

UNSTABLE SYSTEMS IN RELATIVISTIC QUANTUM FIELD THEORY

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Abstract

We show how the state of an unstable particle can be defined in terms of stable asymptotic states. This general definition is used to discuss and to solve some old problems connected with the short-time and large-time behaviour of the non-decay amplitude.

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1 Introduction

The definition of unstable states in quantum mechanics is notoriously difficult[1],[2],[3]. The most usual approach is based on the Breit-Wigner approximation of scattering amplitudes[4] in which the unstable state is associated with a factorizable simple pole in the complex s -channel variable, s being the total momentum squared in s -channel. This approach, however, is not suitable to describe the detailed space-time behaviour of the unstable system. In particular it underscores the deviations from simple exponential behaviour at very short and very large times, which follow from general arguments[5].

The short-time behaviour of an unstable state weakly coupled to the final decay channels has been investigated recently in Ref.[6]. The result has been to underline an unexpected singularity in the short-time expansion of the non-decay amplitude, which makes the deviations from exponential behaviour in this region depend from the formation process of the unstable state itself. The short-time singularity and the corresponding dependence from the formation condition of the unstable state seems to eliminate the so-called Zeno paradox ([7]- [13]) at least for systems described in terms of Relativistic Quantum Field Theory.

In this paper, we present a general, relativistically invariant, definition of the unstable state which allows to elucidate the space-time behaviour of the non-decay amplitude. The definition follows closely the experimental procedure to measure the lifetime of weakly decaying particles (i.e. the so-called impact parameter distribution).

For weakly unstable particles, where perturbation theory can be used, we find that the general quantum-mechanical expression of the decay rate per unit time (the Fermi Golden Rule) has to be modified at short times in a well defined way. In addition, we recover the results of the previous analysis[6], with a much clearer characterization of the non- universal dependence upon the wave-packet shape of the initial particles from which the unstable state was formed.

The space-time behaviour of the non-decay amplitude is specified in terms of S -matrix amplitudes. Adopting for the latter the Breit-Wigner form with a constant width gives, for times larger than the formation time, the familiar exponential behaviour.

Finally, on the basis of unitarity arguments, one can estimate the very large time behaviour where again deviations from the exponential form are to be expected, in the form of a power-law time dependence. It is worth stressing that, in this case, the time behaviour is again a non-universal one, related to the form of the initial state wave packets.

Although not strictly related to the central arguments of the present paper, we also give, in the last section, a simple and general derivation of the so-called Wigner's delay relation[15] for scattering systems, because it fits very naturally within the general spatio-temporal description of the scattering process used in this investigation.

2 Constructing the Unstable State from Scratch

We define as usual the S - and the T - matrices, according to:

$$S_{\alpha\beta} = \langle \alpha; out | \beta; in \rangle = \langle \alpha; in | S | \beta; in \rangle = \langle \alpha; out | S | \beta; out \rangle$$

$$\begin{aligned}
S &= I + iT = \sum_{\delta} |\delta; in\rangle \langle \delta; out| \\
TT^+ &= T^+T = 2ImT
\end{aligned} \tag{1}$$

where α , β and δ denote a suitable set of free particle quantum numbers, e.g. momenta and spin components. However, to describe the space-time evolution of the scattering process it is necessary to introduce wave-packets for the initial states. We consider, therefore, a two-particle state in the far past:

$$|in\rangle = \int d^3p_1 d^3p_2 f(\underline{p}_1) g(\underline{p}_2) |\underline{p}_1, \underline{p}_2; in\rangle \tag{2}$$

We consider, for simplicity, equal mass, spinless, particles and we set ourselves in the center of mass (c.o.m.) frame of reference. We are interested in a situation in which the incoming particles can create a resonant state. We therefore choose the wave packets so that $f(\underline{p})$ is peaked around some momentum \underline{p}_{res} and $g(\underline{p})$ around momentum $-\underline{p}_{res}$ such that the c.o.m. energy is about equal to the resonance mass M :

$$2\omega(\underline{p}_{res}) \approx \sqrt{s} \approx M \tag{3}$$

Wave packets are chosen so as to represent very distantly localized particles, at some large, negative time $t = -T$, and to overlap around the origin of coordinates, at time $t = 0$. Neglecting long-range forces, which can be dealt with separately, wave packets evolve in time freely for all times before collision, up to the last 10^{-23} seconds or so. We consider the state:

$$-iT |in\rangle = (I - S) |in\rangle = |in\rangle - |out\rangle \equiv |R\rangle \tag{4}$$

The state $|out\rangle$, in Eq.[4], represents a two particle state in the distant future in the wave packets f and g , such that they were overlapping around the origin at $t = 0$. Thus the state vector $|R\rangle$ defined by Eq.[4] is the initial state minus the state where nothing happens, i.e. it represents the state of the products of the collision which has taken place around time $t = 0$.

We consider next the amplitude:

$$A(t) = \langle in | T^+ e^{-iHt} T | in \rangle \equiv \langle Rt | R \rangle \tag{5}$$

$A(t)$ represents the overlap of the collision state, when the collision has taken place at time $t = 0$, $|R\rangle$, with the collision state when the collision has taken place at a later time t (the state $\langle in | iT^+ e^{-iHt} \equiv \langle Rt |$). If the collision goes through the formation of an unstable state, $A(t)$ is, apart from a normalization factor, the amplitude for this state to have remained unchanged during time t , i.e. the non-decay amplitude, so that:

$$\begin{aligned}
A_{non-decay}(t) &= \frac{A(t)}{A(0)} \\
P_{non-decay}(t) &= \left| \frac{A(t)}{A(0)} \right|^2
\end{aligned} \tag{6}$$

Eq.[6] is our basic starting point. We write:

$$\langle \underline{p}_3, \underline{p}_4; in | T | \underline{p}_1, \underline{p}_2; in \rangle = (2\pi)^4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2) T(s, t) \tag{7}$$

where s and t are the usual Mandelstam variables. Using the unitarity relation, Eq.[5] can then be rewritten as:

$$\begin{aligned}
A(t) &= \int d^3 p_1 d^3 p_2 d^3 p_3 d^3 p_4 f^*(\underline{p}_3) g^*(\underline{p}_4) f(\underline{p}_1) g(\underline{p}_2) e^{-i(\omega_{p_1} + \omega_{p_2})t} \times \\
&\times (2\pi)^4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2) 2ImT(s, t) = \\
&= \left\langle e^{-i(\omega_{p_1} + \omega_{p_2})t} (2\pi)^4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2) 2ImT(s, t) \right\rangle
\end{aligned} \tag{8}$$

where brackets indicate folding with wave packets. Eq.[8] can be rewritten as:

$$A(t) = e^{-iMt} \int dx e^{-ixt} F(x) \tag{9}$$

with $F(x)$ a positive definite function with a limited-from-below support (corresponding to the combined support of ImT and of the wave packets):

$$F(x) \equiv \left\langle (2\pi)^4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2) \delta(\omega_{q_1} + \omega_{q_2} - M - x) 2ImT(s, t) \right\rangle \tag{10}$$

We add a few comments.

- The definition of the non-decay probability, Eqs.[5],[6], is given in term of S -matrix elements, i.e. it involves only the stable asymptotic states out of which the unstable state is formed (and in which it will, eventually, decay). No assumption is made, in particular, about the existence of a state representing the unstable particle itself (at time $t = 0$), which is indeed a very questionable assumption, certainly not valid beyond perturbation theory. If we had assumed that, we could have written:

$$A(t) = \langle R | e^{-iHt} | R \rangle = \sum_n e^{-iE_n t} |\langle n | R \rangle|^2 \tag{11}$$

The amplitude $A(t)$ in Eq.[5], as shown by Eqs.[9] and [10], shares with the naive amplitude, Eq.[11], the property of being the Fourier-transform of a positive definite function with a limited-from-below support. This property, first stressed by L.A. Khalfin[7] for the naive amplitude Eq.[11], gives rise to the non-exponential behaviour of $A(t)$ at very large times, as discussed in Section 6.

- The above definition of $A(t)$ makes sense only for times t which are larger than either the overlap time of the initial wave packets or the characteristic decay time of the background processes. In turn, this implies that the definition is useful only if the lifetime of the resonance is larger than either these characteristic times.

Since:

$$\Delta t|_{overlap} = \frac{\Delta x}{v} \approx \frac{1}{v\Delta p} \approx \frac{1}{\Delta E} \approx \frac{1}{M} \tag{12}$$

the first condition gives:

$$\Gamma = \frac{1}{\tau} \ll \frac{1}{\Delta t|_{overlap}} \approx M \tag{13}$$

Similarly, background processes (e.g. box- diagram contributions to the scattering process) decay in time with the only time-scale available, that is:

$$\Delta t|_{backgnd} \approx \frac{1}{\sqrt{s}} \approx \frac{1}{M} \tag{14}$$

so that, in conclusion, our definition is suitable for:

$$\Gamma = \frac{1}{\tau} \ll M \quad (15)$$

as intuitively expected.

- If we transform from the c.o.m. to another frame of reference, the time translation gives rise to a non vanishing space translation. The amplitude $A(t)$ transforms into the probability amplitude for the final particles to originate at a non- vanishing distance from the collision point. $P(t)$ corresponds, in this case, to the so-called impact parameter distribution of the decay products of an unstable particle produced with non-vanishing velocity.

3 Perturbation Theory

In this section we show how the previous general scheme works when perturbation theory is reliable. In particular we will discuss the derivation of the Fermi Golden Rule and its modifications at short times, related to the singularities inherent to Relativistic Quantum Field Theory.

We shall study, for definiteness, a system consisting of an unstable scalar particle ϕ , with mass M , decaying into two scalar particles ψ_1 and ψ_2 with masses m_1 and m_2 respectively. We take a decay hamiltonian, H_I , of the simple (super-renormalizable) form:

$$H_I = g \int d^3x \phi \psi_1 \psi_2 \quad (16)$$

The unstable particle ϕ contributes to the elastic scattering amplitude of the two particles ψ_1 and ψ_2 as:

$$S_{fi} \equiv \langle p_3, p_4; out | p_1, p_2; in \rangle = \delta_{fi} + \frac{i\mathcal{M}_{fi}}{\prod_i \sqrt{(2\pi)^3 2\omega_i}} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \quad (17)$$

where:

$$i\mathcal{M}_{fi} = (ig)^2 \frac{i}{P^2 - M^2 + i\varepsilon} - (ig)^2 \frac{i}{(P^2 - M^2 + i\varepsilon)} i\Pi(P^2) \frac{i}{(P^2 - M^2 + i\varepsilon)} \quad (18)$$

and:

$$P = p_1 + p_2 = p_3 + p_4 \quad (19)$$

$$i\Pi(P^2) \equiv \delta M^2 + i^2 \int dx \langle 0 | T(O(x)O(0)) | 0 \rangle \exp(iPx) \quad (20)$$

$$O(x) \equiv g\psi_1(x)\psi_2(x) \quad (21)$$

The counterterm δM^2 in Eq.[20] is adjusted so that:

$$Re\Pi(M^2) = 0 \quad (22)$$

and therefore M is the renormalized mass of the resonant state. The behaviour of $\Pi(P^2)$ around M is then:

$$\Pi(P^2) \underset{P^2 \approx M^2}{\approx} (P^2 - M^2) \text{Re}\Pi'(M^2) + i \text{Im}\Pi(P^2) \quad (23)$$

where $\text{Re}\Pi'(M^2)$ is an ultraviolet divergent parameter related to the wave function renormalization of the unstable particle propagator, which is to be removed by renormalization. It will be clear in a moment that this term does not affect the non-decay amplitude of the unstable state. As for $\text{Im}\Pi(P^2)$, it has the expression:

$$\text{Im}\Pi(P^2) = \frac{1}{2} \sum_n |\langle 0 | O(0) | n \rangle|^2 (2\pi)^4 \delta^{(4)}(P - P_n) \quad (24)$$

The overlap $A(t)$ defined in Eq.[5], then becomes:

$$\begin{aligned} A(t) = \langle Rt | R \rangle &= \int \frac{f^*(p_3)g^*(p_4)f(p_1)g(p_2)}{(2\pi)^6 \sqrt{2\omega_{p_3}2\omega_{p_4}2\omega_{p_1}2\omega_{p_2}}} \times \\ &\times (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \exp[-i(\omega_{p_1} + \omega_{p_2})t] \{-i\mathcal{M}_{fi} - (i\mathcal{M}_{fi})^*\} \end{aligned} \quad (25)$$

We have:

$$-i\mathcal{M}_{fi} - (i\mathcal{M}_{fi})^* = (g)^2 (2\pi) \delta(P^2 - M^2) - 2g^2 \text{Im} \left(\frac{\Pi(P^2)}{(P^2 - M^2 + i\varepsilon)^2} \right) \quad (26)$$

and:

$$\begin{aligned} \text{Im} \left(\frac{\Pi(P^2)}{(P^2 - M^2 + i\varepsilon)^2} \right) &= \\ \text{Im}\Pi(P^2) \text{Re} \frac{1}{(P^2 - M^2 + i\varepsilon)^2} &+ \text{Re}\Pi(P^2) \text{Im} \frac{1}{(P^2 - M^2 + i\varepsilon)^2} \end{aligned} \quad (27)$$

so that:

$$\text{Im} \left(\frac{\Pi(P^2)}{(P^2 - M^2 + i\varepsilon)^2} \right) = \frac{\text{Im}\Pi(P^2)}{(P^2 - M^2)^2} - 2\pi \text{Re}\Pi'(M^2) \delta(P^2 - M^2) \quad (28)$$

Eq.[25] then becomes:

$$\begin{aligned} \langle Rt | R \rangle &= \int \frac{f^*(p_3)g^*(p_4)f(p_1)g(p_2)}{(2\pi)^6 \sqrt{2\omega_{p_3}2\omega_{p_4}2\omega_{p_1}2\omega_{p_2}}} \times \\ &\times (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \exp[-i(\omega_{p_1} + \omega_{p_2})t] \times \\ &\times \left\{ g^2 (1 + 2\text{Re}\Pi'(M^2)) 2\pi \delta(P^2 - M^2) - 2g^2 \frac{\text{Im}\Pi(P^2)}{(P^2 - M^2)^2} \right\} \end{aligned} \quad (29)$$

For simplicity we choose to work in the narrow wave packet approximation. This means that we parametrize the momenta as:

$$\underline{p}_i = \underline{p}_{res\ i} + \underline{k}_i \quad (30)$$

and neglect terms quadratic in \underline{k}_i .

Eq.[29] can therefore be rewritten as:

$$\langle Rt | R \rangle = \langle Rt | R \rangle_1 + \langle Rt | R \rangle_2 \quad (31)$$

with:

$$\begin{aligned} \langle Rt | R \rangle_1 &= 2\pi g^2 (1 + 2Re\Pi'(M^2)) \exp[-iMt] \int \frac{f^*(p_3)g^*(p_4)f(p_1)g(p_2)}{(2\pi)^6 \sqrt{2\omega_{p_3}2\omega_{p_4}2\omega_{p_1}2\omega_{p_2}}} \times \\ &\times (2\pi)^4 \delta^{(3)}(\underline{k}_1 + \underline{k}_2 - \underline{k}_3 - \underline{k}_4) \delta[2M\underline{v} \cdot (\underline{k}_1 - \underline{k}_2)] \delta[\underline{v} \cdot (\underline{k}_3 - \underline{k}_4)] \end{aligned} \quad (32)$$

and:

$$\begin{aligned} \langle Rt | R \rangle_2 &= -2g^2 \exp[-iMt] \int \frac{f^*(p_3)g^*(p_4)f(p_1)g(p_2)}{(2\pi)^6 \sqrt{2\omega_{p_3}2\omega_{p_4}2\omega_{p_1}2\omega_{p_2}}} \times \\ &\times (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \exp[-i[\underline{v} \cdot (\underline{k}_1 - \underline{k}_2)]t] \frac{Im\Pi(P^2)}{[2M\underline{v} \cdot (\underline{k}_1 - \underline{k}_2)]^2} \end{aligned} \quad (33)$$

Using Eq.[27], Eq.[33] becomes:

$$\begin{aligned} \langle Rt | R \rangle_2 &= \\ &= -2\pi g^2 \exp[-iMt] \sum_n |\langle 0 | O(0) | n \rangle|^2 (2\pi)^3 \delta^{(3)}(\underline{P}_n) \frac{\exp -i(E_n - M)t}{4M^2(E_n - M)^2} \times \\ &\times \int \frac{f^*(p_3)g^*(p_4)f(p_1)g(p_2)}{(2\pi)^6 \sqrt{2\omega_{p_3}2\omega_{p_4}2\omega_{p_1}2\omega_{p_2}}} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \delta[M + \underline{v} \cdot (\underline{k}_1 - \underline{k}_2) - E_n] \end{aligned} \quad (34)$$

We can now define α , α_0 , α_1 and β_1 through:

$$\langle Rt | R \rangle_1 \equiv 2\pi g^2 \alpha \exp -iMt \equiv (2\pi g^2)(\alpha_0 + \alpha_1) \exp -iMt \quad (35)$$

and

$$\langle Rt | R \rangle_2 \equiv 2\pi g^2 \beta_1(t) \exp -iMt \quad (36)$$

where α is independent of t , α_0 is order 0 in g^2 , while α_1 and $\beta_1(t)$ are first order in g^2 .

We finally have for the properly normalized non-decay probability defined in Eq.[6]:

$$\begin{aligned} P_{non-decay}(t) &= \frac{|\langle Rt | R \rangle|^2}{|\langle R0 | R \rangle|^2} = \frac{|\langle Rt | R \rangle_1 + \langle Rt | R \rangle_2|^2}{|\langle R0 | R \rangle_1 + \langle R0 | R \rangle_2|^2} \approx \\ &\approx \frac{1 + 2Re[\frac{\alpha_1 + \beta_1(t)}{\alpha_0}]}{1 + 2Re[\frac{\alpha_1 + \beta_1(0)}{\alpha_0}]} \approx 1 + 2Re[\frac{\beta_1(t) - \beta_1(0)}{\alpha_0}] \end{aligned} \quad (37)$$

If, for simplicity, we take real momentum- space wave functions, we get:

$$P_{non-decay}(t) = 1 - \frac{2}{M} \sum_n \frac{|\langle 0 | O(0) | n \rangle|^2}{(E_n - M)^2} (2\pi)^3 \delta^{(3)}(\underline{P}_n) \sin^2[\frac{(E_n - M)}{2}t] H(E_n - M) \quad (38)$$

where:

$$\begin{aligned} H[E_n - M] &\equiv \\ &\equiv \frac{\int \frac{f(p_3)g(p_4)f(p_1)g(p_2)}{(2\pi)^6 \sqrt{2\omega_{p_3}2\omega_{p_4}2\omega_{p_1}2\omega_{p_2}}} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \delta[M - E_n + \underline{v} \cdot (\underline{k}_1 - \underline{k}_2)]}{\int \frac{f(p_3)g(p_4)f(p_1)g(p_2)}{(2\pi)^6 \sqrt{2\omega_{p_3}2\omega_{p_4}2\omega_{p_1}2\omega_{p_2}}} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \delta[\underline{v} \cdot (\underline{k}_1 - \underline{k}_2)]} \end{aligned} \quad (39)$$

is a function which provides a cutoff in energy at small times t , when the energy conservation is not yet active. In fact, due to the finite spread in energy of the wave packets, $H(x)$ rapidly vanishes for large x , while it goes to unity for vanishing x :

$$H[0] = 1 \quad (40)$$

Eq.[38] reproduces the result first found in Ref.[6]. The "form-factor", $H(E_n - M)$, provides the cut-off to the sum over intermediate states which is generally needed to cope with the singular behavior at very small times, as further discussed in the next section. At larger times, t , on the other hand:

$$\left[2 \frac{\sin[\frac{E_n - M}{2}t]}{E_n - M} \right]^2 \rightarrow 2\pi t \delta(E_n - M) \quad (41)$$

and Eq.[38], in virtue of Eq.[40], reduces to the usual Golden Rule formula for the non-decay probability of an unstable particle:

$$P_{non-decay}(t) = 1 - \frac{t}{2M} \sum_n |\langle 0 | O(0) | n \rangle|^2 (2\pi)^4 \delta^{(3)}(\underline{P}_n) \delta[E_n - M] \quad (42)$$

4 The Nature of Short-Time Singularities

As discussed in Ref.[6], the use of perturbation theory without taking into account the formation time of the resonant state, would lead to an expression for $P_{non-decay}(t)$ similar to the one given by Eq.[38] with $H(E_n - M) \equiv 1$, i.e.:

$$P_{non-decay}(t) = 1 - \frac{2}{M} \sum_n \frac{|\langle 0 | O(0) | n \rangle|^2}{(E_n - M)^2} (2\pi)^3 \delta^{(3)}(\underline{P}_n) \sin^2\left[\frac{(E_n - M)}{2}t\right] \quad (43)$$

It was shown in Ref.[6] that the expansion of Eq.[43] in powers of t is usually marred by meaningless ultraviolet divergencies. In this section we discuss in more detail the nature of these short-time singularities. This discussion clarifies the nature of the assumptions needed to derive the so called Zeno paradox ([7]-[13]). In fact the Zeno paradox strongly depends on the quadratic short-time behaviour, usually inferred from the (highly formal) argument:

$$\begin{aligned} P_{non-decay}(t) &= | \langle P | e^{-iHt} | P \rangle |^2 = \\ &= 1 - t^2 (\langle P | H^2 | P \rangle - | \langle P | H | P \rangle |^2) + \dots = \\ &= 1 - \Delta E^2 t^2 + \dots \end{aligned} \quad (44)$$

Expanding the explicit expression of $P_{non-decay}(t)$ given by Eq.[43] in powers of t , we have:

$$\begin{aligned} P_{non-decay}(t) &= 1 - \frac{t^2}{2M} \sum_n |\langle 0 | O(0) | n \rangle|^2 (2\pi)^3 \delta^{(3)}(\underline{P}_n) = \\ &= 1 - \frac{t^2}{2M} \int d\underline{x} \langle 0 | O(\underline{x}, 0) O(0) | 0 \rangle \end{aligned} \quad (45)$$

Eq.[45] shows that in this way we get, indeed, formally, a t^2 short-time behaviour, as in Eq.[44]. The problem is that the operator product appearing in Eq.[45] is not

integrable. In fact, apart from possible logs, we have from the Operator Product Expansion:

$$O(\underline{x}, 0)O(0) \underset{\underline{x} \approx 0}{\approx} \frac{1}{|\underline{x}|^{2d_O}} I + \dots \quad (46)$$

where d_O denotes the dimension of the operator O . In the present example $d_O = 2$ and Eq.[45] suffers from a linear ultraviolet divergence. The situation worsens for more singular (higher dimensional) decay hamiltonians, as discussed in Ref.[6].

These considerations clearly show that there is a short-time ultraviolet singularity which:

- makes ΔE^2 divergent in Eq.[44]
- is smeared, in the particle survival amplitude Eq.[38], over time scales provided by the formation mechanism of the unstable state (the overlap time of the incident wave-packets).

It should be clear that no form factors (in the case of decays involving hadrons, as e.g. proton decay) can cure this divergence which is quite similar to the one measured in deep-inelastic- scattering, due to the singular product of two currents.

In Eq.[38], we can interpret the presence of $H(E_n - M)$ as a cut-off on the energy of detected final decay products (a reasonable requirement for a measuring apparatus), which, by the optical theorem, produces an effective smearing on the particle survival amplitude.

We stress again the conclusions of Ref.[6] concerning the fallacy of the finiteness of ΔE^2 in Eq.[44], on which many of the papers dealing with the Zeno paradox are based.

5 The Exponential Decay

Higher orders in perturbation theory give rise to higher powers of the time, t , in Eq.[42]. For times of the order of the lifetime τ we need to sum up at least the leading terms, to get a meaningful result.

The linear behaviour in time, in Eq.[42], arises from the singular behaviour in $(s - M^2)$ of the self-energy insertion. Thus, higher powers in t shall correspond to repeated self-energy insertions in the lowest-order particle propagator.

More precisely, the leading singular behaviour is given by insertion of the imaginary part of the self-energy evaluated at $s = M^2$. Inserting the successive terms of the expansion of $Im\Pi$ around $s = M^2$ would give rise to less singular terms in $(s - M^2)$, i.e. to a subleading behaviour in time. The same applies to the insertion of $Re\Pi$, since it vanishes at $s = M^2$ by mass renormalization.

Using the above considerations, we find that the sum of the leading terms in time is given by the Breit-Wigner propagator:

$$A(t) = \left\langle e^{-i(\omega_{p_1} + \omega_{p_2})t} (2\pi)^4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2) 2Im[T(s, t)]_{BW} \right\rangle \quad (47)$$

$$T_{BW} = \frac{-1}{s - M^2 + i\Gamma M} \quad (48)$$

so that (compare with Eq.[9]):

$$A(t) = e^{-iMt} \int dx e^{-ixt} \frac{2M\Gamma}{4M^2(x^2 + \frac{\Gamma^2}{4})} G(x) \quad (49)$$

with $G(x)$ defined by:

$$G(x) \equiv \left\langle (2\pi)^4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2) \delta(\omega_{p_1} + \omega_{p_2} - M - x) \right\rangle \quad (50)$$

In the narrow wave packet approximation, Eq.[30], we have:

$$\omega_{p_1} + \omega_{p_2} \approx M + \underline{v} \cdot (\underline{k}_1 - \underline{k}_2) \quad (51)$$

The region where $G(x) \approx \text{constant} \approx G(0)$ is thus limited by:

$$x_{crit} \approx v\Delta p \approx \frac{v}{\Delta x} \approx \frac{1}{\Delta t|_{\text{overlap}}} \gg \Gamma \quad (52)$$

Since the integral in Eq.[49] is dominated by the region $x \approx \Gamma$, we may take $G(x)$ as a constant, to a good approximation. Extending, furthermore, the integration range to $\pm\infty$, we find:

$$A(t) = \frac{\pi G(0)}{M} e^{-iMt} e^{-\frac{\Gamma t}{2}} \quad (53)$$

that is, in conclusion, a pure exponential behaviour for the non-decay probability.

6 Very Large Time Behaviour

The exponential behaviour displayed in Eq.[53] is the result of an approximation[1],[5]. In fact we know from the Riemann-Lebesgue lemma that the asymptotic behaviour of a Fourier transform integral, as the one appearing in Eq.[9], is determined by the points where $F(x)$ or some of its derivatives are singular. If $F(x)$ were continuous together with all its derivatives, the non-decay amplitude $A_{non-decay}(t)$ would vanish faster than any power of $\frac{1}{t}$. The presence of any singularity makes $A_{non-decay}(t)$ vanish not faster than some power of $\frac{1}{t}$. From the expression of $F(x)$, Eq.[10], it is clear that singularities are necessarily present which originate from two possible sources:

1. singularities due to the unitarity of the S -matrix;
2. singularities of the wave function of the initial state.

As for 1), unitarity requires that a resonance pole be located on the second sheet of the complex energy plane, thus implying the existence of at least one branch singularity of the amplitude $T(s, t)$ in Eq.[10].

The presence of this singularity will, however, be ineffective for the large t behaviour of $A_{non-decay}(t)$, unless the resonance location is very close to threshold. In fact, as can be seen from Eq.[10], its contribution is, in general, depressed by the narrowness of the wave packet of the initial state.

On the second point very little can be said. The possible singularities due to the initial wave function have to do with the details of the experimental preparation of the resonant state and are certainly not even under direct control of the experimentalist.

Whatever their origin, the presence of these singularities in $F(x)$ make $A_{non-decay}(t)$ behave asymptotically in a power-like way, rather than exponentially. It must be remarked, however, that nothing general can be said on the onset time of this power-like behaviour because it depends on the detailed structure of the wave function of the initial state.

To get a crude idea of the times where the power law behaviour takes over, we may replace Eq.[48] with a form in which the two particle cut is considered:

$$T = \frac{-1}{s - M^2 + iM\Pi(s)} \quad (54)$$

where, to account for threshold behaviour:

$$\Pi(s) = \Theta(s - 4m^2) \left[\frac{k(s)}{k(M)} \right]^{2l+1} \Gamma \quad (55)$$

and:

$$4k(s)^2 = s - 4m^2 \quad (56)$$

but otherwise we neglect any energy dependence of the matrix element. With these positions, Eq.[49] becomes:

$$A(t) = \exp(-2imt) \int_0 dx \exp(-ixt) [2M\Pi(s)] [(s - M^2)^2 + M^2\Pi(s)^2]^{-1} G(x) \quad (57)$$

with

$$\sqrt{(s)} \equiv x + 2m \quad (58)$$

Neglecting further any variation of the function $G(x)$, which describes the energy spread of the initial wave packets, the asymptotic behaviour of $A(t)$, as $t \rightarrow \infty$, is easily estimated to be:

$$A(t)_{asympt} = C \left[\frac{4m\Gamma}{k(M)} \right]^{l+\frac{5}{2}} (\Gamma t)^{-l-\frac{3}{2}} \quad (59)$$

with:

$$C = \left(\frac{1}{128} \right) \left(\frac{M}{m^2} \right) G(0) \int_0 du \exp(-iu) u^{l+\frac{1}{2}} \quad (60)$$

Comparing with the exponential law, Eq.[53], we see that the power law gets in for a critical time , t_{crit} , which is very large indeed. Numerically:

$$\Gamma t_{crit} \approx 2 \left(l + \frac{5}{2} \right) \left\{ 25 + \ln \left\{ \left[\frac{k(M)}{0.7 MeV} \right] \left[\frac{\tau}{10^{-10} sec} \right] \right\} \right\} \quad (61)$$

This result amply justifies the fact that no deviations from the exponential behaviour have been observed, until now, for long-lived systems[14].

7 Wigner's Delay

Within the formalism described in the present paper, we can readily recover, in a very general way the result, due to Wigner[15] and usually derived within potential

scattering theory, which says that the derivative of scattering phase shift $\delta_l(E)$ is related to the delay time induced by the interaction through:

$$T = 2 \frac{d\delta_l(E)}{dE} \quad (62)$$

In order to prove Eq.[62] let us consider the center of mass scattering of two spinless particles in a state of given orbital angular momentum l , under the inelastic threshold. The incoming state (a wave packet normalized in a finite volume V) is:

$$|f; in\rangle = \sqrt{\frac{(2\pi)^3}{V}} \int dE f(E) |E, l, m; in\rangle \quad (63)$$

where $|E, l, m; in\rangle$ denotes the incoming state of two particles of given total energy E and given angular momentum l, m , normalized as:

$$\langle E', l', m'; in | E, l, m; in\rangle = \frac{V}{(2\pi)^3} \delta_{ll'} \delta_{mm'} \delta(E - E') \quad (64)$$

and

$$\int dE |f(E)|^2 = 1 \quad (65)$$

The amplitude to find, after the interaction, the system in a state which corresponds to the free propagation of the initial state is:

$$\begin{aligned} A &= \langle f; out | f; in\rangle = \\ &= \frac{(2\pi)^3}{V} \int dE' dE f^*(E') f(E) \langle E', l', m'; out | E, l, m; in\rangle = \\ &= \int dE |f(E)|^2 \exp 2i\delta_l(E) \end{aligned} \quad (66)$$

If the system were non interacting, $\delta_l(E) = 0$, and we would have $A = 1$.

In the case of a narrow wave packet, in which $f(E)$ is strongly peaked around a given energy E_0 , we have:

$$A \approx \exp[2i\delta_l(E_0) - 2i\delta'_l(E_0)E_0] \int dE |f(E)|^2 \exp[2i\delta'_l(E_0)E] \quad (67)$$

We can now ask what is the probability to find the system, after the interaction, in a state which is the free propagation of the initial state delayed by a time T . Such a time- translated state is given by:

$$\begin{aligned} |f, T; out\rangle &= \exp iHT \sqrt{\frac{(2\pi)^3}{V}} \int dE f(E) |E, l, m; out\rangle = \\ &= \sqrt{\frac{(2\pi)^3}{V}} \int dE f(E) \exp[iET] |E, l, m; out\rangle \end{aligned} \quad (68)$$

where, as usual, H denotes the full hamiltonian of the interacting system.

We have, in this case:

$$\begin{aligned} A(T) &= \langle f, T; out | f; in\rangle \approx \\ &\approx \exp[2i\delta_l(E_0) - 2i\delta'_l(E_0)E_0] \int dE |f(E)|^2 \exp iE[2\delta'_l(E_0) - T] \end{aligned} \quad (69)$$

From Eq.[69] it follows that for $T = 2\delta'_l(E_0)$, the probability becomes 1:

$$|A(T = 2\delta'_l(E_0))|^2 = 1 \quad (70)$$

which proves the Wigner delay relation, Eq.[62].

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