} , \tag{32}$$

while $\mathcal{D}_{ijkl}, \nu = e^{\phi}/V^{4/N}$ and $\tilde{g}_{ij} = V^{-2/N} g_{ij}$ are inert. Eq. (31) is manifestly invariant under these combined transformations. Ironically, we find that the description of the type IIB three-brane, claimed to be a singlet under $SL(2,\mathbb{Z})$, requires the introduction of a doublet of wrapping charges, while the description of the (p,q) string only requires a singlet of wrapping charges n^{ij} (together with the doublet m,n). Moreover, the action of $SL(2,\mathbb{Z})_\tau$ duality looks very different from the electric-magnetic duality considered in Ref. [14]. It would be very interesting to understand Eq. (31) from brane dynamics.

In this letter, we have obtained the explicit form of contributions of D-brane instantons to a particular coupling, namely R^4 in $N = 8$ type II superstrings. This coupling is related by supersymmetry to terms with sixteen fermions, and therefore can only receive contributions from BPS states breaking one half of the supersymmetry. Starting from the D0-brane contribution, we found by T-duality contributions from D-branes wrapping supersymmetric cycles of the compactification manifold, and obtained a definite rule for weighting these objects. Several points would deserve clarification.*Primo*, it is not clear why the Russian doll structure we exhibited, in which Dp-branes are fluxes in D$(p+2)$-branes, instantons in D$(p+4)$-branes, etc., does not further break supersymmetry. *Secundo*, it seems that for describing the bound states of these objects only the $U(1)$ gauge field on the world-volume is relevant, whereas these bound states should in principle correspond to normalizable ground states of a $U(N)$ gauge theory on the world-volume. *Tertio*, the rule for summing higher-brane contributions, namely summing over the cycles $n^{ijk\cdots}$ on which the p-brane wraps, appears to differ from the rule for the fundamental string, where one sums over the winding numbers N^i_α modulo the action of the mapping class group $SL(p+1,\mathbb{Z})$ $(p = 1)$. The two rules differ by a Jacobian which involves arithmetic functions of the $n^{ijk\cdots}$, and in this sense, using the fundamental string rule in the Dp-brane case would be unnatural, while it is natural in the (p, q) string case. *Finally*, one should ask how these rules carry over to more general couplings. Couplings related by supersymmetry will definitely exhibit the same instanton contributions, but with different vertex insertions modifying the prefactor in Eq. (4). Couplings related by supersymmetry to terms with more than 16 fermions will receive contributions from BPS instantons breaking more that half of the supersymmetries, together with the present ones. It would be interesting to obtain the prefactors of these instanton effects from a careful string computation. Eventually, one would also like to reproduce these results from Matrix Theory[15].

Furthermore, we have shown that U-duality forces the inclusion of extra states beyond the familiar D-branes. On the type IIA side, these states appear as D2-branes boosted along the eleventh dimension, whereas on the type IIB they are obtained by a $SL(2,\mathbb{Z})_\tau$ duality transformation from the D3-branes. A distinct feature of these states is that their contributions scale as e^{-1/λ^2}, and are much smaller than the usual D-branes. It would however be very interesting to elucidate their nature.

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