

CERN-TH/97-263

Hamilton-Jacobi Approach to Pre-Big Bang Cosmology at long-wavelengths

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Abstract

We apply the long-wavelength approximation to the low-energy effective string action in the context of Hamilton-Jacobi theory. The Hamilton-Jacobi equation for the effective string action is explicitly invariant under scale factor duality. We present the leading order, general solution of the Hamilton-Jacobi equation. The Hamilton-Jacobi approach yields a solution consistent with the one obtained by Lagrange formalism. The momentum constraints take an elegant, simple form. Furthermore, this general solution reduces to the quasi-isotropic one, if the evolution of the gravitational radiation is neglected. Duality transformation for the general solution is written as a coordinate transformation in an abstract field space.

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*This work is dedicated to K. Atılan.

†After 10 Sep. 1997.

I. INTRODUCTION

In this paper, we apply the long wavelength approximation to the low-energy effective string action in the context of Hamilton-Jacobi theory.

The correct theory of quantum gravity is generally believed to be string theory. The low energy effective action of string theory has the form of a Brans-Dicke (BD) action with parameter $\omega_{BD} = -1$. Duality symmetries play an important role in string theory. In the Pre-Big Bang scenario [1] scale factor duality associates inflationary solutions to non-inflationary ones, [2]. By this property it has become an alternative to standard inflationary cosmology.

Long-wavelength gravity (gradient expansion) has proved to be a fruitful method for studying (slightly) inhomogeneous fields. It is a significant improvement over those of homogeneous minisuperspace. The main idea of this scheme is that when the scale of spatial variations of the fields are larger than the Hubble radius, one can solve the equations neglecting the second-order spatial gradients. It was first introduced by E. M. Lifschitz and I. M. Khalatnikov [3]. Later, Tomita developed this approximation as the Anti-Newtonian scheme [4]. For General Relativity, it was formulated either directly in terms of lagrange equations [5], or in the framework of Hamilton-Jacobi equation [6,7]. It was also studied for the BD theory in the Hamilton-Jacobi framework [8].

Recently, an inhomogeneous version of Pre-Big Bang cosmology has been investigated using the lagrange equations [9,10]. In this scenario, the universe, with very perturbative (i.e. weak coupling and very small curvature) but otherwise arbitrary initial conditions is followed towards the Big Bang singularity in the future. Quasi-homogeneous regions, which exhibit Pre-Big Bang behaviour, eventually fill almost all of space, and within these regions the Universe appears homogeneous, flat, and isotropic [9].

The purpose of the present paper is two-fold. First we solve the long-wavelength problem for low-energy string cosmology in the framework of Hamilton-Jacobi (HJ) equation. We also show that the HJ equation is invariant under scale factor duality (SFD) transformation.

We work directly in the physical string frame. The HJ approach has importance for quantum cosmology since it is the lowest order equation in the WKB approximation of Wheeler-De Witt equation. Secondly, following Salopek [6], we write the momentum constraints in a simple, elegant form.

In section II, we write the action in the hamiltonian form and we derive the equations for the fields and for their conjugate momenta. The action also gives rise to Hamiltonian and the momentum constraints. For completeness, we also give a brief summary of the canonical transformations via the generating functional technics. In section III, we write the Hamilton-Jacobi equation, and we show that it is invariant under SFD transformation. The solutions represent a universe evolving towards a Big Bang singularity in the future. Section IV includes the most general solution near a singularity. In this section, the generating functional is taken as a function of both the dilaton and the metric. This dependence is chosen in such a way that will effectively decompose the gravitational momentum tensor into a trace contribution and a traceless part. Furthermore, with the help of this choice, the Hamilton-Jacobi equation reduces to that of massless scalar fields. We write the momentum constraint in a simple form. The momentum constraints state that the generating functional is invariant under spatial coordinate transformations. We write them in terms of the new canonical variables. Then in section V, we present a quasi-isotropic solution of Pre-Big Bang cosmology. The general solution reduces to the quasi-isotropic one, if the evolution of the gravitational radiation is neglected. In section VI, we briefly discuss duality transformation for the general solution. We represent the evolution of the universe in a space of fields where the duality transformation can be written as the transformation of an angle in a suitable plane.

II. HAMILTON FORMALISM AND THE CANONICAL TRANSFORMATIONS

The low-energy effective string action, in the string frame, is

$$\Gamma = \int e^{-\phi} (R + g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi) \sqrt{-g} d^4x \quad , \quad (1)$$

where R is the scalar curvature, ϕ is the dilaton. We set to zero the antisymmetric tensor field $B_{\mu\nu}$. The HJ equation for this action can be obtained using the ADM formalism in which the space-time is foliated by space-like hypersurfaces. In the ADM formalism the metric is parametrized as

$$ds^2 = -N^2(t)dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \quad , \quad (2)$$

where N and N^i are the lapse and shift functions respectively, and γ_{ij} is the 3-metric. We work in the synchronous gauge $N = 1$, $N^i = 0$.

The action written in hamiltonian form becomes,

$$\Gamma = \int (\pi^{ij} \dot{\gamma}_{ij} + \pi^\phi \dot{\phi} - N\mathcal{H} - N^i \mathcal{H}_i) d^4x \quad , \quad (3)$$

where N , and N^i act as lagrange multipliers. Their variation gives rise to the hamiltonian constraint

$$\begin{aligned} \mathcal{H} = & \gamma^{-1/2} e^\phi [\pi^{ij} \pi^{kl} \gamma_{ik} \gamma_{jl} + \frac{1}{2} (\pi^\phi)^2 + \pi \pi^\phi] \\ & - \gamma^{1/2} e^{-\phi} R - \gamma^{1/2} e^{-\phi} \gamma_{ij} \partial_i \phi \partial_j \phi + 2\gamma^{1/2} \Delta e^{-\phi} = 0 \quad , \end{aligned} \quad (4)$$

and to the momentum constraints

$$\mathcal{H}_i = -2(\gamma_{ik} \pi^{kj})_{,j} + \pi^{kl} \gamma_{kl,i} + \pi^\phi \phi_{,i} = 0 \quad . \quad (5)$$

Variation with respect to the canonical variables yield the evolution equations

$$-2K_{ij} = \frac{1}{N} (\dot{\gamma}_{ij} - N_{i;j} - N_{j;i}) = \gamma^{-1/2} e^\phi (2\pi^{kl} \gamma_{ik} \gamma_{jl} + \gamma_{ij} \pi^\phi) \quad , \quad (6)$$

$$\frac{1}{N} (\dot{\phi} - N^i \phi_{,i}) = \gamma^{-1/2} e^\phi (\pi^\phi + \pi) \quad , \quad (7)$$

$$\frac{1}{N} \left[\dot{\pi}^i_j - (N^m \pi^i_j)_{,m} + N^i_{,m} \pi^m_j - N^m_{,j} \pi^i_m \right] = \frac{1}{2} \gamma^{-1/2} e^\phi \delta^i_j \left[\pi^{mn} \pi^{kl} \gamma_{mk} \gamma_{nl} + \frac{1}{2} (\pi^\phi)^2 + \pi \pi^\phi \right] \quad , \quad (8)$$

$$\frac{1}{N} \left[\dot{\pi}^\phi - (N^i \pi^\phi)_{,i} \right] = -\gamma^{-1/2} e^\phi \left[\pi^{ij} \pi^{kl} \gamma_{ik} \gamma_{jl} + \frac{1}{2} (\pi^\phi)^2 + \pi \pi^\phi \right] \quad . \quad (9)$$

Here K_{ij} is the extrinsic curvature, which is the relevant object in lagrange formalism.

The basic idea of canonical transformations is well known [6]. One defines new fields, which we denote by tilde, so that Hamilton's equations are preserved. This implies that the new action has the same form as the original except that it may have a total time derivative added to it,

$$\Gamma = \int (\tilde{\pi}^{ij} \dot{\tilde{\gamma}}_{ij} + \tilde{\pi}^\phi \dot{\tilde{\phi}} - N\tilde{\mathcal{H}} - N^i \tilde{\mathcal{H}}_i) d^4x + \int \dot{\mathcal{S}} dt \quad , \quad (10)$$

where \mathcal{S} is a functional which depends on the old and the new field variables. It is assumed that \mathcal{S} does not depend on time explicitly. Applying the chain rule,

$$\dot{\mathcal{S}} = \int \left[\frac{\delta \mathcal{S}}{\delta \phi(x)} \dot{\phi}(t, x) + \frac{\delta \mathcal{S}}{\delta \tilde{\phi}(x)} \dot{\tilde{\phi}}(t, x) + \frac{\delta \mathcal{S}}{\delta \gamma_{ij}(x)} \dot{\gamma}_{ij}(t, x) + \frac{\delta \mathcal{S}}{\delta \tilde{\gamma}_{ij}(x)} \dot{\tilde{\gamma}}_{ij}(t, x) \right] d^3x \quad , \quad (11)$$

and comparing equation (11) with (10), one derives the canonical transformation linking the various variables,

$$\mathcal{H}(x) = \tilde{\mathcal{H}}(x) \quad , \quad \mathcal{H}_i(x) = \tilde{\mathcal{H}}_i(x) \quad (12)$$

$$\begin{aligned} \pi^\phi(x) &= \frac{\delta \mathcal{S}}{\delta \phi(x)} \quad , \quad \pi^{ij} = \frac{\delta \mathcal{S}}{\delta \gamma_{ij}(x)} \quad , \\ \tilde{\pi}^\phi(x) &= -\frac{\delta \mathcal{S}}{\delta \tilde{\phi}(x)} \quad , \quad \tilde{\pi}_{ij} = -\frac{\delta \mathcal{S}}{\delta \tilde{\gamma}_{ij}(x)} \quad . \end{aligned} \quad (13)$$

The new variables, denoted by a tilde, will be chosen so that the new Hamiltonian density vanishes strongly. Therefore they are constant in time.

III. HAMILTON-JACOBI EQUATION

The HJ equation is given by the hamiltonian constraint (4) after expressing the momenta through equation (13). In the long-wavelength limit we neglect the last three terms since they involve two spatial derivatives. We thus obtain the HJ equation in the approximate term

$$\gamma^{-1/2} e^\phi \left[\frac{\delta \mathcal{S}}{\delta \gamma_{ij}} \frac{\delta \mathcal{S}}{\delta \gamma_{kl}} \gamma_{ik} \gamma_{jl} + \frac{1}{2} \left(\frac{\delta \mathcal{S}}{\delta \phi} \right)^2 + \gamma_{ij} \frac{\delta \mathcal{S}}{\delta \gamma_{ij}} \frac{\delta \mathcal{S}}{\delta \phi} \right] = 0 \quad . \quad (14)$$

The HJ equation is interesting because of its intimate relation to quantum gravity. The Wheeler-De Witt equation, and the momentum constraint for the effective string action are given by

$$\mathcal{H}\Psi = 0 \quad , \quad \mathcal{H}_i\Psi = 0 \quad , \quad (15)$$

where the canonical commutation relations,

$$\begin{aligned} [\gamma_{ij}(x), \pi^{kl}(x')] &= \frac{i}{2}(\delta_i^k\delta_j^l + \delta_j^k\delta_i^l)\delta(x-x') \quad , \\ [\phi(x), \pi^\phi(x')] &= i\delta(x-x') \quad , \end{aligned} \quad (16)$$

are used. If we consider the WKB approximation, we get the HJ equation at the lowest order.

The HJ equation takes a simpler form by using, instead of ϕ , the shifted dilaton $\Phi = \phi - \ln \gamma^{1/2}$. We then find

$$e^\Phi \left[Tr \left(\gamma \frac{\delta \mathcal{S}}{\delta \gamma} \gamma \frac{\delta \mathcal{S}}{\delta \gamma} \right) - \frac{1}{4} \left(\frac{\delta \mathcal{S}}{\delta \Phi} \right)^2 \right] = 0 \quad . \quad (17)$$

Equation (17) is invariant under SFD transformation, which is defined as,

$$[\gamma] \longrightarrow [\gamma]^{-1} \quad , \quad \Phi \longrightarrow \Phi \quad . \quad (18)$$

Notice that, although the HJ equation is SFD invariant, the momentum constraints are not.

We can write the momentum constraint, via the equations (5) and (13)

$$\mathcal{H}_i = -2 \left(\gamma_{ik} \frac{\delta \mathcal{S}}{\delta \gamma_{kj}} \right)_{,j} + \frac{\delta \mathcal{S}}{\delta \gamma_{kl}} \gamma^{kl,i} + \frac{\delta \mathcal{S}}{\delta \phi} \phi_{,i} = 0 \quad . \quad (19)$$

They state that the generating functional is invariant under spatial diffeomorphisms.

IV. GENERAL SOLUTION NEAR A SINGULARITY

A. Ansatz for the Generating Functional

In this section, we investigate the full classical long-wavelength problem of the low energy string cosmology. The generating functional \mathcal{S} is assumed to be a function of both the scalar

field and the gravitational field. Adapting an ansatz used by Salopek [6], the dependence on the gravitational field is chosen in a specific way, without losing the generality of the solution.

We choose the ansatz given below for the lowest order generating functional,

$$\mathcal{S}^o = -2 \int e^{-\frac{3}{2}(\phi-\bar{\phi})} H(\phi, h_{ij}; \bar{\phi}, \tilde{h}_{ij}) \sqrt{\gamma} d^3x \quad , \quad (20)$$

where $h_{ij} = \gamma^{-1/3} \gamma_{ij}$, and $\tilde{h}_{ij} = \tilde{\gamma}^{-1/3} \tilde{\gamma}_{ij}$ are the unimodular conformal three-metric. Note that we introduced the $e^{-\frac{3}{2}\bar{\phi}}$ factor only for convenience, as it will become clear in the following. Since

$$\frac{\partial H}{\partial \gamma_{ij}} = \gamma^{-1/3} \left[\frac{\partial H}{\partial h_{ij}} - \frac{1}{3} \frac{\partial H}{\partial h_{kl}} h_{kl} h^{ij} \right] \quad , \quad (21)$$

We can write the HJ equation in terms of the conformal metric h_{ij} ,

$$H^2 = \frac{8}{3} \frac{\partial H}{\partial h_{ij}} \frac{\partial H}{\partial h_{kl}} \left(h_{ik} h_{jl} - \frac{1}{3} h_{ij} h_{kl} \right) + \frac{4}{3} \left(\frac{\partial H}{\partial \phi} \right)^2 \quad . \quad (22)$$

This is referred to as the separated HJ equation.

Inserting (20) in (13) we find the new and the old momenta

$$\pi^{ij} = -2\gamma^{1/2} e^{-\frac{3}{2}(\phi-\bar{\phi})} \left[\frac{1}{2} \gamma^{ij} H + \gamma^{-1/3} \left(\frac{\partial H}{\partial h_{ij}} - \frac{1}{3} \frac{\partial H}{\partial h_{kl}} h_{kl} h_{ij} \right) \right] \quad , \quad (23)$$

$$\tilde{\pi}^{ij} = 2\gamma^{1/2} \tilde{\gamma}^{-1/3} e^{-\frac{3}{2}(\phi-\bar{\phi})} \left[\frac{\partial H}{\partial \tilde{h}_{ij}} - \frac{1}{3} \frac{\partial H}{\partial \tilde{h}_{kl}} \tilde{h}_{kl} \tilde{h}_{ij} \right] \quad , \quad (24)$$

$$\pi^\phi = -2\gamma^{1/2} e^{-\frac{3}{2}(\phi-\bar{\phi})} \left[-\frac{3}{2} H + \frac{\partial H}{\partial \phi} \right] \quad , \quad (25)$$

$$\tilde{\pi}^\phi = 2\gamma^{1/2} e^{-\frac{3}{2}(\phi-\bar{\phi})} \left[\frac{3}{2} H + \frac{\partial H}{\partial \bar{\phi}} \right] \quad . \quad (26)$$

The trace of the gravitational momentum is proportional to H,

$$\pi = \pi^i_i = -3\gamma^{1/2} e^{-\frac{3}{2}(\phi-\bar{\phi})} H \quad . \quad (27)$$

and this is proportional to the integrand in equation (20).

The specific choice for the dependence of the function H on the metric, through the combination $h_{ij} = \gamma^{-1/3} \gamma_{ij}$, effectively decomposes the gravitational momentum tensor into a trace contribution and a traceless part which describes the evolution of gravitational radiation. Similarly, the new gravitational momentum tensor is traceless, $\tilde{\pi}^{ij} \tilde{\gamma}_{ij} = 0$.

Furthermore, we attempt the following solution to the separated HJ equation (22)

$$H(\phi, h_{ij}; \tilde{\phi}, \tilde{h}_{ij}) \equiv H(\phi, \tilde{\phi}, z) \quad , \quad (28)$$

where z is defined as [6],

$$z^2 = \frac{1}{2} \text{Tr} \left[\ln \left([h][\tilde{h}]^{-1} \right) \ln \left([h][\tilde{h}]^{-1} \right) \right] \quad . \quad (29)$$

Here $[h]$ and $[\tilde{h}]^{-1}$ are matrices with components h_{ij} and \tilde{h}^{ij} respectively. This variable may be thought of as the “distance”, in field space, between the old conformal metric h_{ij} and the new one \tilde{h}_{ij} , [6], [11]. We do not lose any information, because six constants of integration have been introduced through \tilde{h}_{ij} , which are sufficient to describe the dynamics of the gravitational field.

In terms of this variable, the separated Hamilton-Jacobi equation reduces to that of massless scalar fields z and ϕ ,

$$H^2 = \frac{4}{3} \left[\left(\frac{\partial H}{\partial z} \right)^2 + \left(\frac{\partial H}{\partial \phi} \right)^2 \right] \quad , \quad (30)$$

where, we used

$$\frac{\partial H}{\partial h_{ij}} = \frac{\partial H}{\partial z} \frac{\partial z}{\partial h_{ij}} = \frac{1}{2} \frac{\partial H}{\partial z} z^{-1} \left[[h]^{-1} \ln \left([h][\tilde{h}]^{-1} \right) \right]^{ij} \quad . \quad (31)$$

Similarly

$$\frac{\partial H}{\partial \tilde{h}_{ij}} = \frac{\partial H}{\partial z} \frac{\partial z}{\partial \tilde{h}_{ij}} = -\frac{1}{2} \frac{\partial H}{\partial z} z^{-1} \left[[h]^{-1} \ln \left([h][\tilde{h}]^{-1} \right) [h][\tilde{h}]^{-1} \right]^{ij} \quad . \quad (32)$$

Equations (31) and (32) yield

$$h_{ij} \frac{\partial H}{\partial h_{ij}} = 0 \quad , \quad \tilde{h}_{ij} \frac{\partial H}{\partial \tilde{h}_{ij}} = 0 \quad . \quad (33)$$

Therefore the equations for the momenta become

$$\pi^{ij} = -2\gamma^{1/2}e^{-\frac{3}{2}(\phi-\tilde{\phi})} \left[\frac{1}{2}\gamma^{ij}H + \gamma^{-1/3}\frac{\partial H}{\partial h_{ij}} \right] , \quad (34)$$

$$\tilde{\pi}^{ij} = 2\gamma^{1/2}\tilde{\gamma}^{-1/3}e^{-\frac{3}{2}(\phi-\tilde{\phi})}\frac{\partial H}{\partial \tilde{h}_{ij}} , \quad (35)$$

$$\pi^\phi = -2\gamma^{1/2}e^{-\frac{3}{2}(\phi-\tilde{\phi})} \left[-\frac{3}{2}H + \frac{\partial H}{\partial \phi} \right] , \quad (36)$$

$$\tilde{\pi}^\phi = 2\gamma^{1/2}e^{-\frac{3}{2}(\phi-\tilde{\phi})} \left[\frac{3}{2}H + \frac{\partial H}{\partial \tilde{\phi}} \right] . \quad (37)$$

Note that the new gravitational momentum is related to the old one through the reciprocity relation

$$\pi^{ij}\gamma_{jl} = \frac{1}{3}\pi\delta_l^i + \tilde{\pi}^{ij}\tilde{\gamma}_{jl} . \quad (38)$$

We will also need the equations for z and its momentum. One can write them in a manner similar to those of the dilaton.

$$\dot{z} = \gamma^{-1/2}e^\phi(\pi^z + \pi) , \quad (39)$$

$$\pi^z = -2\gamma^{1/2}e^{-\frac{3}{2}(\phi-\tilde{\phi})} \left[-\frac{3}{2}H + \frac{\partial H}{\partial z} \right] . \quad (40)$$

It will become clear, below, that these equations are consistent and provide the correct evolution for z .

B. Momentum Constraint

The momentum constraints admit a simple expression through the solution (28), [6]. The gravitational momentum tensor can be decomposed into a trace and a traceless part which we denote by an overbar

$$\pi^{ij} = \frac{1}{3}\pi\gamma^{ij} + \bar{\pi}^{ij} . \quad (41)$$

The momentum constraints become

$$\mathcal{H}_i = -\frac{2}{3}\pi_{,i} - 2(\bar{\pi}^{jl}\gamma_{li})_{,j} + \pi^{kl}\gamma_{kl,i} + \pi^\phi\phi_{,i} = 0 \quad . \quad (42)$$

Using equation (27), the generating functional can be written in terms of the trace of the gravitational momentum

$$\mathcal{S} = \frac{2}{3} \int \pi \left(\phi(x), h_{ij}(x); \tilde{\phi}(x), \tilde{h}_{ij}(x) \right) d^3x \quad . \quad (43)$$

Therefore the new and the old canonical variables can be expressed as partial derivatives of π . The spatial derivative of π can be written as

$$\pi_{,i} = \frac{3}{2}\pi^{kl}\gamma_{kl,i} - \frac{3}{2}\tilde{\pi}^{kl}\tilde{\gamma}_{kl,i} + \frac{3}{2}\pi^\phi\phi_{,i} - \frac{3}{2}\tilde{\pi}^\phi\tilde{\phi}_{,i} \quad . \quad (44)$$

If we substitute this in to the equation (42), using the reciprocity relation, we can write the momentum constraint in terms of the new variables

$$\tilde{\mathcal{H}}_i = -2(\tilde{\gamma}_{ik}\tilde{\pi}^{kj})_{,j} + \tilde{\pi}^{kl}\tilde{\gamma}_{kl,i} + \tilde{\pi}^\phi\tilde{\phi}_{,i} = 0 \quad . \quad (45)$$

Here one effectively performs a legendre transformation between the new and the old variables.

The evolution equations for the new variables are given by the new action which was written in equation (10)

$$\tilde{\Gamma} = \int (\tilde{\pi}^{ij}\dot{\tilde{\gamma}}_{ij} + \tilde{\pi}^\phi\dot{\tilde{\phi}} - N^i\tilde{H}_i)d^4x \quad . \quad (46)$$

One can easily see that, if the shift function N_i vanishes, then the new canonical variables are independent of time, but they can depend on space coordinates. They are restricted by the momentum constraint (45). We can write the momentum constraint in terms of \tilde{h}_{ij} ,

$$\tilde{\mathcal{H}}_i = -2(\tilde{\gamma}^{1/3}\tilde{h}_{ik}\tilde{\pi}^{kj})_{,j} + \tilde{\gamma}^{1/3}\tilde{\pi}^{kl}\tilde{h}_{kl,i} + \tilde{\pi}^\phi\tilde{\phi}_{,i} = 0 \quad . \quad (47)$$

Since the theory does not depend on the parametrization of the spatial coordinates, one may write the momentum constraint in terms of a covariant derivative with respect to \tilde{h}_{ij} ,

$$\tilde{\mathcal{H}}_i = -2(\tilde{\gamma}^{1/3}\tilde{\pi}_i^j)_{;j} + \tilde{\pi}^\phi\tilde{\phi}_{,i} = 0 \quad . \quad (48)$$

C. Solution

Solution of the equation (30) is given by

$$H = -\frac{2}{3t_o e^{\tilde{\phi}}} \exp \left\{ \frac{\sqrt{3}}{2} \left[(\phi - \tilde{\phi})^2 + (z - \tilde{z})^2 \right]^{1/2} \right\} , \quad (49)$$

where a tilde refers to initial value of the corresponding variable. The initial value of H is chosen in order to have a Pre-Big Bang behaviour and t_o is an arbitrary constant [9]. Its meaning will become apparent below. Here one should note the rotational symmetry of the solution.

Using the evolution equations for ϕ and z (7), (39), and the equations for their conjugate momentum (36), (40), we find

$$\dot{\phi} = -2e^{\phi} e^{-\frac{3}{2}(\phi - \tilde{\phi})} \frac{\partial H}{\partial \phi} , \quad (50)$$

$$\dot{z} = -2e^{\phi} e^{-\frac{3}{2}(\phi - \tilde{\phi})} \frac{\partial H}{\partial z} . \quad (51)$$

If we use the new variables x and y defined as

$$x = \frac{\sqrt{3}}{2}(\phi - \tilde{\phi}) , \quad y = \frac{\sqrt{3}}{2}(z - \tilde{z}) , \quad r^2 = x^2 + y^2 , \quad (52)$$

then we find

$$\dot{x} = \frac{1}{t_o} \frac{x}{\sqrt{x^2 + y^2}} e^{-\frac{1}{\sqrt{3}}x + \sqrt{x^2 + y^2}} , \quad (53)$$

$$\dot{y} = \frac{1}{t_o} \frac{y}{\sqrt{x^2 + y^2}} e^{-\frac{1}{\sqrt{3}}x + \sqrt{x^2 + y^2}} . \quad (54)$$

Because of rotational symmetry, it is natural to use the polar coordinates in the (ϕ, z) plane. Then one finds that the angular coordinate is constant in time. It depends only on the spatial coordinates. The radial coordinate is given by

$$r = \frac{-1}{1 - \frac{1}{\sqrt{3}} \cos \varphi} \ln \left(1 - \frac{t}{\tilde{t}} \right) , \quad \tilde{t} = \frac{t_o}{1 - \frac{1}{\sqrt{3}} \cos \varphi} . \quad (55)$$

We find, using $x = r \cos \varphi$, $y = r \sin \varphi$ and equation (52),

$$\phi = \tilde{\phi} + \beta \ln \left(1 - \frac{t}{\tilde{t}} \right) \quad , \quad \beta = \frac{-\frac{2}{\sqrt{3}} \cos \varphi}{1 - \frac{1}{\sqrt{3}} \cos \varphi} \quad , \quad (56)$$

$$z = \frac{-\frac{2}{\sqrt{3}} \sin \varphi}{1 - \frac{1}{\sqrt{3}} \cos \varphi} \ln \left(1 - \frac{t}{\tilde{t}} \right) \quad . \quad (57)$$

Here notice that $\tilde{z} = 0$ by definition. In the (φ, z) -plane circles concentric with the origin corresponds to constant H surfaces. The evolution of the fields ϕ and z at a fixed spatial point are given by the rays originating from the origin. They remain orthogonal to the uniform H surfaces everytime. One can see, using equations (34) and (36), that the momenta for the gravitational field and the dilaton are constant in time. But they can have spatial dependence,

$$\pi_j^i = \lambda_j^i(x) \quad . \quad (58)$$

The evolution of the unimodular conformal metric h_{ij} is given by

$$\dot{h}_{ij} = -4e^\phi e^{-\frac{3}{2}(\phi - \tilde{\phi})} \frac{\partial H}{\partial h_{kl}} h_{ki} h_{lj} \quad , \quad (59)$$

Here equations (6), (34) and (36) are used. At this point, by a direct application of the chain rule, one can check that the equations (39) and (40) for \dot{z} and π^z are consistent such that they lead to correct expression for the evolution of z . We can find the evolution of the unimodular conformal metric using equation (59) and the solution (57) for z ,

$$\left[\ln \left([h][\tilde{h}]^{-1} \right) \right] = z [p(x)] \quad . \quad (60)$$

Here the matrix $[p(x)]$ satisfies

$$Tr ([p][p]) = 2 \quad , \quad Tr ([p]) = 0 \quad . \quad (61)$$

Hereafter, in order to avoid repetition, we are going to write the results without going into the details of algebraic manipulations. Using equation (34), we find

$$\frac{1}{\lambda} [\lambda] = \frac{1}{3} \left(I + \frac{\sqrt{3}}{2} \sin \varphi [p] \right) \quad (62)$$

At this stage, eliminating $[p]$ in favor of $[\lambda]$, we define the matrix $[\alpha]$,

$$[\alpha(x)] = I - \frac{2}{1 - \frac{1}{\sqrt{3}} \cos \varphi} \frac{1}{\lambda} [\lambda] \quad . \quad (63)$$

Then the trace of $[\alpha]$ is

$$\alpha = \frac{1 - \sqrt{3} \cos \varphi}{1 - \frac{1}{\sqrt{3}} \cos \varphi} \quad . \quad (64)$$

The matrix $[\alpha]$ can be simplified further as below. Equation (60) yields

$$[h(t, x)] = \exp \left\{ 2 \left([\alpha] - \frac{1}{3} \alpha I \right) \ln \left(1 - \frac{t}{\tilde{t}} \right) \right\} [\tilde{h}(x)] \quad . \quad (65)$$

If we replace equations (34) and (36) in equation (6), then we find the determinant of the metric evolves according to

$$\gamma = \left(\frac{\lambda \tilde{t} e^{\tilde{\phi}}}{3 - \alpha} \right)^2 \left(1 - \frac{t}{\tilde{t}} \right)^{2\alpha} \quad . \quad (66)$$

Then we find the evolution of the metric $\gamma_{ij} = \gamma^{1/3} h_{ij}$,

$$[\gamma(t, x)] = \exp \left\{ 2[\alpha(x)] \ln \left(1 - \frac{t}{\tilde{t}} \right) \right\} [\tilde{\gamma}(x)] \quad , \quad (67)$$

where

$$[\tilde{\gamma}(x)] = \left(\frac{\lambda \tilde{t} e^{\tilde{\phi}}}{3 - \alpha} \right)^{2/3} [\tilde{h}(x)] \quad . \quad (68)$$

It is possible to introduce local coordinates in which the matrix $[\alpha]$ and the metric are diagonal for discussion of duality below. One can also write, arranging equations (56) and (57), the scalar field ϕ , and z in terms of trace α . Here, it is illuminating to find the extrinsic curvature. We obtain, using equation (6),

$$[K] = \frac{[\alpha]}{\tilde{t} - t} \quad , \quad (69)$$

where $[K]$ is the matrix representation of the extrinsic curvature, with entries K_j^i . This yields the expansion rate of the universe as

$$K = -\frac{1}{2} \frac{\dot{\gamma}}{\gamma} = \frac{\alpha}{\tilde{t} - t} \quad . \quad (70)$$

Meanwhile we can write H as,

$$H = -\frac{3-\alpha}{3\tilde{t}e^{\tilde{\phi}}}\left(1-\frac{t}{\tilde{t}}\right)^{-\frac{3-\alpha}{2}}. \quad (71)$$

K is proportional to $e^{-\frac{\phi}{2}}H$. Equation (64) yields $-\sqrt{3} \leq \alpha \leq \sqrt{3}$. The condition for quasi-homogeneous regions to undergo superinflation, $\alpha < 0$, corresponds to the region $\cos\varphi > 1/\sqrt{3}$ in the (ϕ, z) -plane [9]. The maximal rate of expansion is reached for $\alpha = -\sqrt{3}$. This corresponds to the quasi-isotropic case $\varphi = 0$, as explained in the next chapter. $[\alpha]$ and the matrix of gravitational momentum $[\lambda]$ are related as

$$\frac{1}{\lambda}[\lambda] = \frac{1}{3-\alpha}(I - [\alpha]) \quad (72)$$

The relations (61) and (62) yields

$$\frac{1}{\lambda^2}\lambda_j^i\lambda_i^j = \frac{1}{6}(3 - \cos^2\varphi) \quad (73)$$

and this gives the condition

$$\frac{1}{3} \leq \frac{1}{\lambda^2}\lambda_j^i\lambda_i^j \leq \frac{1}{2} \quad (74)$$

Here, the lower limit corresponds to the quasi-isotropic case. Therefore we have a Kasner-like solution,

$$Tr([\alpha][\alpha]) = 1 \quad , \quad \beta = -1 + \alpha \quad (75)$$

Momenta π^ϕ and π^z are related to angle φ and trace of the gravitational momentum λ as follows

$$\pi^\phi = \left(-1 + \frac{1}{\sqrt{3}}\cos\varphi\right)\lambda \quad , \quad \pi^z = \left(-1 + \frac{1}{\sqrt{3}}\sin\varphi\right)\lambda \quad (76)$$

and they satisfy

$$\left(\pi^\phi + \lambda\right)^2 + \left(\pi^z + \lambda\right)^2 = \frac{1}{3}\lambda^2 \quad (77)$$

Initial momentum for z , ϕ , and the gravitational field are

$$\tilde{\pi}^\phi = \pi^\phi \quad , \quad \tilde{\pi}^z = \pi^z \quad (78)$$

$$\tilde{\pi}_j^i = \lambda_j^i - \frac{1}{3}\delta_j^i\lambda \quad (79)$$

V. QUASI-ISOTROPIC SOLUTION

In this section we consider a quasi-isotropic space. This is a special case of the general solution, as it is explained below. If we use the quasi-isotropic ansatz

$$\mathcal{S}^o = -2 \int e^{-\frac{3}{2}\phi} H(\phi) \sqrt{\gamma} d^3x \quad , \quad (80)$$

the Hamilton-Jacobi equation reduces to

$$H^2 = \frac{4}{3} \left(\frac{\partial H}{\partial \phi} \right)^2 \quad . \quad (81)$$

The momentum constraint (diffeomorphism invariance)

$$H_{,i} = \frac{\partial H}{\partial \phi} \phi_{,i} \quad , \quad (82)$$

is automatically satisfied by this ansatz. Meanwhile the equations of motion for the fields are

$$\dot{\phi} = -2e^{-\frac{1}{2}\phi} \frac{\partial H}{\partial \phi} \quad , \quad \dot{\gamma}_{ij} = e^{-\frac{1}{2}\phi} \left[H - 2 \frac{\partial H}{\partial \phi} \right] \gamma_{ij} \quad . \quad (83)$$

They immediately yield a quasi-isotropic solution $\gamma_{ij} = a^2(\phi) h_{ij}(x)$. Here

$$a^2 = \exp \left\{ -\frac{1 - \sqrt{3}}{2} \int \frac{H}{\partial H / \partial \phi} d\phi \right\} \quad , \quad (84)$$

and h_{ij} is the seed metric. If we solve the equations and the constraints explicitly, we find, for the scalar field and the gravitational field

$$\phi = \frac{2}{1 - \sqrt{3}} \ln \left(1 - \frac{t}{\tilde{t}} \right) \quad , \quad (85)$$

$$[\gamma] = \left(1 - \frac{t}{\tilde{t}} \right)^{-\frac{2}{\sqrt{3}}} [h(x)] \quad . \quad (86)$$

Here t_o is rescaled by a factor of $(1 - 1/\sqrt{3})^{-1}$. We obtain, using equation (13) and the solution, that the momentum of the gravitational field and the scalar field are constant independent of space-time coordinates. Furthermore the traceless part of the gravitational momentum is zero and evolution of the gravitational radiation is neglected. H is given by

$$H = \frac{2}{\sqrt{3}(1-\sqrt{3})} \frac{1}{\tilde{t}} \left(1 - \frac{t}{\tilde{t}}\right)^{\frac{\sqrt{3}}{1-\sqrt{3}}} . \quad (87)$$

Meanwhile we obtain for the extrinsic curvature,

$$[K] = -\frac{1}{\sqrt{3}} \frac{1}{\tilde{t}-t} I . \quad (88)$$

This yields, for the expansion rate of the quasi-isotropic universe,

$$K = -\frac{\sqrt{3}}{\tilde{t}-t} . \quad (89)$$

We expect that the general solution contains the quasi-isotropic one as a special case. If, in the general solution, we consider the case $\varphi = 0$, then traceless part of $[\alpha]$ and the momentum $[\lambda]$ disappear. They contain only trace parts which are independent of space coordinates. The unimodular conformal metric $[h]$ becomes independent of time. One should notice that, we introduced the factor $e^{-\frac{3}{2}\tilde{\phi}}$ in the generating function by hand, only for convenience. As a result the metric $[\gamma]$ and the scalar field ϕ reduce to those of the quasi-isotropic space. Similarly, the extrinsic curvature reduces to the corresponding quasi-isotropic one. This correspondance can be checked explicitly, by putting $\phi = 0$ in the general solution.

VI. DUALITY

In this section we briefly discuss the duality property of the general solution. It is apparent, from the form of the solution, that the transformation $\alpha_a \longrightarrow \alpha'_a = -\alpha_a$ (for all a) generates a dual solution. This yields $\alpha \longrightarrow \alpha = -\alpha$ for the trace of the matrice $[\alpha]$. We can perform this as a transformation of the angular coordinate φ in the (ϕ, z) -plane

$$\cos(\pi - \varphi') = \frac{\cos \varphi - \frac{\sqrt{3}}{2}}{1 - \frac{\sqrt{3}}{2} \cos \varphi} . \quad (90)$$

One can easily check that, this transformation is equivalent to $\alpha \longrightarrow \alpha'$, using equation (64) explicitly. The dual solution is of the same form. This transformation is well defined since

$-1 < \cos \varphi' < 1$, and $-\sqrt{3} < \alpha' < \sqrt{3}$. Transformation of the radius can be found by using equation (55). Using $x = r \cos \varphi$ and $y = r \sin \varphi$ we find the known result

$$\phi \longrightarrow \phi' = \phi - \ln \gamma \quad (91)$$

However, z does not experience any change except an additional constant contribution (remember $\tilde{z} = 0$ by definition). This can be seen easier if it is written in terms of α .

We can decompose the gravitational momentum tensor into a trace contribution and a traceless part as

$$\frac{1}{\lambda}[\lambda] = \frac{1}{3}I + [q] \quad . \quad (92)$$

Then we find that the traceless part $[q]$ transforms as

$$[q'] = \frac{-\frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{1 - 6|q|^2}}{-1 - \frac{\sqrt{3}}{2}\sqrt{1 - 6|q|^2}}[q] \quad , \quad (93)$$

under the duality transformation. Here $|q|^2 = Tr([q][q])$. We should also impose the momentum constraints simultaneously.

Meanwhile, equation (90) yields that the dual of the quasi-isotropic solution is again a quasi-isotropic one. However it is not contained in the superinflationary section of the plane.

VII. CONCLUSION AND SUMMARY

In this paper, we applied the long-wavelength approximation to low-energy effective string action in the context of Hamilton-Jacobi theory. The Hamilton-Jacobi equation for the effective string action in four dimensions is invariant under SFD transformation. However the momentum constraints are not invariant under this transformation. Long-wavelength gravity (gradient expansion) has proved to be a fruitful method for studying slightly inhomogeneous cosmology. It is a significant improvement over those of homogeneous minisuperspace.

We presented leading order solution of HJ equation. This is the most general solution near a singularity. We solved the HJ equation including the evolution of the gravitational

radiation. In order to do this we performed a transformation to new canonical variables where the Hamiltonian density vanishes strongly. Therefore, the new variables are constant in time, if the shift function vanishes. However, they depend on the spatial coordinates. In the separated Hamilton-Jacobi equation, the gravitational degrees of freedom can be reduced to that of a single massless scalar field. However, the gravitational field is fundamentally different from massless scalar fields. For example, it carries spin angular momentum, and the momentum constraints restrict the longitudinal modes of the gravitational momentum tensor. Then we presented the quasi-isotropic solution. The general solution contains this one as a special case. In this case, gravitational radiation is neglected. The Hamilton-Jacobi approach yields a result consistent with the one derived by using the Lagrange equations directly, [9].

In the Hamilton-Jacobi approach, we can simply represent physically important cases in the (ϕ, z) -plane. Constant H surfaces are circles concentric with the origin. The evolution of the fields ϕ and z , at a fixed spatial point are given by trajectories originating from the origin. These trajectories remain orthogonal to uniform H surfaces everytime. The region $\cos\varphi > 1/\sqrt{3}$ corresponds to superinflationary solutions. Meanwhile, we obtain a quasi-isotropic universe on the ϕ -axis. The Big Bang instant corresponds to a point at infinity. The momentum constraints admit a simple expression in terms of the new canonical variables. This form is useful for general discussions. We also performed the duality transformation as a transformation of angle in (ϕ, z) -plane, $\varphi \rightarrow \varphi'$. We have to impose the momentum constraints simultaneously. However this transformation and its relation to the momentum constraint needs further clarification.

Acknowledgments

I would like to thank CERN Theory Division and especially G. Veneziano for proposing this problem, and for helpful discussions. I also thank TÜBİTAK (Scientific and Technical Research Council of Turkey) for providing financial support during this study.

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