Octonionic Selfduality for SuperMembranes[*]

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Abstract

In this work we study the recently introduced octonionic duality for membranes. Restricting the self - duality equations to seven space dimensions, we provide various forms for them which exhibit the symmetries of the octonionic and quaternionic structure. These forms may turn to be useful for the question of the integrability of this system. Introducing a consistent quadratic Poisson algebra of functions on the membrane we are able to factorize the time dependence of the self - duality equations. We further give the general linear embeddings of the three dimensional system into the seven dimensional one using the invariance of the self-duality equations under the exceptional group G_2.

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1.   Introduction.

M theory is the leading candidate for the unification of all superstring theories in their perturbative and non-perturbative sector. This theory contains $N = 1$, 11-dimensional supergravity and at least a sector of supermembranes and their magnetic duals the superfive branes[1, 2]. These extended objects exist as solitons of the eleven dimensional supergravity and they are distinguished from the fundamental superbranes as solitonic superbranes.[3]

Most of the recent work on compactifications of M theory is concentrated on a unified "proof" of various non-perturbative dualities of superstring theories connecting their strong and weak coupling sectors or small with large volumes of the compactifying manifolds[1, 2]. There is a line of attack from the point of view of the 11-d superbranes which either uses double dimensional reduction to connect with type IIA or heterotic superstrings, or using purely classical worldvolume dualities of the superbranes which happens miraculously to explain non - perturbative phenomena, (dualities of superstring theories)[4]. From the point of view of superstring theories, supermembranes in non-compact 11 dimensions correspond to the strong coupling regime of the superstrings[1].

Many basic questions concerning supermembrane theories have not been answered today. A top priority issue, is the derivation of the eleven dimensional supergravity theory as a low energy effective action of the supermembrane. To do this one needs to understand the quantum mechanics of the supermembrane, that is, define a sensible perturbation theory. This is an extremely hard problem for two reasons. First, the moduli space of three dimensional Remannian metrics is largely unknown and a representation theory of the three dimensional diffeomorphism group or (in the light-cone gauge) of the area preserving diffeomorphism group of the supermembrane is lacking. Second, – unlike the string where in the light cone gauge the theory becomes an infinite set of free transverse oscillators – in the supermembrane case, the light cone gauge does not solve fully the constraints and there is

no coupling constant in the interaction term. Fortunately, the Hamiltonian in the light cone gauge is of Yang-Mills (YM) type with gauge group the area preserving diffeomorphisms of the membrane[5, 6, 7].

Another issue is the following. During the compactifications from eleven dimensions to ten, one is freezing an infinite number of string degrees of freedom of the supermembrane and considers only the Kaluza-Klein dilatonic modes which are supposed to be the infinite tower of superstring solitons which complete the duality picture. It could be possible that taking into account in a controlable way the interaction of the remaining string excitations of the supermembrane, one could define a perturbation theory[8]. Recently, an old mode regularization of supermembrane through $SU(N)$ matrix super YM mechanics has been re-incarnated as possible candidate model for M theory[9]

Another possible approach to define a perturbative expansion for the 11 dimensional supermembrane is to study various compactifications of the 11-d supergravity where the classical supermembrane has very simple dynamics (it can be even static-stretched) and then around these classical solutions study the quantum excitations of the supermembrane. In this way one hopes to get a classical state which could be used as a quantum vacuum state for the membrane. One test would be to find in the excitation spectrum of the supermembrane the 11-d supergravity multiplet around the classical background. The problem is that one has to preserve in one way or another the $N = 1$, 11-d sypersymmetry during these compactifications[3, 4]. Following old work in the compactification of 11-d supergravity on the seven sphere[10] there is a recent activity on octonionic solitons for strings and supermembranes[11]. In this work, specific background field configurations of the compactified supergravity on seven sphere considered as various fiberbundles which are coupled through their singularities to supermembrane sources.

In this work, we want to move in different direction which exploits some aspects of the non perturbative structure of the supermembrane vacuum in flat space time, studying classi-

cal Euclidean time equations which describe quantum tunneling processes between classical configurations of the supermembrane which could be considered as vacua of different topological sectors. Although there is an extensive work [3], where essentially the background field equations are solved, as far as we know, the question of the Euclidean membrane as an extended object connecting different topological sectors has not been addressed except in [12, 13, 14]. The topological charge and the Bogomol'nyi bound known from supersymmetric YM theory, can be extended to Euclidean supermembranes in $(4+1)$[12, 13] and, as has been shown recently, in $(8+1)$-dimensions[14]. In section 2 we recall the main results of the works[12, 13] where the self dual bosonic membrane in $(2+1)$ and $(4+1)$ dimensions has been introduced. In section 3, the generalization by[14] in $(8+1)$ dimensions is described in a compact form and possible factorizations of the time dependence are discussed. In section 4, the same equations in octonionic and quartenionic representations are introduced which exhibit specific properties of the self-duality equations. Finally, is section 5 the general formulation of embedding the three dimensional solutions into seven dimensions is described and the constraint equations are derived. Some examples of specific embeddings of the $(4+1)$-dimensional system to $(8+1)$ dimensions are also analysed.

2. SU(N) Yang Mills and Membranes.

To start, we recall that it has been known since sometime that the supermembrane Hamiltonian in the light-cone gauge is a very close relative of Yang-Mills (YM) theories in the gauge $A_0 = 0$ and in one space dimension less [5, 6]. To describe in some more detail this relationship, we restrict our discussion to the bosonic part of the Hamiltonian of the supermembrane in the light cone gauge and to spherical topology for the membrane[6, 15, 13]. In reference[15] using results of[5], it was pointed out that in the large N-limit, $SU(N)$ YM theories have, at the classical level, a simple geometrical structure with the SU(N) matrix potentials $A_\mu(X)$ replaced by c-number functions of two additional coordinates θ, ϕ

of an internal sphere S^2 at every space-time point, while the $SU(N)$ symmetry is replaced by the infinite dimensional algebra of area preserving diffeomorhisms of the sphere S^2 called SDiff(S^2). The $SU(N)$ fields ($N \times N$ matrices)

$$A_\mu(X) = A_\mu^\alpha(X)t^\alpha,$$

$$t^\alpha \in SU(N),$$

$$\alpha = 1, 2, \ldots, N^2 - 1, \quad \mu = 0, 1, \ldots d - 1 \tag{1}$$

in the large N - limit become c-number functions of an internal sphere S^2,

$$A_\mu(X, \theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} A_\mu^{lm}(X)Y_{lm}(\theta, \phi), \tag{2}$$

where $Y_{lm}(\theta, \phi)$ are the spherical harmonics on S^2. The local gauge transformations

$$\delta A_\mu = \partial_\mu \omega + [A_\mu, \omega], \quad \omega = \omega^\alpha t^\alpha, \tag{3}$$

and

$$\delta F_{\mu\nu} = [F_{\mu\nu}, \omega], \tag{4}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{5}$$

are replaced by

$$\delta A_\mu(X, \theta, \phi) = \partial_\mu \omega(X, \theta, \phi) + \{A_\mu, \omega\}, \tag{6}$$

$$\delta F_{\mu\nu}(X, \theta, \phi) = \{F_{\mu\nu}, \omega\}, \tag{7}$$

where

$$F_{\mu\nu}(X, \theta, \phi) = \partial_\mu A_\nu - \partial_\nu A_\mu + \{A_\mu, A_\mu\}, \tag{8}$$

and the Poisson bracket on S^2 is defined as follows:

$$\{f, g\} = \frac{\partial f}{\partial \phi} \frac{\partial g}{\partial \cos \theta} - \frac{\partial g}{\partial \phi} \frac{\partial f}{\partial \cos \theta} \tag{9}$$

So the commutators are replaced by Poisson brackets according to

$$\lim_{N \to \infty} N[A_\mu, A_\nu] = \{A_\mu, A_\nu\} \tag{10}$$

Then the YM action in the large-N limit becomes [15]

$$S_\infty = \frac{1}{16\pi g^2} \int_{S^2} d\Omega \int d^4 X F_{\mu\nu}(X, \theta, \phi) F^{\mu\nu}(X, \theta, \phi), \tag{11}$$

where

$$g = \lim_{N\to\infty} \frac{g_N}{N^{3/2}} \tag{12}$$

This large-N limit of $SU(N)$ YM theories was found by making use of the relation between the SU(N=2 s+1) algebra in a particular basis (up to spin s SU(2)- tensor $N \times N$- matrices) and $SDiff(S^2)$ in the basis of the spherical harmonics $Y_{lm}(\theta, \phi)$. In the present day language, this $SDiff(S^2)$ YM theory corresponds to the effective theory of infinite number, $N \to \infty$, $d - 1$ dimensional Dirichlet branes[16, 17]. Similar considerations hold for membranes of different topologies, torus, double torus etc[18]. Here we note that the recently proposed matrix theory which is claimed to be the long seeked formulation of M theory, is nothing but the SU(N) supersymmetric YM mechanics which was used as a consistent truncation of the supermembrane[5, 6, 19].

The above considered large N-limit, is a very specific one which depends on the appropriate basis of $SU(N)$ generators convenient for the topology of the membrane and it has nothing to do, at least in a direct way, with the planar approximation of YM theories. Also, it is different from the large N-limit used in Matrix theory.

In the case of the spherical membranes the $SDiff(S^2)$ YM- theory describes the dynamics of an infinite number of $D0$- branes forming a topological 2- sphere. In the light cone gauge the transverse coordinates X_i, $(i, 1..., 9)$ of the 11-d bosonic part of the supermembrane satisfy the following equations

$$\ddot{X}_i = \{X_k, \{X_k, X_i\}\} \quad i, k = 1, \ldots 9 \tag{13}$$

where summation over repeated indices is implied. The corresponding Gauss law which is the generator of the $SDiff(S^2)$ group is given by the constraint

$$\{X_i, \dot{X}_i\} = 0 \tag{14}$$

In ref[12] Euclidean bosonic membranes in 3 Dimensional target space have been introduced defining the topological charge density to be

$$\Omega(X) = \frac{1}{3!}\epsilon^{abc} f_{ijk} X_a^i X_b^j X_c^k \tag{15}$$

where, a, b, c run from 1 to 3 and i, j, k from 1 to d space time dimensions.

$$X_a^i = \partial_{\xi_a} X^i \tag{16}$$

and $\xi_{1,2,3}$ are the worldvolume coordinates. The self-duality equations were defined as

$$P_i^a = \pm\frac{1}{2}\epsilon^{abc} f_{ijk} X_b^j X_c^k \tag{17}$$

Here P_i^a are the canonical momenta,

$$P_i^a = T\frac{\delta}{\delta X_a^i}(Det[X_a^i X_b^i]))^{1/2} \tag{18}$$

The self duality equations for the case $d = 3$ and $f_{ijk} = \epsilon_{ijk}$ were shown to satisfy both the constraints and the equations of motion. Solutions were given for the case of sphere and torus. In reference[13] 3-d Euclidean self duality equations in the light cone gauge (that is 4+1-dimensional target space) for the bosonic part of the supermembrane could be written in analogy with the 3-d Nahm equations of self dual BPS YM- monopoles. In the light cone gauge this means that one had to fix 6 of the 9 transverse coordinates to be constants. This constraint solves the second order equations (13) for the 6 coordinates. Then the self duality equations are

$$\dot{X}_i = \frac{1}{2}\epsilon_{ijk}\{X_j, X_k\} \quad i,j = 1, 2, 3 \tag{19}$$

The self duality equations solve automatically the second order Euclidean time equations as well as the Gauss law due to the Jacobi identity for the ϵ symbol and its well known properties. The above system has a Lax pair and infinite number of conservation laws[13]. In order to see this, first we rewrite eqs.(19) in the form

$$\dot{X}_+ = i\{X_3, X_+\}, \quad \dot{X}_- = i\{X_3, X_-\}, \quad \dot{X}_3 = \frac{1}{2}i\{X_+, X_-\}, \tag{20}$$

where

$$X_\pm = X_1 \pm iX_2 \tag{21}$$

There exists a linear system corresponding to (20) which is the following

$$\dot{\psi} = L_{X_3+\lambda X_-}\psi, \quad \dot{\psi} = L_{\frac{1}{\lambda}X_+ - X_3}\psi, \tag{22}$$

where the differential operators L_f are defined as

$$L_f \equiv i\left(\frac{\partial f}{\partial \phi}\frac{\partial}{\partial \cos\theta} - \frac{\partial f}{\partial \cos\theta}\frac{\partial}{\partial \phi}\right). \tag{23}$$

The compatibility condition of (22) is

$$[\partial_t - L_{X_3+\lambda X_-}, \partial_t - L_{\frac{1}{\lambda}X_+ - X_3}] = 0, \tag{24}$$

from which, comparing the two sides for the coefficients of the powers $\frac{1}{\lambda}, \lambda^0, \lambda^1$ of the spectral parameter λ, we find (20). From the linear system (22) using the inverse scattering method, one could in principle construct all solutions of the self-duality equations.

Specific solutions could be obtained due to the existence of an $SU(2)$ subalgebra of $SDiff(S^2)$ which happens to be its only finite dimensional subalgebra. Using this $SU(2)$ subalgebra, for spherically symmetric solutions it can be shown that the system reduces to the Toda $SU(2)$ equations. Another method to find solutions of the integrable system (20) has been proposed in[20] where the system is linearized by considering the target space variables as worldvolume variables and vice-versa. More recently, there have been discussions of the same issue in papers[14, 21]. In ref[22] the connection with self dual Einstein equations has been discussed. Before closing the section, we would like to note that the Euclidean membrane configurations which are solutions of the selfduality equations are expected to interpolate between classical vacuum configurations of the membrane that is, points or strings. Also the case of membrane is the first in the series of extended objects which there is a gauge principle to define the interactions and the possibility arises for topology change through gauge interactions. The case of string has an ad hoc interaction

which is not enforced uniquely by any gauge principle. Moreover, the classical vacua of string are points.[12]

3.   The octonionic structure of the self duality equations

An obvious way to generalize duality for super p-branes, is to use Poincaré duality. For the fundamental supermembranes in particular, this has been done by Duff et al[23] and it has been exploited later, proving various conjectures of string-string, string-membrane and membrane-membrane dualities[24, 25, 26]. Another type of duality has been investigated recently[14, 27] which is based on the existence of the last real division algebra, the octonionic or Cauley algebra[28]. The work of[14] is based on the similarity between the supermembrane and the super YM theories refered previously and the work on 8-dimensional YM instantons many years ago by[29]. Another way of considering the work of ref[14] is as extension of the quaternionic case[13] using the possibility to define a cross product of two vectors in 8 dimensions through the multiplication rule of octonions.

In this section, we restrict the self duality equations of [14] to seven dimensions by choosing fixed values for 8 and 9 membrane coordinates. Then, the the self duality equations[14] become

$$\dot{X}_i = \frac{1}{2}\Psi_{ijk}\{X_j, X_k\} \tag{25}$$

where Ψ_{ijk} is the completely antisymmetric tensor which defines the multiplications of octonions[28]. The Gauss law results automatically by making use of the Ψ_{ijk} cyclic symmetry

$$\{\dot{X}_i, X_i\} = 0 \tag{26}$$

The Euclidean equations of motion are obtained as follows

$$\ddot{X}_i \ = \ \frac{1}{2}\Psi_{ijk}\left(\{\dot{X}_j, X_k\} + \{X_j, \dot{X}_k\}\right) \tag{27}$$

$$= \ \{X_k, \{X_i, X_k\}\} \tag{28}$$

where use has been made of the identity

$$\Psi_{ijk}\Psi_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} + \phi_{ijlm} \tag{29}$$

and of the cyclic property of the symbol ϕ_{ijlm}[28].

As in the case of the 3-d system we may try to factorize the time dependence. We assume the following factorization.

$$X_i = Z_{ij}(t)f_j(\xi) \tag{30}$$

Then, from Eq.(25) we obtain

$$\dot{Z}_{im}f_m = \frac{1}{2}\Psi_{ijk}Z_{jl}Z_{kn}\{f_l, f_n\} \tag{31}$$

We observe that if we make the Ansatz for the 7×7 matrix

$$\dot{Z}_{im}(t)\Psi_{mln} = \Psi_{ijk}Z_{jl}(t)Z_{kn}(t) \tag{32}$$

then the equation

$$f_i = \frac{1}{2}\Psi_{ijk}\{f_j, f_k\} \tag{33}$$

is automatically satisfied, while at the same time we have succeeded to disentangle the time dependence from the self-duality equation. Therefore, the problem is reduced to find solutions for $f_i(\xi)$ and Z_{kl} equations seperately.

Another equivalent form of the previous equation for the matrices Z_{ij} is

$$\dot{Z}_{ij} = \frac{1}{6}\Psi_{ikl}\Psi_{jmn}Z_{km}Z_{ln} \tag{34}$$

In the case of diagonal matrices $Z_{ij} = \delta_{ij}R_j(t)$, we have

$$\dot{R}_i = \frac{1}{6}\Psi_{ikl}^2 R_k R_l \tag{35}$$

We make now some observations about the symmetries of Eqs(25,33). If X_i is a solution of (25) then for every matrix R of the group G_2 which is a subgroup of $SO(7)$ then

$$Y_i = R_{ij}X_j \tag{36}$$

is automatically a solution of the same equation because the elements of G_2 preserve the structure constants Ψ_{ijk}. In components,

$$\Psi_{ijk}R_{kl} = \Psi_{imn}R_{mj}R_{nl} \tag{37}$$

The above relation shows the way to define G_2 group elements starting from two orthonormal 7-vectors. The equation is obviously covariant under $SDiff(S2)$ transformations. One can define combined G_2 and $SDiff(S2)$ transformations to get $SO(3)$ spherically symmetric solutions since $SO(3)$ can be realised as a subalgebra of $SDiff(S2)$.

We note that in principle it is possible to look for non-linear symmetries of the self-duality equations, generalizing (36)

$$Y_i = f_i(X) \tag{38}$$

where $f_i(X)$ must satisfy the equation

$$\Psi_{ijk}\frac{\partial f_k}{\partial X_l} = \Psi_{imn}\frac{\partial f_m}{\partial X_j}\frac{\partial f_n}{\partial X_l} \tag{39}$$

In the following we examine the self consistency of Eq.(33). Multiplying by Ψ_{ilm}, we get

$$\Psi_{ilm}f_i = \{f_l, f_m\} + \frac{1}{2}\phi_{lmjk}\{f_j, f_k\} \tag{40}$$

Then, since the Poisson brackets satisfy the Jacobi identity, the above equation is constrained to satisfy the identity

$$\frac{1}{3}\phi_{ijkl}f_l = \Psi_{ijn}\{f_m, f_k\} + \text{cyclic perm. of}(ijk) \tag{41}$$

This system of equations is exactly the same as in (33).

Another check for the self consistency of f_i equations can be found as follows. Define the tensors

$$X^{ij}{}_{kl}(u) = \Delta^{ij}{}_{kl} + \frac{u}{4}\phi^{ij}{}_{kl} \tag{42}$$

where $\Delta^{ij}{}_{kl} = \frac{1}{2}(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k)$ and the symbol $\phi^{ij}{}_{kl} \equiv \phi_{ijkl}$. Then, equations(33) can be written as follows

$$\Psi_{ijk} f_k = X^{ij}{}_{lm}(2)\{f_l, f_m\} \tag{43}$$

Using now the algebra of the $X^{ij}{}_{kl}(u)$ tensors discussed in detail in the Appendix we can prove that both the identities

$$\Psi_{ijk} X^{jk}{}_{lm}(-1) = 0 \tag{44}$$

and

$$X^{ij}{}_{mn}(-1) X^{mn}{}_{kl}(2) = 0 \tag{45}$$

hold, and this terminates the second consistency check.

4. Octonionic and quaternionic formulation of the self-duality equations.

The octonionic or Cauley alrgebra, is the appropriate structure to organize the seven self duality equations[28, 29]. The octonionic units o_i satisfy the algebra

$$o_i o_j = -\delta_{ij} + \Psi_{ijk} o_k \tag{46}$$

where $i = 1, \ldots, 7$ are the 7 octonionic imaginary units with the property

$$\{o_i, o_j\} = -2\delta_{ij} \tag{47}$$

We choose the multiplication table[28]

$$\Psi_{ijk} = \begin{cases} 1 & 2 & 4 & 3 & 6 & 5 & 7 \\ 2 & 4 & 3 & 6 & 5 & 7 & 1 \\ 3 & 6 & 5 & 7 & 1 & 2 & 4 \end{cases} \tag{48}$$

In terms of these units an octonion can be written as follows

$$X = x_0 o_0 + \sum_{i=1}^{7} x_i o_i \tag{49}$$

with o_0 the identity element. The conjugate is

$$\bar{X} = x_0 o_0 - \sum_{i=1}^{7} x_i o_i \tag{50}$$

The octonions over the real numbers can also be defined as pairs of quaternions

$$X = (x_1, x_2) \tag{51}$$

where $x_1 = x_1^\mu \sigma_\mu$, $x_2 = x_2^\mu \sigma_\mu$ and the indices μ run from 0 to 3, while $x_{1,2}^0$ are real numbers and $x_{1,2}^i$, $i = 1, 2, 3$ are imaginary numbers. Finally, σ_0 is the Identity 2×2 matrix and σ_i are the three standard Pauli matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{52}$$

If $q = (q_1, q_2)$ and $r = (r_1, r_2)$ are two octonions, the multiplication law is defined as

$$q * r \equiv (q_1, q_2) * (r_1, r_2) = (q_1 r_1 - \bar{r}_2 r_2, r_2 q_1 + q_2 \bar{r}_1) \tag{53}$$

where $q_1 = q_1^0 + q_1^i \sigma_i$ and $\bar{q}_1 = q_1^0 - q_1^i \sigma_i$. One can also define a conjugate operation for an octonion as

$$\bar{q} \equiv \overline{(q_1, q_2)} = (\bar{q}_1, -q_2) \tag{54}$$

and we get the possibility to define the norm and the scalar product q and r

$$q\bar{q} = (q_1 \bar{q}_1 + \bar{q}_2 q_2, 0) \tag{55}$$

$$= \sum_{\mu=0}^{3} \left(x_1^{\mu 2} + x_2^{\mu 2} \right) \tag{56}$$

$$< q|r > = \frac{1}{2} (q\bar{r} + \bar{q}r) \tag{57}$$

In terms of the above formalism, the self duality equations can be written as follows

$$\dot{X} = \frac{1}{2} \{X, X\} \tag{58}$$

where now $X = X^i o_i$ with $i = 1, \cdots 7$ and the Poisson bracket for two octonions is defined as

$$\{X, Y\} = \frac{\partial X}{\partial \xi_1} \frac{\partial Y}{\partial \xi_2} - \frac{\partial X}{\partial \xi_2} \frac{\partial Y}{\partial \xi_1} \tag{59}$$

Using now (51), and the multiplication rule (53) we can write eq(58) as follows

$$\dot{x}_1 = \frac{1}{2} \left(\{x_1, x_1\} + \{\bar{x}_2, x_2\} \right) \tag{60}$$

$$\dot{x}_2 = -\{x_2, x_1\} \equiv \{x_2, \bar{x}_1\} \tag{61}$$

where $x_1 = x_1^\mu \sigma_\mu$, $x_2 = x_2^\mu \sigma_\mu$. Defining the octonionic units

$$\begin{aligned}
o_0 &= (1, 0) & o_1 &= (\imath \sigma_1, 0) & o_2 &= (\imath \sigma_2, 0) & o_3 &= (-\imath \sigma_3, 0) \\
o_4 &= (0, 1) & o_5 &= (0, \imath \sigma_3) & o_6 &= (0, \imath \sigma_2), & o_7 &= (0, \imath \sigma_1)
\end{aligned} \tag{62}$$

we easily check, that the chosen multiplication table for the octonions (48) is satisfied and the seven coordinates X_i are grouped now as follows ($x_1^0 = 0$)

$$x_1^i = \imath X_1, \imath X_2, \imath X_3 \tag{63}$$

$$x_2^\mu = X_4, \imath X_7, \imath X_6, \imath X_5 \tag{64}$$

and

$$x_1 = \begin{pmatrix} X_3 & X_1 - \imath X_2 \\ X_1 + \imath X_2 & -X_3 \end{pmatrix} \tag{65}$$

$$x_2 = \begin{pmatrix} X_4 + \imath X_5 & X_6 + \imath X_7 \\ -(X_6 - \imath X_7) & X_4 - \imath X_5 \end{pmatrix} \tag{66}$$

The organization in (65,66) of the seven X_i components obtained from the quaternionic formulation will prove very useful to identify specific classes of solutions as we will see in the next section.

5.   Embeddings of the three dimensional system.

An obvious observation is that any three dimensional solution is also a solution of the seven dimensional system discussed here. There are various ways however to embed a three dimensional solution to the seven dimensional system. In this section, we discuss solutions of the self duality equations where the coordinates X_i are linear functions of the $SU(2)$ basis of functions on the sphere, which are the components of the unit vector in 3 dimensions written in spherical coordinates[13]

$$\{e_a, e_b\} = -\epsilon_{abc} e_c \tag{67}$$

Thus, our Ansatz is

$$X_i(\xi_1, \xi_2, t) = A_i^a(t) e_a(\xi_1, \xi_2) \tag{68}$$

and implies a generalised form of Nahm's equations

$$\dot{A}_i^a = -\frac{1}{2} \Psi_{ijk} A_j^b A_k^c e_{abc}, \tag{69}$$

where a, b, c take the values $1, 2, 3$. This Ansatz contains all the embeddings of the three dimensional system with $SU(2)$ symmetry which can be written explicitely as a G_2 rotation R_{ij} of a seven-vector with non zero components the first three.

$$A_i^a = R_{ij} B_j^a \tag{70}$$

where B_j^a is defined through the three dimensional $SU(2)$ solution

$$B_i^a = (T_1^a, T_2^a, T_3^a, 0, 0, 0, 0) \tag{71}$$

Here the matrix T_a^b , $a, b = 1, 2, 3$ satisfies the three dimensional Nahm equations.

Let us now present some simple cases: The grouping of coordinates in relations (65,66) is suggestive in order to write the self duality equations in terms of the complex coordinates $X_\pm = X_1 \pm \imath X_2$,, $Y_\pm = X_4 \pm \imath X_5$, and $Z_\pm = X_6 \pm \imath X_7$. In terms of the later the system can

be written as follows

$$\dot{X}_+ = \imath(\{X_3, X_+\} + \{Y_+, Z_-\}) \tag{72}$$

$$\dot{Y}_+ = \imath(\{Y_+, X_3\} + \{X_+, Z_+\}) \tag{73}$$

$$\dot{Z}_+ = \imath(\{X_3, Z_+\} + \{X_-, Y_+\}) \tag{74}$$

$$\dot{X}_3 = \imath\frac{1}{2}(\{X_+, X_-\} + \{Z_+, Z_-\} - \{Y_+, Y_-\}) \tag{75}$$

We can easily obtain some simple solutions of the system in five or seven dimensions. In five dimensions in particular, we set $X_+ = \imath Y_-$ and $Z_+ = 0$. Then, we find that the system is reduced in the three dimensional case[13] with the identifications

$$X_\pm \to A_\pm/\sqrt{2} \ , \quad X_3 \to A_3 \tag{76}$$

Another solution which embeds every solution of the 3 dimensional case in seven dimensions, can be obtained by the identifications $X_+ = Z_+ = \imath Y_-$. This solution is reduced to that of the three dimensional one[13] with the following rescaling

$$X_\pm \to A_\pm/\sqrt{3} \ , \quad X_3 \to A_3 \tag{77}$$

An explicit construction shows that the two solutions are connected with the orthogonal transformation $|\xi_7> = \mathcal{O}|\xi_3>$ where the matrix $\mathcal{O}$ is

$$\mathcal{O} = \begin{pmatrix} a & 0 & 0 & -b & 0 & b & a \\ 0 & a & 0 & 0 & -b & a & -b \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & -b & -a & b \\ a & 0 & 0 & -b & 0 & -b & -a \\ a & 0 & 0 & 2b & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 2b & 0 & 0 \end{pmatrix} \tag{78}$$

where $a = \frac{1}{\sqrt{3}}$ and $b = \frac{1}{\sqrt{6}}$, $<\xi_3| = (X_1, X_2, X_3, 0, 0, 0, 0)$ and $|\xi_7>$ stands for the seven dimensional vector.

We conclude this work by summarizing our results. The relation of the octonionic algebra with quaternions gives a useful formulation of the self - duality equations which extends in a natural way the three dimensional system and the corresponding generalized Nahm's equations for $SDiffS_2$. By intoducing in the place of $SU(2)$ algebra of functions on the sphere, a quadratic algebra of seven functions with G_2 symmetry, we succeeded to factorize the time dependence in a simple way which may facilitate the study of solutions of the self duality equations. Although the general system of self duality equations in seven dimensions does not seem to have a Lax pair, at least in a direct way, due to the non-associativity of the octonionic algebra, it may happen that there is a generalization of the zero-curvature condition under which this system is integrable. In the case of three dimensions the restriction of the solutions to the $SU(2)$ subalgebra of functions on the spherical membrane reduces the problem to the study of Nahm's SU(2) equations. In the same way, in seven dimensions the introduction of the quadratic algebra of functions on the sphere reduces the problem to the generalization of Nahm's equations with similar scaling properties with respect to time. This gives indications that the specific system maybe relevant for the study of monopole type of configurations of membranes.

The relevance of the self duality membrane equations in seven dimensions for the spectrum of instantons of the 11-dimensional supermembrane is an open problem as well as the number of supersymmetries surviving these solutions.

6. Appendix

In this Appendix we derive the properties of the tensors $X^{ij}{}_{kl}(u)$ used in section 3 to make consistency checks of our Ansatz. Consider the generalized matrices $\mathcal{P}^{ij}{}_{kl}(u,v)$.

$$\mathcal{P}^{ij}{}_{kl}(u,v) = u\Delta^{ij}{}_{kl} + \frac{v}{4}\phi^{ij}{}_{kl} \tag{79}$$

Using the properties

$$\Delta^{ij}{}_{kl}\Delta^{mn}{}_{ij} = \Delta^{mn}{}_{kl} \tag{80}$$

$$\phi^{ij}{}_{mn}\phi^{mn}{}_{kl} = 8\left(\Delta^{ij}{}_{kl} + \frac{1}{4}\phi^{ij}{}_{kl}\right) \tag{81}$$

$$\Delta^{ij}{}_{kl}\phi^{kl}{}_{mn} = \phi^{ij}{}_{mn} \tag{82}$$

we derive the following multiplication rule

$$\mathcal{P}^{ij}{}_{kl}(u_1,v_1)\mathcal{P}^{ij}{}_{kl}(u_2,v_2) = \mathcal{P}^{ij}{}_{kl}(u_3,v_3) \tag{83}$$

where

$$u_3 = u_1 u_2 + \frac{v_1 v_2}{2}$$

$$v_3 = u_1 v_2 + u_2 v_1 + \frac{v_1 v_2}{2}$$

We observe that this is a group structure which can be realised as a subgroup of the general linear group in two dimensions through the matrices

$$G(u,v) = \begin{pmatrix} u & v/2 \\ v & u + v/2 \end{pmatrix} \tag{84}$$

For the existence of the inverse, one should restrict the parameters u,v inside the angular regions

$$v = u \tag{85}$$

$$v = -2u \tag{86}$$

For the case of $u = 1$, we restrict to $\mathcal{P}^{ij}{}_{kl}(1, v) \equiv X^{ij}{}_{kl}(v)$ matrices. Using the above, we find the multiplication law

$$
\begin{aligned}
X^{ij}{}_{kl}(u)X^{mn}{}_{ij}(v) &= (1 + \frac{uv}{2})\Delta^{ij}{}_{kl} + \frac{1}{4}(u + v + \frac{uv}{2})\phi^{ij}{}_{kl} \\
&\equiv (1 + \frac{uv}{2})X^{mn}{}_{kl}(w)
\end{aligned}
\tag{87}
$$

where

$$
w = \frac{u + v + uv/2}{1 + uv/2}
\tag{88}
$$

Form the properties of the symbol Ψ_{ijk}, we find

$$
\Psi_{ijk}X^{jk}{}_{lm}(u) = (1 + u)\Psi_{ilm}
\tag{89}
$$

so for $u = -1$, the $X^{jk}(u)$ antisymmetric matrices satisfy the costraint for the G_2 algebra

$$
\Psi_{ijk}X^{jk}{}_{lm}(-1) = 0
\tag{90}
$$

We finally observe the following interesting projective properties for the end points of the group parameter u

$$
X^{ij}{}_{mn}(u)X^{mn}{}_{kl}(2) = (1 + u)X^{ij}{}_{kl}(2)
\tag{91}
$$

$$
X^{ij}{}_{mn}(u)X^{mn}{}_{kl}(-1) = (1 - \frac{u}{2})X^{ij}{}_{kl}(-1)
\tag{92}
$$

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