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Scaling Laws for Dynamic Aperture due to Chromatic Sextupoles

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Abstract

Scaling laws for the dynamic aperture due to chromatic sextupoles are investigated. The problem is addressed in a simplified lattice model containing $4N$ identical cells and one linear betatron phase shifter to break the overall cell-lattice symmetry. Two families of chromatic sextupoles are used to compensate the natural chromaticity. Analytical formulae for the dynamic aperture as a function of the number of cells and of the cell length are found and confirmed through computer tracking.

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1 Introduction

The new generation of hadron colliders should reach a center-of-mass energy of the order of 20 to 100 TeV. Since they should use superconducting dipoles with a maximum field of the order of 10 Tesla, their total length should be of 100 km or more [1, 2].

A sound design of the regular cell is of paramount importance both for the accelerator performance and cost. It is well known that the dynamic aperture depends quite strongly on the cell length and on the number of cells. In fact, long cells provide a moderate focusing of the circulating beam and a large value of the dispersion. Therefore, the chromaticity sextupole are more effective to compensate the chromaticity and less perturbing for long-term transverse beam stability. On the other hand, the larger beam dimension makes more harmful the effect of the field-shape imperfections of the superconducting magnets. On the contrary, short cells require stronger chromatic sextupoles but provide a stronger focusing and consequently a lower sensitivity to the field-shape imperfections. The parameters of the regular cell have been optimized in the past with computer tracking simulations [3, 4].

In order to understand the physical limitations to the dynamic aperture in the design of future large colliders, it would be highly desirable to have scaling laws that relate the dynamic aperture with the linear parameters of the lattice (i.e., the cell length, the cell phase advance and the number of cells). Indeed, it is well known that an analytical evaluation of the dynamic aperture for a generic lattice with nonlinearities is a very hard task, which has not been [5, 6] accomplished yet. In fact, excluding the trivial case where a low-order unstable resonance is dominant, in general the dynamic aperture is determined by a very intricate interplay of high order resonances [5, 6]. For this reason, one usually considers analytical quality factors such as the resonance strength or the detuning as indicators of a good or bad dynamic aperture [7, 8].

In this paper we restrict ourselves to the analysis of the dynamic aperture limitations due to chromatic sextupoles for a lattice whose free parameters are the cell length and the number of cells. In other words, we neglect not only the nonlinearities due to multipolar errors in the superconducting magnets, but also the chromatic effects of the insertion quadrupoles. Even with such a drastic simplification, the evaluation of the dynamic aperture cannot be done analytically. Nevertheless, assuming that the dynamic aperture is dominated by the first nonzero nonlinear term, we can rescale lattices with different linear parameters to the same lattice. From that, we can derive an analytical law that gives the dependence of the dynamic aperture on the number of cells and on the cell length. The assumption of the dominance of the first significant nonlinear order is justified *a posteriori* through plain tracking for several different configurations of the lattice model.

The main result of the paper is that, according to the obtained scaling law, it is possible to increase the length of a machine keeping fixed the dynamic aperture due to the chromaticity of the regular cells. This result is by far insufficient to fix the parameters of the regular cell. However, it allows one to establish the upper limit for the cell length, above which the beam stability is insufficient no matter how good is the field in the superconducting magnets.

The plan of the paper is the following: in section 2 we introduce the lattice model that we use for both the analytical and the numerical simulations. In section 3 we give an analytical derivation of the scaling law, and in section 4 we present the numerical results.

2 The lattice model

Although the lattice of a large collider is mostly made of regular cells, a simple test lattice that is only made of cells is an inappropriate model; in fact, the very strong symmetry of the lattice implies a large cancellation of the non-linear effects. Indeed, our lattice model is made of $4N$ regular cells and one insertion. There are chromatic sextupoles close to each cell-quadrupole. Any other nonlinear field, as the field-shape errors in the superconducting magnets, is neglected. The insertion is considered linear and its chromatic contribution is disregarded. Its description in the Courant-Snyder coordinates is a rotation by a constant angle $2\pi\nu$. The cell phase advance is fixed to $\pi/2$. Since we use $4N$ cells, we have a perfect cancellation of the first order effect of the chromatic sextupoles. If the number of cells is not a multiple of 4, but the machine has a large number of cells, one can show that the first order sextupolar terms is negligible with respect to the second order, that scales with N.

In the plane transverse to the orbit, the single-particle motion is described by the coordinates (x, y) and the associated momenta (p_x, p_y), whereas s is the coordinate along the orbit. The synchrotron motion is supposed to be uncoupled, therefore the betatron motion is described by an s-dependent Hamiltonian with two degrees of freedom. Fixing a section of the machine s_0, we evaluate the one-turn map in the four-dimensional phase space (x, p_x, y, p_y) [5].

2.1 Linear Map

Each FODO cell contains one focusing and one defocusing quadrupole, plus dipoles and drifts. We approximate the dipoles by drifts with a thin deflection in the middle, and the quadrupoles by thin lenses. Let L be the length of the half-cell, and K_1 be the integrated gradient of the quadrupole. The transfer matrix relative to the horizontal betatron motion is

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ -K_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ K_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 + K_1 L & 2L + L^2 K_1 \\ -LK_1^2 & 1 - K_1 L - L^2 K_1^2 \end{pmatrix}. \tag{1}$$

We restrict ourselves to a phase advance of 90 degrees per cell: the ratio between the quadrupole gradient and the semicell length is:

$$K_1 = \frac{\sqrt{2}}{L}. \tag{2}$$

Therefore the cell matrix reads

$$\mathbf{M} = \begin{pmatrix} 1 + \sqrt{2} & (2 + \sqrt{2})L \\ -\frac{2}{L} & -1 - \sqrt{2} \end{pmatrix}. \tag{3}$$

We can now work out the dependence of the optical functions on the distance s to the focusing quadrupole:

$$\beta_x(s) = (2 + \sqrt{2})L - s(2 + 2\sqrt{2}) + \frac{2s^2}{L}$$

$$
\begin{aligned}
\alpha_x(s) &= \frac{d\beta_x}{ds} = -2 - 2\sqrt{2} + \frac{4s}{L} \\
\mu_x(s) &= \int_0^s \frac{1}{\beta_x(l)} dl = arctan\left(\frac{2s}{L} - 1 - \sqrt{2}\right) + \frac{3}{8}\pi
\end{aligned} \tag{4}
$$

The beta and the alpha functions in the quadrupoles are

$$
\begin{aligned}
\beta_{xf} &= (2+\sqrt{2})L \qquad\qquad \beta_{xd} = (2-\sqrt{2})L \\
\alpha_{xf} &= \mp(1+\sqrt{2}) \qquad\qquad \alpha_{xd} = \pm(\sqrt{2}-1)
\end{aligned} \tag{5}
$$

The upper sign holds at the entrance and the lower sign at the exit of the quadrupoles. Let $\mathbf{T}$ be the Courant-Snyder transformation

$$
\begin{pmatrix} \hat{x} \\ \hat{p}_x \end{pmatrix} = \mathbf{T} \begin{pmatrix} x \\ p \end{pmatrix} \qquad \text{where} \qquad \mathbf{T} = \begin{pmatrix} \frac{1}{\sqrt{\beta_x}} & 0 \\ -\gamma_x & \sqrt{\beta_x} \end{pmatrix}. \tag{6}
$$

The cell matrix in the Courant-Snyder coordinates is a rotation of 90 degrees:

$$
\hat{\mathbf{M}} = \mathbf{TMT}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{7}
$$

In the following, we denote the transfer maps in the Courant-Snyder coordinates by the hat.

2.2 Nonlinear elements

In this section we derive the expressions for the sextupoles using the map formalism. We consider the sextupoles as nonlinear elements of zero length and of integrated gradient K_2 (kick approximation); let β_x be the beta function in the sextupole: then, the 2D horizontal transfer map of the sextupole kick reads

$$
\mathbf{K} = \begin{pmatrix} x \\ p_x + \frac{K_2 x^2}{2} \end{pmatrix}. \tag{8}
$$

The same map in the Courant-Snyder variables reads

$$
\hat{\mathbf{K}} = \mathbf{TKT}^{-1} = \begin{pmatrix} \hat{x} \\ \hat{p}_x + \beta_x^{3/2} \frac{K_2 \hat{x}^2}{2} \end{pmatrix}. \tag{9}
$$

2.3 Chromatic correction

The chromaticity due to the quadrupoles in the cell is

$$
\begin{aligned}
Q_x^{'quad} &= -\frac{1}{4\pi} K_1(\beta_f - \beta_d) = -\frac{1}{\pi} \\
Q_y^{'quad} &= -\frac{1}{4\pi} K_1(\beta_f - \beta_d) = -\frac{1}{\pi}.
\end{aligned} \tag{10}
$$

The dispersion in the dipoles is evaluated through the equation

$$
\begin{pmatrix} D_f \\ 0 \\ 1 \end{pmatrix} = \mathbf{M}_{1/2} \begin{pmatrix} D_d \\ 0 \\ 1 \end{pmatrix}, \tag{11}
$$

where $\mathbf{M}_{1/2}$ is the half cell matrix; we approximate the rectangular dipole matrix for the bending angle $\phi = 2\pi/4N \ll 1$ according to

$$D = \begin{pmatrix} 1 & L & L\phi/2 \\ 0 & 1 & \phi \\ 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

where ϕ is the deflection angle of each of the $4N$ cells. The semicell matrix reads

$$M_{1/2} = \begin{pmatrix} 1 - K_1 L/2 & L & L\phi/2 \\ -K_1^2 L/4 & 1 + K_1 L/2 & \phi + \phi K_1 L/4 \\ 0 & 0 & 1 \end{pmatrix}, \tag{13}$$

and therefore the dispersion is

$$\begin{aligned} D_f &= L\phi\left(2 + \frac{\sqrt{2}}{2}\right) = \frac{L\pi}{2N}\left(2 + \frac{\sqrt{2}}{2}\right) \\ D_d &= L\phi\left(2 - \frac{\sqrt{2}}{2}\right) = \frac{L\pi}{2N}\left(2 - \frac{\sqrt{2}}{2}\right). \end{aligned} \tag{14}$$

The chromaticity is usually corrected through two families of sextupoles placed close to the quadrupoles; we assume that the chromatic sextupoles are inside the quadrupoles, and therefore their chromatic contribution is

$$\begin{aligned} Q_x'^{sex} &= \tfrac{1}{4\pi}(K_{2f}\beta_{xf}D_f - K_{2d}\beta_{xd}D_d) \\ Q_y'^{sex} &= \tfrac{1}{4\pi}(K_{2f}\beta_{xd}D_f - K_{2d}\beta_{xf}D_d). \end{aligned} \tag{15}$$

Setting the sum of the quadrupole and of the sextupole chromaticities to zero, we obtain the value of the sextupole integrated gradients

$$\begin{aligned} K_{2f} &= \frac{1}{LD_f} = \frac{N}{\pi L^2}\frac{4}{4+\sqrt{2}} \\ K_{2d} &= \frac{1}{LD_d} = \frac{N}{\pi L^2}\frac{4}{4-\sqrt{2}}. \end{aligned} \tag{16}$$

3 Analytical derivation of scaling laws

3.1 First order one-turn map

The one-turn map of the analysed model is the composition of the 4N identical maps of the cell plus the phase shifter. We first assume that one of the two families of chromatic sextupoles can be neglected: therefore each cell contains only one nonlinearity; we carry out the computation in the Courant-Snyder coordinates, where the cell matrix is a rotation by $\pi/2$, the phase shifter is a rotation by $2\pi\nu$, and the sextupole map is given by Eq. (9):

$$\hat{\mathbf{M}} = \mathbf{R}(2\pi\nu)[\mathbf{R}(\pi/2)\hat{\mathbf{K}}]^{4N}. \tag{17}$$

One can explicitly carry out the computations for the linear part and for the first significant order of the nonlinear terms:

$$\hat{\mathbf{M}} = \mathbf{R}(2\pi\nu)\begin{pmatrix} \hat{x} - 8N\beta^3\frac{K_2^2}{2}\hat{p}_x\hat{x}^2 + O(x^4) \\ \\ \hat{p}_x - 8N\beta^3\frac{K_2^2}{2}\hat{p}_x^2\hat{x} + O(x^4) \end{pmatrix}. \tag{18}$$

Due to compensation between sextupoles separated by a phase of π, there is no second order term (powers of coordinates), and no first order term (power in the gradient).

3.2 Scaling laws

We assume that the dynamic aperture is determined by the first significant nonlinear term. Under this rather strong assumption (that will be verified *a posteriori* through tracking) one can derive an analytic scaling law: we rescale the phase space $\hat{X} = \sigma\hat{x}$ and $\hat{P} = \sigma\hat{p}_x$ to obtain

$$\begin{pmatrix} \hat{X}' \\ \\ \hat{P}'_x \end{pmatrix} = \mathbf{R}(2\pi\nu) \begin{pmatrix} \hat{X} - 8N\beta^3 \frac{K_2^2}{2\sigma^2} \hat{P}_x \hat{X}^2 + O(\hat{X}^4) \\ \\ \hat{P} - 8N\beta^3 \frac{K_2^2}{2\sigma^2} \hat{P}_x^2 \hat{X} + O(\hat{X}^4) \end{pmatrix}. \tag{19}$$

Setting $N\beta^3 \frac{K_2^2}{8\sigma^2} = 1$, all the lattices characterized by different N and L have the same dynamic aperture in the $\hat{X}, \hat{P}_x$ space. Since $(\hat{x}, \hat{p}_x) = (\hat{X}, \hat{P}_x)/\sigma$, in the Courant-Snyder space, the dynamic aperture $\hat{A}$ is $\propto \sigma^{-1}$. Therefore, in the physical space $x = \sqrt{\beta}\hat{x}$, the dynamic aperture A scales according to

$$A \propto \frac{\sqrt{\beta}}{\sigma} = \frac{1}{\sqrt{N}\beta K_2} \tag{20}$$

Since the beta function and the sextupole gradient K_2 depend on N and L according to

$$\beta \propto L \qquad\qquad K_2 \propto NL^{-2} \tag{21}$$

one finds out the following dependence on L and N

$$A(N, L) \propto \frac{L}{N^{3/2}} \tag{22}$$

If we want to increase the length of the machine $2\pi R = 8LN$ by a factor κ, keeping the same dynamic aperture, we must

- Increase the cell length $L \Longrightarrow \kappa^{3/4} L$
- Increase the number of cells $N \Longrightarrow \kappa^{1/4} N$

E. Keil suggested to work out the scaling law for the dynamic aperture in Courant and Snyder coordinates. Indeed, we are intersted to evaluate the invariant acceptance $\epsilon = \hat{A}^2 E$, where E is the beam energy. We have

$$\hat{A} \propto \sigma^{-1} = \frac{2\sqrt{2}}{\sqrt{N}\beta^{3/2} K_2} \tag{23}$$

$$N = \frac{2\pi R}{L} = \frac{2\pi B\rho}{LB}\frac{R}{\rho} \propto \frac{2\pi E}{LB}\frac{R}{\rho} \tag{24}$$

Where R is the average machine radius, ρ the radius of curvature and B the magnetic field intensity. Using (21), we can easily find:

$$\epsilon \propto \frac{LE}{N^3} \propto \frac{L^4 B^3}{E^2} \tag{25}$$

Therefore, the invariant acceptance ϵ is constant, at constant B, for $L \propto \sqrt{E}$.

It must be pointed out that these scaling laws are obtained using the following assumptions

- Only one family of chromatic sextupoles is considered: the influence of the weaker chromatic sextupole on the dynamic aperture is neglected.
- The dynamic aperture is dominated by the first significant nonlinear order.
- The dynamic aperture in the four dimensional phase space has the same behaviour as the dynamic aperture in the two dimensional phase space.

In the following section we will verify these hypotheses through direct tracking.

4 Numerical results

We have considered the following models:

- A 2D lattice made up of $4N$ cells plus a phase shifter, with a single sextupole for each cell. We fix the sextupole gradient and the cell length, and we increase N to check the dependence of the dynamic aperture on the number of cells. We use this model to verify the scaling law (22), i.e., that the dynamic aperture is dominated by the first significant order of the map.
- A 2D lattice made up of $4N$ cells plus a phase shifter, with two sextupoles families for each cell to correct the chromaticity. The sextupole gradients are fixed by the chromatic condition, and the free parameters are the cell length and the number of cells. We use this model to verify that the second family of sextupoles does not change the scaling law.
- The same lattice as before, but in 4D. This is done to verify the effect of the nonlinear coupling between the two planes (x, p_x) and (y, p_y).

The dynamic aperture is computed using the techniques outlined in Ref. [9]. In Fig. 1 we plot the dynamic aperture as a function of the number of cells N for a lattice with semicell length $L = 50$ m, sextupole gradient $K_2 = 1\ m^{-1}$ and linear tune $\nu_x = 0.28$. The tracking data (empty circles) agree very well with the scaling law (solid line). In Fig. 2 we plot the dynamic aperture as a function of the length of the machine $2\pi\rho = 8LN$, for a 2D lattice with chromatic correction. We selected three different semicell lengths $L = 12.5, 50, 200$ m and we varied the machine length from 1 to 1000 km. Also in this case the tracking data (empty circles) agree very well with the scaling law (solid line). In Fig. 3 we plot the same case of Fig. 2, but using a 4D model. The results are very similar to the previous case.

The main result of the numerical computations is that the scaling law derived for the extremely simplified model (only one sextupole family, 2D motion) holds also for the more realistic case (two sextupole families, 4D motion).

5 Conclusions

We have considered a simplified lattice model where the dynamic aperture is dominated by the chromatic sextupoles; we have shown that one can analytically derive the dependence of the dynamic aperture on the number of cells N and on the cell length L. The dynamic aperture turns out to be proportional to $NL^{-3/2}$. This formula shows that it is possible to increase the total length of a chromaticity corrected machine while conserving a fixed dynamic aperture.

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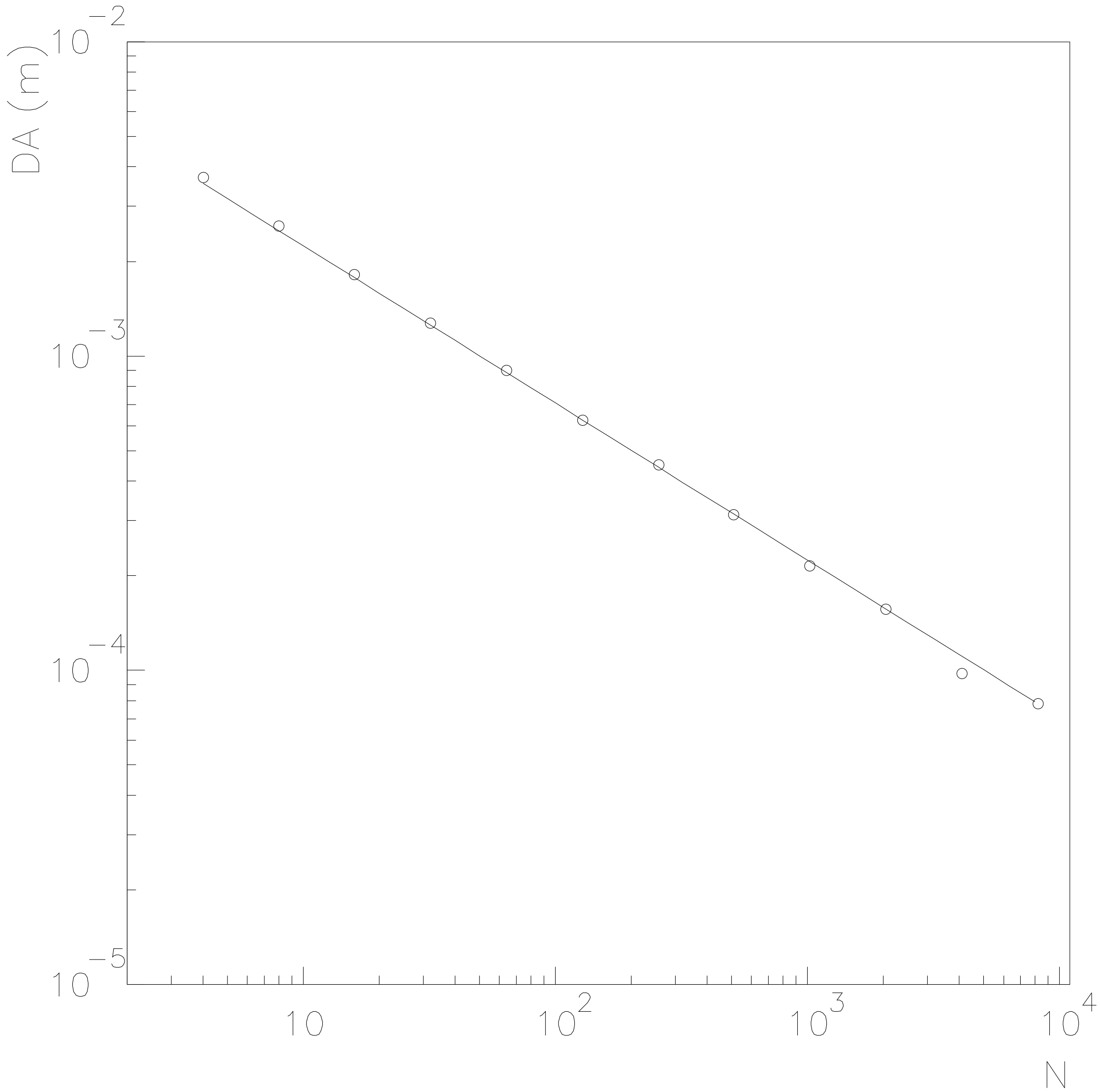

Figure 1: Dynamic aperture normalized at beta maximum versus number of cells for a lattice with one sextupole for each cell. Interpolation with the theoretical law $D(N) \propto N^{-1/2}$ is shown.

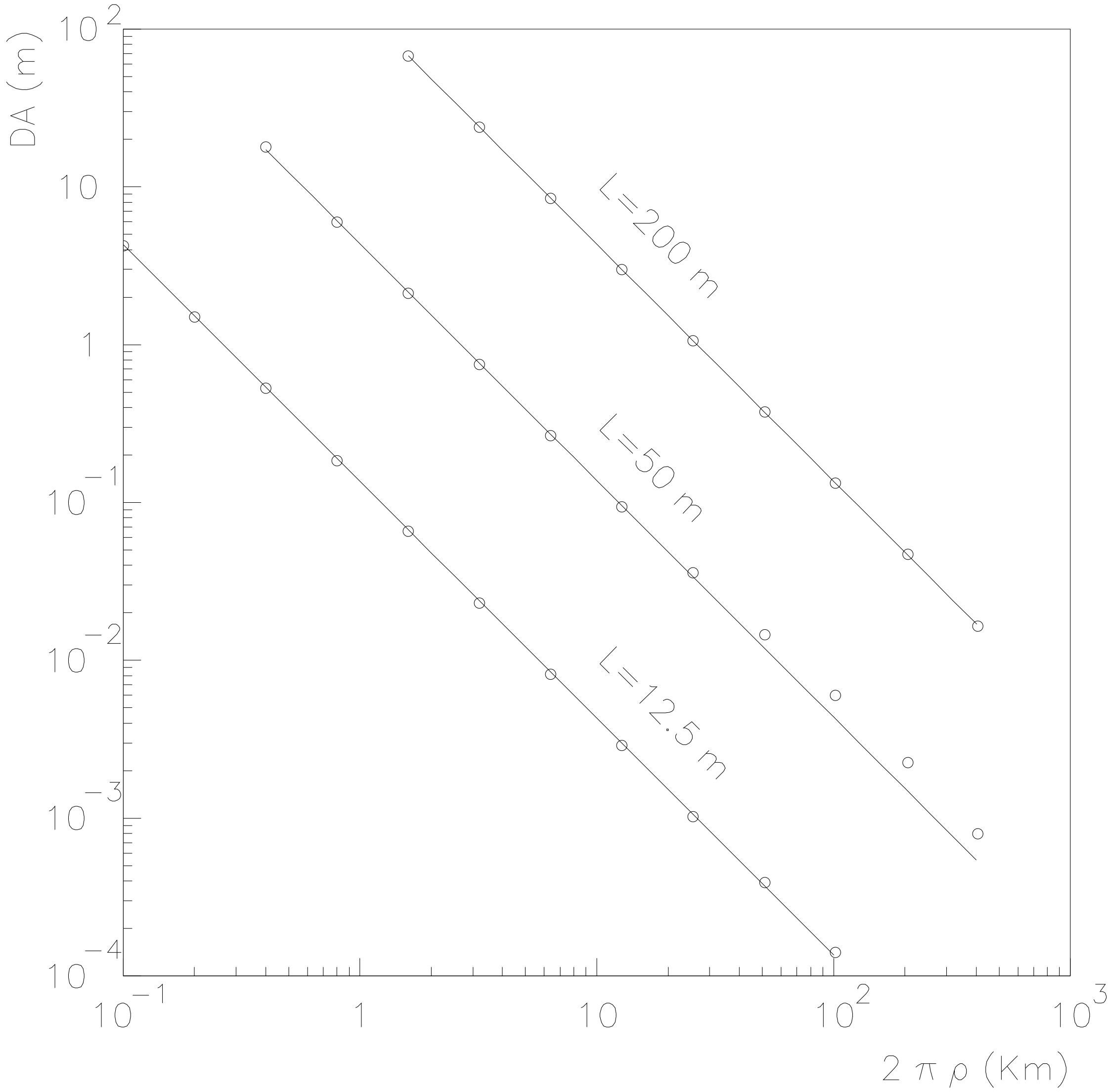

Figure 2: Dynamic aperture normalized at beta maximum versus the machine length for a 2D lattice with two chromatic sextupole families. Different semicell lengths are shown. Interpolations with the theoretical law $D(N) \propto N^{-3/2}$ are shown.

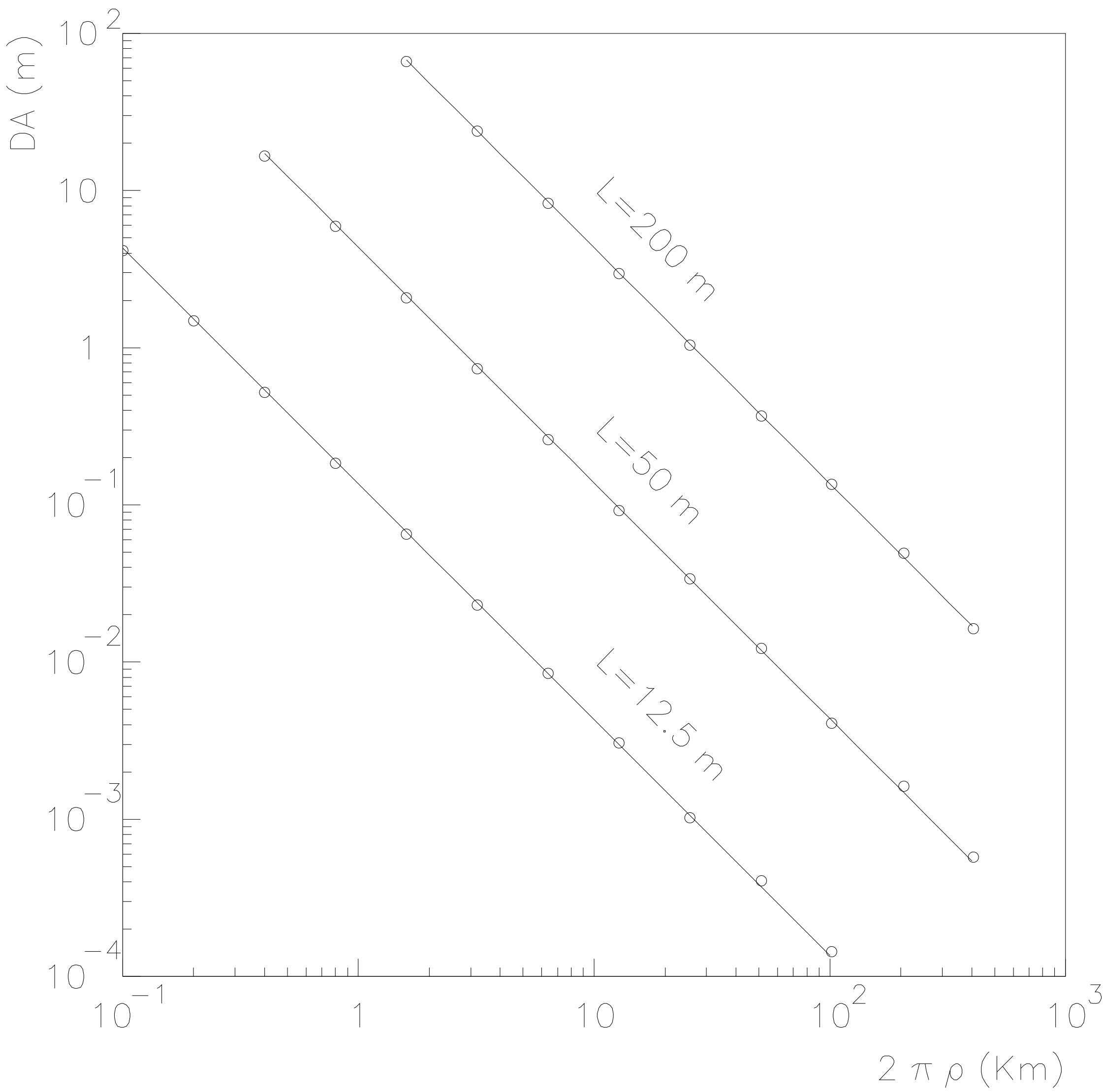

Figure 3: Dynamic aperture normalized at beta maximum versus the machine length for a 4D lattice with two chromatic sextupole families. Different semicell lengths are shown. Interpolations with the theoretical law $D(N) \propto N^{-3/2}$ are shown.