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The gluon contribution to the polarized structure function g_2

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Abstract

We compute the structure function g_2 for a gluon target in perturbative QCD at order α_S. We show that its first moment vanishes, as predicted by the Burkhardt-Cottingham sum rule.

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A computation of the structure function g_2 for a target quark at order α_S in perturbative QCD has been performed in ref. [1]. The interest in that computation was mainly driven by the possibility of performing a direct test of the validity of the Burkhardt-Cottingham (BC) sum rule [2] in perturbative QCD. The BC sum rule states that

$$\int_0^1 dx\, g_2(x, Q^2) = 0 \tag{1}$$

on the basis of general arguments about the analytic structure in the complex $\nu \equiv Q^2/(2x)$ plane of the amplitude whose imaginary part gives g_2 (a detailed discussion on the derivation and the validity of the BC sum rule is given in ref. [3]). It is therefore interesting to check whether eq. (1) is valid in perturbative QCD. The result of ref. [1] (later confirmed in ref. [4]) is that the first moment of g_2 actually vanishes at order α_S for a massive target quark (at leading order, g_2 itself vanishes for a target quark).

In this note we present an analogous calculation for a gluon target. We will show that also in this case the BC moment vanishes in a wide class of regularization schemes for the collinear singularities.

Structure functions are defined starting from the Fourier transform of the forward matrix element of the product of two electromagnetic currents between polarized states:

$$W^{\mu\nu}(p, q, s) = \frac{1}{4\pi} \int d^4x\, e^{iqx} \langle p, s | J^\mu(x) J^\nu(0) | p, s \rangle, \tag{2}$$

where p and s are the target momentum and spin four-vectors, respectively. In the case of a target gluon, the tensor $W^{\mu\nu}$ is given by

$$W^{\mu\nu}(p, q, s) = W^{\mu\nu\rho\sigma}(p, q) \epsilon_\rho \epsilon^*_\sigma, \tag{3}$$

where the gluon polarization vector ϵ carries the dependence on s. Only the antisymmetric part of $W^{\mu\nu}$ is relevant for the computation of polarized structure functions; we have

$$iW_A^{\mu\nu} = \frac{1}{2}(W^{\mu\nu} - W^{\nu\mu}) = \frac{1}{2}(W^{\mu\nu\rho\sigma} - W^{\nu\mu\rho\sigma})\epsilon_\rho \epsilon^*_\sigma = W^{\mu\nu\rho\sigma} \frac{1}{2}(\epsilon_\rho \epsilon^*_\sigma - \epsilon_\sigma \epsilon^*_\rho), \tag{4}$$

where in the last step we used the symmetry of $W^{\mu\nu\rho\sigma}$ under the simultaneous exchanges $\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma$. We are therefore interested in computing the antisymmetric part of the gluon polarization density matrix $\rho_{\rho\sigma} = \epsilon_\rho \epsilon^*_\sigma$. To do this, we assume that the gluon in the initial state is off the mass shell, $p^2 \neq 0$; this allows us to define a longitudinal spin vector for the gluon:

$$s^\alpha = \lambda N \left(p^\alpha - \frac{p^2}{pq} q^\alpha \right), \tag{5}$$

which satisfies the transversity condition $ps = 0$; λ is the gluon helicity, and the normalization factor N (real and positive) is related to s^2 by

$$N^2 = -\frac{s^2}{p^2\tilde{\beta}^2}, \tag{6}$$

where

$$\tilde{\beta} = \sqrt{1 - \frac{p^2q^2}{(pq)^2}} = \sqrt{1 + \frac{4p^2x^2}{Q^2}}, \tag{7}$$

with the usual definitions $Q^2 = -q^2$, $x = Q^2/(2pq)$. We see that we can choose N so that $|s^2| = 1$, provided s^2 and p^2 have opposite signs. We can now write the antisymmetric component of $\epsilon_\rho\epsilon^*_\sigma$ as a function of p and s:

$$\frac{1}{2}(\epsilon_\rho\epsilon^*_\sigma - \epsilon_\sigma\epsilon^*_\rho) = -\frac{i}{2\sqrt{|p^2|}}\,\epsilon_{\rho\sigma\alpha\beta}p^\alpha s^\beta; \tag{8}$$

the normalization is fixed by the condition that, for $p^2 \to 0^+$, the imaginary part of ρ_{12} is equal to $-\lambda/2$, where λ is the gluon helicity.

The structure functions g_1 and g_2 are conventionally defined by means of the following, general parametrization of $iW_A^{\mu\nu}$:

$$iW_A^{\mu\nu} = \frac{i\sqrt{|p^2|}}{pq}\epsilon^{\mu\nu\rho\sigma}q_\rho\left[g_1(x,Q^2)s_\sigma + g_2(x,Q^2)\left(s_\sigma - \frac{qs}{pq}p_\sigma\right)\right], \tag{9}$$

where s is now a generic spin vector (not necessarily longitudinal, since the gluon is off the mass shell). One can define projectors $P^i_{\mu\nu}$, $i = 1,2$ such that

$$P^i_{\mu\nu}W_A^{\mu\nu} = g_i \quad (i = 1,2). \tag{10}$$

One possible choice is the following:

$$P^1_{\mu\nu} = P^{-1}\epsilon_{\mu\nu\alpha\beta}\,q^\alpha\left(p^\beta + \frac{qs}{s^2}\frac{p^2}{pq}s^\beta\right) \tag{11}$$

$$P^2_{\mu\nu} = P^{-1}\epsilon_{\mu\nu\alpha\beta}\,p^\alpha\left(q^\beta - \frac{qs}{s^2}s^\beta\right), \tag{12}$$

with

$$P = 2\sqrt{|p^2|}qs\left[1 - \frac{p^2q^2}{(pq)^2}\left(1 - \frac{(qs)^2}{q^2s^2}\right)\right]. \tag{13}$$

Notice that $P = 0$ even for $p^2 \neq 0$ if s is purely longitudinal, as one can see using eqs. (5) and (6); we must assume that s has a transverse component until we have projected out the structure functions, which are by construction independent of s. Collecting everything together, we finally obtain

$$g_1(x,Q^2) = -P^{-1}\epsilon^{\mu\nu\gamma\delta}\, q_\gamma \left(p_\delta + \frac{qs}{s^2}\frac{p^2}{pq}s_\delta\right)\frac{1}{2\sqrt{|p^2|}}\,\epsilon^{\rho\sigma\alpha\beta}p_\alpha s_\beta W_{\mu\nu\rho\sigma}(p,q) \quad (14)$$

$$g_2(x,Q^2) = -P^{-1}\epsilon^{\mu\nu\gamma\delta}\, p_\gamma \left(q_\delta - \frac{qs}{s^2}s_\delta\right)\frac{1}{2\sqrt{|p^2|}}\,\epsilon^{\rho\sigma\alpha\beta}p_\alpha s_\beta W_{\mu\nu\rho\sigma}(p,q). \quad (15)$$

Even before explicitly computing $W_{\mu\nu\rho\sigma}$, we can check that g_1 and g_2 given by eqs. (14) and (15) are indeed independent of s, as they should. In fact, it is easy to prove that only a term proportional to $g_{\mu\rho}p_\nu q_\sigma$ gives a non-zero contribution to g_1, while in the case of g_2 the only surviving term is proportional to $g_{\mu\rho}q_\nu q_\sigma$. Inserting these terms in eqs. (14) and (15), the s-dependence is seen to cancel against the factor P^{-1}.

We now proceed to compute $W_{\mu\nu\rho\sigma}(p,q)$. The calculation in this case is much simpler than in the quark case, because at order α_s there are no loop diagrams that contribute to the relevant amplitude. For this reason, no ultraviolet or soft divergences are involved. We have

$$W_{\mu\nu\rho\sigma} = \frac{1}{4\pi}\frac{1}{N_c^2-1}\sum_{colour}\int d\phi^{(2)}\left(A^{(1)}_{\mu\rho}+A^{(2)}_{\mu\rho}\right)\left(A^{(1)}_{\nu\sigma}+A^{(2)}_{\nu\sigma}\right)^*, \quad (16)$$

where $N_c = 3$ is the number of colours, and $d\phi^{(2)}$ is the two-body phase space. The amplitudes $A^{(1)}$, $A^{(2)}$ correspond to the diagrams of fig. 1. The singularities that arise when an on-shell gluon radiates massless quarks in the collinear configuration may be regularized either by a non-zero gluon virtuality p^2, or by a non-vanishing quark mass m. We will keep both p^2 and m different from zero at this level, and we will discuss later the behaviour of our results in the limit $p^2, m^2 \to 0$. In the photon-gluon centre-of-mass frame, the momentum k of the produced quark is

$$k = \frac{E}{2}(1, 0, \beta\sin\theta, \beta\cos\theta), \quad (17)$$

where

$$E^2 = (p+q)^2 = p^2 + Q^2\frac{1-x}{x} \quad (18)$$

and

$$\beta = \sqrt{1-\frac{4m^2}{E^2}}. \quad (19)$$

The two-body phase space $d\phi^{(2)}$ takes the form

$$d\phi^{(2)} = \frac{\beta}{16\pi}d\cos\theta. \quad (20)$$

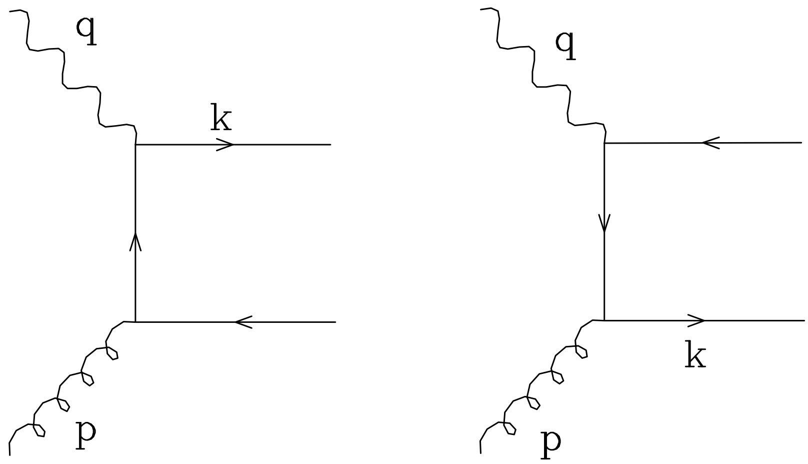

Figure 1: *Diagrams contributing to $W^{\mu\nu}$ for a gluon target.*

The denominators of the virtual quark propagators appearing in the amplitude are given by

$$(k-q)^2 - m^2 = -pq\left(1+\beta\tilde{\beta}\cos\theta\right) \tag{21}$$

$$(k-p)^2 - m^2 = -pq\left(1-\beta\tilde{\beta}\cos\theta\right). \tag{22}$$

The phase-space integration is therefore singular for $\cos\theta \to \pm 1$ when $\beta = \tilde{\beta} = 1$, or equivalently $p^2 = m^2 = 0$. The calculation is straightforward (we have performed it with the help of the algebraic manipulation program MACSYMA); the $\cos\theta$ integration can easily be performed by observing that, after inserting eq. (16) in eqs. (14) and (15), the numerator of the integrand expression is a degree-2 polynomial in the invariants kq, kp and ks. Terms proportional to powers of kq and kp can be integrated immediately, since their dependence on $\cos\theta$ is given explicitly by eqs. (21) and (22). Terms containing powers of ks can also be expressed in terms of integrals containing only kq and kp. Consider for example

$$I^\mu = \int d\phi^{(2)}\, f(kp,kq)\, k^\mu, \tag{23}$$

where $f(kp,kq)$ is a generic scalar function. The result must be a linear combination of q^μ and p^μ:

$$I^\mu = A\, q^\mu + B\, p^\mu, \tag{24}$$

and the scalar coefficients A and B can be obtained by solving the system

$$A\, q^2 + B\, pq = q_\mu I^\mu \tag{25}$$

$$A\,pq + B\,p^2 = p_\mu I^\mu, \tag{26}$$

so that, finally,

$$s_\mu I^\mu = \int d\phi^{(2)}\, f(kp, kq)\, ks = A\,qs. \tag{27}$$

Terms proportional to $(ks)^2$ can be treated in a similar way. Therefore, all phase-space integrals are of the type

$$\int_{-1}^{1} d\cos\theta\,(1+\beta\tilde{\beta}\cos\theta)^a(1-\beta\tilde{\beta}\cos\theta)^b, \tag{28}$$

with a and b integers between -2 and 2. We obtain the following results:

$$\begin{aligned} g_1 \;=\; & -\frac{e^2\alpha_S}{8\pi}\frac{1}{\tilde{\beta}^4 x}\Bigg[\frac{\beta}{1-\beta^2\tilde{\beta}^2}\,(4\tilde{\beta}^4x^2 - 8\beta^2\tilde{\beta}^2x^2 - 8\tilde{\beta}^2x^2 + 12x^2 - 2\beta^2\tilde{\beta}^4x \\ & -6\tilde{\beta}^4x + 8\beta^2\tilde{\beta}^2x + 12\tilde{\beta}^2x - 12x - \tilde{\beta}^6 + 2\beta^2\tilde{\beta}^4 + 3\tilde{\beta}^4 - 2\beta^2\tilde{\beta}^2 - 5\tilde{\beta}^2 + 3) \\ & -\frac{L}{2\tilde{\beta}}(4\tilde{\beta}^4x^2 - 8\tilde{\beta}^2x^2 + 12x^2 - 4\tilde{\beta}^4x + 12\tilde{\beta}^2x - 12x - \tilde{\beta}^6 + 3\tilde{\beta}^4 - 5\tilde{\beta}^2 + 3)\Bigg] \end{aligned} \tag{29}$$

$$\begin{aligned} g_2 \;=\; & -\frac{e^2\alpha_S}{8\pi}\frac{1}{\tilde{\beta}^4 x}\Bigg[\frac{\beta}{1-\beta^2\tilde{\beta}^2}\,(8\beta^2\tilde{\beta}^2x^2 + 4\tilde{\beta}^2x^2 - 12x^2 + 2\beta^2\tilde{\beta}^4x \\ & +2\tilde{\beta}^4x - 8\beta^2\tilde{\beta}^2x - 8\tilde{\beta}^2x + 12x - 2\beta^2\tilde{\beta}^4 - \tilde{\beta}^4 + 2\beta^2\tilde{\beta}^2 + 4\tilde{\beta}^2 - 3) \\ & -\frac{L}{2\tilde{\beta}}(4\tilde{\beta}^2x^2 - 12x^2 - 8\tilde{\beta}^2x + 12x - \tilde{\beta}^4 + 4\tilde{\beta}^2 - 3)\Bigg], \end{aligned} \tag{30}$$

where e is the electric charge of the produced quark in units of the positron charge, α_S is the strong coupling, and

$$L = \log\frac{1+\beta\tilde{\beta}}{1-\beta\tilde{\beta}}. \tag{31}$$

The collinear singularities are collected in the factor L, which diverges logarithmically when both m^2 and p^2 go to zero. The structure function g_1 was first computed in ref. [5] for $m = 0$, $p^2 < 0$. The general case $m^2 \neq 0, p^2 \neq 0$ was considered in ref. [6]. Our result for g_1, eq. (29), is different from the analogous formula obtained in ref. [6]. The origin of this discrepancy is the fact that the operator used in ref. [6] to obtain g_1 from $W_A^{\mu\nu}$ actually projects out the desired structure function only in the limit $p^2 \to 0$, while p^2 is kept non-zero elsewhere at this stage. However, the final result of ref. [6] is correct in the physically interesting limit, as we shall see later. Equation (30), on the other hand, is a new result.

For $Q^2 \to \infty$ with

$$r = \frac{-p^2}{m^2} \tag{32}$$

fixed, we find

$$g_1 = -\frac{e^2\alpha_S}{4\pi}\left[\frac{4rx^3 - 6rx^2 + 2rx - 4x + 3}{rx^2 - rx - 1} - (2x-1)L\right] \tag{33}$$

$$g_2 = +\frac{e^2\alpha_S}{4\pi}\left[\frac{6rx^3 - 10rx^2 + 4rx - 4x + 3}{rx^2 - rx - 1} - (2x-1)L\right]. \tag{34}$$

In this limit, L takes the form

$$L = \log\frac{Q^2}{m^2} - \log\left(rx^2 + \frac{x}{1-x}\right). \tag{35}$$

It is interesting to notice that the terms proportional to L, which contains the collinear divergence, cancel in the sum $g_1 + g_2$.

As a test of the correctness of our calculation, we can check that we reproduce the known results for g_1. Indeed, eq. (33) gives

$$r = 0: \quad g_1 = \frac{e^2\alpha_S}{4\pi}\left[-4x + 3 + (2x-1)\left(\log\frac{Q^2}{m^2} - \log\frac{x}{1-x}\right)\right], \tag{36}$$

$$\int_0^1 dx\, g_1(x, Q^2) = 0; \tag{37}$$

$$r \to \infty: \quad g_1 = \frac{e^2\alpha_S}{4\pi}\left[-4x + 2 + (2x-1)\left(\log\frac{Q^2}{-p^2} - \log x^2\right)\right], \tag{38}$$

$$\int_0^1 dx\, g_1(x, Q^2) = -\frac{e^2\alpha_S}{4\pi}, \tag{39}$$

which coincide, for example, with the results of ref. [6]. The fact that different choices of the regularization scheme lead to different results for g_1 (and in particular for its first moment) has important physical implications, and has been widely discussed in the literature [7].

We now turn to the structure function g_2. From eq. (34) we get

$$r = 0: \quad g_2 = \frac{e^2\alpha_S}{4\pi}\left[4x - 3 - (2x-1)\left(\log\frac{Q^2}{m^2} - \log\frac{x}{1-x}\right)\right], \tag{40}$$

$$r \to \infty: \quad g_2 = \frac{e^2\alpha_S}{4\pi}\left[6x - 4 - (2x-1)\left(\log\frac{Q^2}{-p^2} - \log x^2\right)\right]. \tag{41}$$

The first moment of g_2 vanishes in both cases; actually, it is easy to prove, using eq. (34), that

$$\int_0^1 dx\, g_2(x,Q^2) = 0 \tag{42}$$

for all values of r. Therefore we conclude that the BC sum rule is satisfied also by the gluon contribution to g_2 at order α_S, within the class of regularization schemes we have adopted.

Having computed $g_2(x,Q^2)$, its n-th moment

$$g_2^n = \int_0^1 dx\, x^{n-1}\, g_2(x,Q^2), \tag{43}$$

can be obtained for any n. In the two cases $r=0$ and $r\to\infty$, the n^{th} moment of g_2 is given by

$$r=0: \quad g_2^n = \frac{e^2\alpha_S}{4\pi}\left[\frac{4}{n+1} - \frac{3}{n} + \frac{1}{n^2} - \frac{n-1}{n(n+1)}\left(\log\frac{Q^2}{m^2} - S(n)\right)\right] \tag{44}$$

$$r\to\infty: \quad g_2^n = \frac{e^2\alpha_S}{4\pi}\left[\frac{6}{n+1} - \frac{4}{n} - \frac{4}{(n+1)^2} + \frac{2}{n^2} - \frac{n-1}{n(n+1)}\log\frac{Q^2}{-p^2}\right], \tag{45}$$

where $S(n) = \sum_{k=1}^n 1/k$. Once again, we see that both expressions vanish when $n=1$.

The computation presented here is equivalent to the calculation of gluon coefficient functions in the light-cone operator product expansion of $W_A^{\mu\nu}$; in that case, the quantities that are directly computed are odd moments of the coefficient functions; in the case of g_2, only moments for $n \geq 3$ are obtained, and therefore no direct test of the BC sum rule can be performed. Such a calculation was performed in ref. [5] in the case $m=0$, which corresponds to our eq. (45). The two results are in agreement.

In conclusion, we have performed a calculation of the structure function g_2 for a target gluon. We have considered various regularization schemes for the collinear divergences, and we have found that in all of them the first moment of g_2 vanishes as expected.

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