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BPS States in $N = 3$ Superstrings

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ABSTRACT

The $N = 3$ string models are special solutions of the type II perturbative string theories. We present explicit expressions for the helicity supertraces, which count the number of the perturbative BPS multiplets. Assuming the non-perturbative duality ($S \leftrightarrow T$) of the heterotic string on T^6 and type II on $K_3 \times T^2$ valid in $N = 4$ theories, we derive the $N = 3$ non-perturbative BPS mass formula by “switching off” some of the $N = 4$ charges and “fixing” to special values some of the $N = 4$ moduli. This operation corresponds to a well-defined Z_2 projection acting freely on the compactification manifold. The consistency of this projection and the precise connection of the $N = 4$ and $N = 3$ BPS spectrum is shown explicitly in several type II string constructions. The heterotic $N = 3$ and some asymmetric type II constructions turn out to be non-perturbative with the S moduli fixed at the self-dual point $S = i$. Some of the non-perturbative $N = 3$ type II are defined in the context of F-theory.

The bosonic sector of the $N = 3$ string effective action is also presented. This part can be useful for the study of 4d black holes in connection with the asymptotic density of BPS states in $N = 3$ string theory.

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1 Introduction

Recently significant progress has been made in the study of the non-perturbative aspects of string theory. One of the major steps in this direction has been the identification of the solitonic states, predicted through BPS mass formulae, with the D-brane states in string theory [1]. The counting of these states, through an open string construction with Dirichlet boundary condition, has been shown to give expressions for the black hole entropy, in a number of cases from the microscopic point of view, which match the macroscopic description. As a result, apart from having possible interesting phenomenological applications, these developments also have a potential to solve the Hawking paradox for the black-hole information loss.

A major requirement in the study of the non-perturbative aspects of string theory is the preservation of a part of the original supersymmetry. This, in many cases, protects the physical quantities from receiving quantum corrections, through non-renormalization theorems and a semi-classical analysis is sufficient for getting exact results.

However, the major thrust of these studies has been restricted to the case of $N = 4$ [2, 3] and $N = 2$ [4] space-time supersymmetries. The $N = 4$ string theories provide the simplest non-trivial possibilities, since the only allowed massless matter multiplets in this case are the vectors. The perturbative moduli space is parametrized by a coset, $SO(6, 22)/SO(6) \times SO(22)$. This coset gives a complete classification of the perturbative string spectrum with $N = 4$ supersymmetry. There are, in addition, the axion-dilaton moduli as well, which parametrize a coset space, $SU(1, 1)/U(1)$. The $SU(1, 1)$ symmetry mixes the electric charges with the magnetic ones and turns the weak-coupling string theory into a strong-coupling one. The BPS states of the $N = 4$ theory preserve either one-half or one-quarter of the supersymmetry. For instance, the extremal black holes of [5] are examples of BPS states preserving one-half of the supersymmetry. On the other hand, $N = 2$ supersymmetric theories have a much richer moduli structure which can among other things account for the confinement in the supersymmetric gauge theories.

In this paper, we discuss some aspects of a similar study for $N = 3$ superstring theories [6]. Although, these theories have attracted much less interest, compared with the ones discussed above, they seem to possess in some ways interesting features of both of $N = 2$ and $N = 4$ theories. Since the physical degrees of freedom in a vector multiplet of an $N = 3$ theory are the same as those in an $N = 4$ one, the moduli space of the two theories share similar universal properties. In particular, the number of vectors in the spectrum still determines the scalar manifold uniquely. Due to a similarity in vector multiplets in the two cases, the $N = 3$ theories with global supersymmetries get automatically extended to $N = 4$. This is one of the reasons for comparatively less attention being paid to them. On the other hand, the supergravity sector of an $N = 3$ theory resembles that of the $N = 2$ case, as there are no scalars in this sector in both cases. The full moduli space for $N = 3$ theories for n matter

multiplets, at a generic point, has the form $SU(3, n)/U(1) \times SU(3) \times SU(n)$. The supergravity couplings therefore clearly distinguish the $N = 4$ and $N = 3$ cases. The present study highlights some of these similarities and differences. In particular, in section 2 of this paper, we study the particle spectrum of $N = 3$ string theory with an emphasis on the BPS states. We present the $N = 3$ constructions and derive the expressions for various helicity-supertraces that count the number of these multiplets.

In the $N = 2$ models, it is known that, depending on the details of the string construction, the dilaton belongs either to a vector or to a hyper-multiplet [9]. In the $N = 3$ case also, as we will discuss, there are two allowed projections for constructing models. The first possibility is to carry out a Z_2 projection either in the heterotic or the $(4, 0)$ type II models, in which the dilaton is projected out. This is possible, provided one is at a self-dual point on the space of string coupling for $N = 4$. Such theories therefore are only defined at a non-perturbative level. However, we would like to emphasize that our projection can be used to find out both the massless and the massive BPS spectra in these cases. One can also use a Z_2 projection in the type II models with $(2, 2)$ supersymmetry; in this case the dilaton survives the projection and belongs to the vector multiplet. We show the exact connection between these two projections, through a known prescription for the construction of type II dual pairs [10]. The action of the projections on the 16 extra right-moving coordinates of the heterotic string theory is also obtained by examining the transformation of the twisted sector states in the original $(2, 2)$, type II, model.

Our results therefore provide an example of duality in the $N = 3$ context. This is similar to the F-theory/heterotic string duality [11] discussed in the literature. There are many known examples in the F-theory side, which are only non-perturbatively defined [12], but can still be shown to be dual to a heterotic string construction. In those cases, as in ours, the coupling constants are frozen to a fixed value in the orbifold limit. Following a similar line of study, it may also be possible to define the $N = 3$ model as a geometric compactification of a “hidden” theory, which would allow us to study their non-perturbative aspects.

After the identification of the projection in the heterotic (or $(4, 0)$ type II) models, the BPS mass formula for $N = 3$ string theory is obtained through a projection of the $N = 4$ formula. We show that the expressions for the BPS formula for the $N = 4$ string have a unique truncation, which defines the $N = 3$ case, irrespective of whether the BPS states preserve the $1/2$ or the $1/4$ of the original supersymmetry. This uniqueness is explained by the fact that the states of the $N = 3$ strings are classified by a single central charge, whereas in the $N = 4$ case there are two such charges. As a result, the projection acting on both these states gives the unique short-multiplet of an $N = 3$ theory. This multiplet preserves $1/4$ of the original $N = 4$ supersymmetry. The final mass formula is $U(3, n)$ -invariant and formally has a structure similar to the one used in writing down the black-hole entropy [13].

The derivation of the BPS mass formula in theories with lesser supersymmetries, obtained by projection of a theory with a larger number of supersymmetries, has been

discussed earlier in a different context [14]. In our case, these projections can also be used to obtain the number of BPS states with given quantum numbers, from the knowledge of the degeneracy of such states in the original $N = 4$ theory.

Another application of our results is to write down the $N = 3$ string effective action. In this paper we present the bosonic part of this effective action. The effective action is also invariant under a $U(3, n)$ symmetry, which can be used as a solution-generating technique for these models. In particular the black-hole solutions for $N = 3$ theories, carrying 14 electric charges, can be obtained along a line similar to the one in [5] for the $N = 4$ case.

2 $N = 3$ Constructions

2.1 (2,1)–Type II Models

We now start by presenting the type II $N = 3$ string models [6] and write down their partition function. All the models of [6] have been obtained by applying projections to the $N = 8$ string theories, which preserve modular invariance as well as the conformal symmetries on the string worldsheet. The $N = 8$ model is described in the bosonic language, in the light-cone gauge, by 8 worldsheet left/right-moving bosonic and fermionic coordinates, $\psi_i^{L,R}$ and $X_i^{L,R}$ ($i = 1, \dots, 8$). In our notation, the coordinates $\psi_\mu^{L,R}$ and $X_\mu^{L,R}$ ($\mu = 1, 2$) represent the space-time degrees of freedom, whereas the remaining ones correspond to the internal degrees of freedom. This bosonic description will be appropriate for the asymmetric orbifold construction in section 3 where we will discuss various issues related to the non-perturbative BPS spectrum.

In the fermionic construction [7], [8], $X_i^{L,R}$'s ($i = 3, \dots, 8$) are replaced by a pair of Majorana–Weyl spinors $\omega_a^{L,R}$ and $y_a^{L,R}$, ($a = 1, \dots, 6$). To follow the standard notation of the fermionic construction [6], we also rename the internal components of the field $\psi_i^{L,R}$'s as $\chi_a^{L,R}$'s. The construction of string models amounts to a choice of boundary conditions for these fermions, which satisfies local and global consistency requirements. The $N = 8$ model, constructed in this manner, have four space-time supersymmetries originating from the left-moving sector and another four from the right-moving sector. In the language of the fermionic construction [7], this model is constructed by introducing three basis sets, namely F , which contains all the left- and the right-moving fermions:

$$F = [\psi_\mu^L, \chi_a^L, y_a^L, \omega_a^L \mid \psi_\mu^R, \chi_a^R, y_a^R, \omega_a^R] \quad (\mu = 1, 2; a = 1, \dots, 6), \quad (2.1)$$

and the basis sets S and \bar{S} , which contain only eight left or right-moving fermions:

$$S = [\psi_\mu^L, \chi_a^L] \quad \bar{S} = [\psi_\mu^R, \chi_a^R]. \quad (2.2)$$

Four of the gravitinos of the $N = 8$ model belong to sector S and the other four to \bar{S} . Then by applying appropriate projections one obtains the $N = 3$ superstring model with gauge groups of various ranks.

First, to keep the discussion simple, we avoid those models that make a contribution to the massless particle spectrum from the twisted sectors of new basis sets. As a result, all the states in these models are a subset of those in the $N = 8$ case. For example, the first projection for constructing an $N = 3$ model with three matter multiplets, is specified by a choice of fermion basis b_3^H :

$$b_3^H = [\psi_\mu^L, \chi_{1,2}^L, y_{3,\dots,6}^L, y_1^L, \omega_1^L \mid \psi_\mu^R, \chi_{1,2}^R, y_{3,\dots,6}^R, y_1^R, \omega_1^R]. \quad (2.3)$$

This applies a left–right-symmetric Z_2 projection in the planes defined by the coordinates $\chi_{3,4}$ and $\chi_{5,6}$. It breaks half of the supersymmetries in both the sides, by projecting out two of the gravitinos from each of the sectors S and \bar{S} . The resulting model has $N = 4$ supersymmetry with 12 $U(1)$ gauge fields. The local structure of the moduli space is parametrized by a coset:

$$\frac{SU(1,1)}{U(1)} \times \frac{SO(6,6)}{SO(6) \times SO(6)}. \quad (2.4)$$

In terms of orbifold construction, b_3^H acts without fixed points, making the twisted sector states heavy.

A second projection on the $N = 8$ construction yielding a $N = 3$ model specified by the basis set:

$$b_3^h = [\psi_\mu^L, y_{1,2,3}^L, \omega_4^L, \chi_{5,6}^L, y_5^L, \omega_5^L \mid y_5^R, \omega_5^R]. \quad (2.5)$$

This asymmetric projection, which acts as a Z_2 twist on the planes defined by $\chi_{1,2}^L$ and $\chi_{3,4}^L$, breaks another one-half supersymmetry from the left-moving sector by projecting out one more gravitino from S . The resulting model has six $U(1)$ gauge fields and the structure of the moduli space is now given by the coset structure:

$$\frac{SU(3,3)}{U(1) \times SU(3) \times SU(3)'}. \quad (2.6)$$

The massless states of the model constructed above come from various sectors of the original $N = 8$ theory. Among these, the bosonic ones are in the sectors classified by the NS–NS sector ϕ and the R–R sector $S\bar{S}$. These massless states can be arranged in various representations of the subgroups of $SU(3,3)$, the symmetry group of the moduli deformations; the subgroup $SU(3,3;Z) \subset SU(3,3)$ defines the conjectured \mathcal{U} -duality group for this $N = 3$ string construction. For convenience, we decompose the compact subgroups $SU(3)$ and $SU(3)'$ of $SU(3,3)$ as

$$SU(3) \rightarrow U(1) \times SU(2)_R, \quad \text{and} \quad SU(3)' \rightarrow U(1) \times SU(2)'_R. \quad (2.7)$$

The group $SU(3,3)$ can also be decomposed as

$$SU(3,3) \rightarrow SU(1,1) \times SU(2,2). \quad (2.8)$$

The massless scalars from the NS–NS sector then are:

(i) dilaton and axion, which parametrize the coset

$$\frac{SU(1,1)}{U(1)} , \quad (2.9)$$

(ii) eight scalars parametrizing

$$\frac{SO(2,4)}{SO(2)_L \times SO(4)_R} \equiv \frac{SU(2,2)}{SU(2)_R \times SU(2)'_R \times U(1)_L} . \quad (2.10)$$

This sector also provides two $U(1)$ gauge fields that transform as a vector of $SO(2)_L$. We have four additional $U(1)$ gauge fields and four complex scalars transforming as $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$ of $SU(2)_R \times SU(2)'_R$ from the R–R sector.

$N = 3$ string models with a gauge sector of various other ranks, n , have been presented in [6]. For $n > 3$ they involve projections whose twisted sectors give extra contributions to the massless spectrum.

The $SU(3,3+8)$ model

An $n = 11$ model constructed in [6] uses the projections

$$b_{11}^H = [\psi_\mu^L, \chi_{1,2}^L, y_{3,\dots,6}^L \mid \psi_\mu^R, \chi_{1,2}^R, y_{3,\dots,6}^R], \quad (2.11)$$

$$b_{11}^h = [\psi_\mu^L, y_{1,2,3,4}^L, \chi_{5,6}^L, y_{5,6}^L, \omega_{5,6}^L \mid y_{5,6}^R, \omega_{5,6}^R], \quad (2.12)$$

and

$$T = [y_{5,6}^L, \omega_{5,6}^L \mid y_{5,6}^R, \omega_{5,6}^R]. \quad (2.13)$$

The basis T factorizes the (5,6)-torus with independent boundary conditions. The modular invariant partition function for the above $N = 3$ model is:

$$Z^{\text{string}} = \frac{1}{\text{Im}\tau \eta^2 \bar{\eta}^2} \frac{1}{4} \sum_{(H,G,h,g)} \frac{1}{4} \sum_{(\gamma,\delta,\gamma',\delta')} e^{i\pi(\gamma'g + \delta'h + gH)} Z_L Z_R , \quad (2.14)$$

where $Z_{L,R}$ are themselves the products of contributions from worldsheet fields $\psi_\mu^{L,R}$, $\chi_a^{L,R}$, $\omega_a^{L,R}$ and $y_a^{L,R}$ written in terms of Riemann theta functions. The boundary conditions of these fields as specified by the various indices in the sum above. We then have

$$Z_L = Z_L^{\psi\chi} Z_L^\omega Z_L^y, \quad Z_R = Z_R^{\psi\chi} Z_R^\omega Z_R^y , \quad (2.15)$$

with

$$Z_L^{\psi\chi} = \frac{1}{2} \sum_{(a,b)} \frac{(-)^{a+b+ab}}{\eta^4} \theta_{[b]}^a \theta_{[b+g]}^{[a+h]} \theta_{[b-G-g]}^{[a-H-h]} \theta_{[b+G]}^{[a+H]} , \quad (2.16)$$

$$Z_L^\omega = \frac{1}{\eta^3} \theta_{[\delta]}^{[\gamma]} \theta_{[\delta]}^{[\gamma]} \theta_{[\delta']}^{[\gamma']} , \quad (2.17)$$

$$Z_L^y = \frac{1}{\eta^3} \theta_{[\delta+g]}^{[\gamma+h]} \theta_{[\delta-G-g]}^{[\gamma-H-h]} \theta_{[\delta'+G]}^{[\gamma'+H]} . \quad (2.18)$$

Similarly the contributions of the right-moving fermions is given as

$$Z_R^{\psi\chi} = \frac{1}{2} \sum_{\bar{a}, \bar{b}=0}^1 \frac{(-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}}}{\bar{\eta}^4} \bar{\theta}_{[\bar{b}]}^{[\bar{a}]} \bar{\theta}_{[\bar{b}-G]}^{[\bar{a}-H]} \bar{\theta}_{[\bar{b}+G]}^{[\bar{a}+H]} , \quad (2.19)$$

$$Z_R^\omega = \frac{1}{\bar{\eta}^3} \bar{\theta}_{[\delta]}^{[\gamma]} \bar{\theta}_{[\delta]}^{[\gamma]} \bar{\theta}_{[\delta']}^{[\gamma']} , \quad (2.20)$$

$$Z_R^y = \frac{1}{\bar{\eta}^3} \bar{\theta}_{[\delta]}^{[\gamma]} \bar{\theta}_{[\delta-G]}^{[\gamma-H]} \bar{\theta}_{[\delta'+G]}^{[\gamma'+H]} . \quad (2.21)$$

The partition function written in eqs. (2.14)–(2.21) is only one of the many possibilities arising from a choice of the GSO projection in the string construction. One can modify these GSO projections by introducing modular invariant phases (discrete torsions), which appear as coefficients of various terms in the partition function. In our case we can choose one of the following phases (or any product of them):

$$e^{i\pi(\gamma g + \delta h + gh)} , \quad e^{i\pi(\gamma' g + \delta' h + gh)} , \quad e^{i\pi(\gamma G + \delta H + GH)} , \quad e^{i\pi(\gamma' G + \delta' H + GH)} . \quad (2.22)$$

The projection b_{11}^H in the partition function is represented by the twists H and G , while b_{11}^h is represented by h and g . It can also be checked that the partition function of the original $N = 8$ construction is reproduced from above by setting $H = G = h = g = 0$ in the arguments of theta functions. We will now use these results to obtain an expression of the generating function that counts the number of BPS states in perturbative $N = 3$ string theory.

2.2 Perturbative BPS States

We now study the spectrum of the BPS states for the $N = 3$ model constructed above. As mentioned before, the $N = 3$ supersymmetry algebra allows only one independent central charge. As a result it possesses two allowed representations for the non-vanishing value of this central charge. The long one is a 2^6 -dimensional complex representation of $SO(12)$ Clifford algebra and the short one is a complex representation of dimension 2^4 [15], [16]. A vacuum configuration in a long multiplet representation breaks supersymmetry completely. Since the short ones are annihilated by one of the supersymmetry generators, they preserve a part of the supersymmetry. The generating functions that count the number of $N = 3$ BPS states are given as

trace formula over the supermultiplets. In the $N = 3$ case, the helicity-generating function, defined as

$$Z_R(y) = \text{str } y^{2\lambda} = \text{Tr } (-)^{2\lambda} y^{2\lambda}, \quad (2.23)$$

with λ denoting the helicity of the states within a multiplet R , is given for a long multiplet by

$$Z_{\text{long}}(y) = z_{[j]} (1 - y)^3 (1 - 1/y)^3. \quad (2.24)$$

In eq. (2.24)

$$z_{[j]} = (-)^{2j} \frac{y^{2j+1} - y^{-2j-1}}{y - 1/y} \quad \text{massive}, \quad (2.25)$$

$$z_{[j]} = (-)^{2j} (y^{2j} + y^{-2j}) \quad \text{massless}, \quad (2.26)$$

for a particle of spin j . For short multiplets, preserving one of the supersymmetries, the generating function has a form:

$$Z_{\text{short}} = 2z_{[j]} (1 - y)^2 (1 - 1/y)^2. \quad (2.27)$$

The extra factor of 2 in (2.27) is due to the fact that the central charge is necessarily non-zero in this case and implies a doubling of the representations. The ‘‘helicity supertrace’’ over a supermultiplet R is defined as:

$$B_{2n}(R) \equiv \text{str } \lambda^{2n} = \text{Tr}_R [(-)^{2\lambda} \lambda^{2n}], \quad (2.28)$$

and can be obtained from the generating functions $Z_R(y)$ as:

$$B_{2n}(R) = \left(y^2 \frac{d}{dy^2} \right)^{2n} Z_R(y)|_{y=1}. \quad (2.29)$$

For $N = 3$ supersymmetry, B_n ($n < 4$) all vanish, B_4 is non-zero only for short multiplets, and B_6 is non-zero for both long and short ones. A direct computation of these quantities gives, for $N = 3$ massless multiplets:

$$\begin{aligned} B_4(\text{vector}) &= \frac{3}{2}, & B_4(\text{sugra}) &= \frac{15}{2}, & \text{and} \\ B_6(\text{vector}) &= \frac{15}{8}, & B_6(\text{sugra}) &= \frac{525}{8}. \end{aligned} \quad (2.30)$$

In string theory, the above expressions are further extended to include the infinite tower of massive states, by defining a modified partition function [16]:

$$Z^{\text{string}}(v, \bar{v}) = \text{Tr } q^{L_0} \bar{q}^{\bar{L}_0} e^{2\pi i v \lambda_L - 2\pi i \bar{v} \lambda_R}. \quad (2.31)$$

Such modifications to the partition function have been studied earlier in order to obtain exact solutions of string theory in the background of physical magnetic fields and

to investigate the associated phase-transition phenomena [17], [16]. In that context the quantities v and \bar{v} play the role of the background magnetic field. The physical helicity is given by $\lambda = \lambda_L + \lambda_R$ and the generating function $Z_R(y)$ for the $N = 3$ supermultiplets is obtained through an identification $y = e^{i\pi(v+\bar{v})}$.

The helicity supertrace B_{2n} in the string case can then be derived from the generating function $Z^{\text{string}}(v, \bar{v})$ by defining

$$Q = \frac{1}{2\pi i} \frac{\partial}{\partial v}, \quad \bar{Q} = -\frac{1}{2\pi i} \frac{\partial}{\partial \bar{v}}, \quad (2.32)$$

then

$$B_{2n}^{\text{string}} = \text{str} [\lambda^{2n}] = (Q + \bar{Q})^{2n} Z^{\text{string}}(v, \bar{v}) |_{(v = \bar{v} = 0)}. \quad (2.33)$$

An explicit expression for $Z^{\text{string}}(v, \bar{v})$ for the $N = 3$ string model of section 2.1 is given by an expression that is similar to the $v = \bar{v} = 0$ case presented in (2.14)-(2.21), and has a form:

$$Z^{\text{string}}(v, \bar{v}) = \frac{1}{\eta^2 \bar{\eta}^2} \frac{1}{4} \sum_{(H, G, h, g)} \frac{1}{4} \sum_{(\gamma, \delta, \gamma', \delta')} e^{i\pi(\gamma'g + \delta'h + gH)} \xi(v) \bar{\xi}(\bar{v}) Z'_L Z'_R, \quad (2.34)$$

where

$$\xi(v) = \prod_1^{\infty} \frac{(1 - q^n)^2}{(1 - q^n e^{2\pi i v})(1 - q^n e^{-2\pi i v})} = \frac{\sin \pi v}{\pi} \frac{\theta'_1}{\theta_1(v)}, \quad (2.35)$$

is an even function of v : $\xi(v) = \xi(-v)$. The expressions for $Z'_{L,R}$ in terms of the individual contributions of the worldsheet fields, ψ , χ , ω and y , namely $Z'^{\psi\chi}_{L,R}$, $Z'^{\omega}_{L,R}$, $Z'^y_{L,R}$, are also identical to the one in (2.15). However, the expressions for $Z'^{\psi\chi}_{L,R}$ are now modified by a change in the argument of the theta function:

$$Z'^{\psi\chi}_L = \frac{1}{2} \sum_{a,b} \frac{(-)^{a+b+ab}}{\eta^4} \theta_{[b]}^{[a]}(v) \theta_{[b+g]}^{[a+h]} \theta_{[b-G-g]}^{[a-H-h]} \theta_{[b+G]}^{[a+H]} \quad (2.36)$$

and

$$Z'^{\psi\chi}_R = \frac{1}{2} \sum_{\bar{a}, \bar{b}} \frac{(-)^{\bar{a}+\bar{b}+\bar{a}\bar{b}}}{\bar{\eta}^4} \bar{\theta}_{[\bar{b}]}^{[\bar{a}]}(\bar{v}) \bar{\theta}_{[\bar{b}]}^{[\bar{a}]} \bar{\theta}_{[\bar{b}-G]}^{[\bar{a}-H]} \bar{\theta}_{[\bar{b}+G]}^{[\bar{a}+H]}. \quad (2.37)$$

Modifications in the expression for $Z^{\text{string}}(v, \bar{v})$ in eq. (2.34), with respect to the one in eq. (2.14), arise from the change in the contributions of the worldsheet fermions $\psi_{\mu}^{L,R}$ and bosons $X_{\mu}^{L,R}$, which represent the space-time degrees of freedom. These modifications, due to fermions, are absorbed in theta functions through an additional argument (v) and (\bar{v}). The modifications in the oscillator contributions from X are taken into account through the extra factors $\xi(v)$ and $\bar{\xi}(\bar{v})$.

To compute the quantities B_{2n} , one can observe that the sum over indices (a, b) and (\bar{a}, \bar{b}) in eq. (2.14) involves only the worldsheet fields ψ_{μ} and χ_a through the

terms $Z_{L,R}^{\psi_X}$ in the partition function. We now sum over these indices and use the Riemann identity of theta functions to write

$$\begin{aligned} Z_L^{\psi_X} &= \frac{1}{\eta^4} \theta_{[1]}^{[1]} \left(\frac{v}{2} \right) \theta_{[1-g]}^{[1-h]} \left(\frac{v}{2} \right) \theta_{[1+G+g]}^{[1+H+h]} \left(\frac{v}{2} \right) \theta_{[1-G]}^{[1-H]} \left(\frac{v}{2} \right) , \\ Z_R^{\psi_X} &= \frac{1}{\bar{\eta}^4} \bar{\theta}_{[1]}^{[1]} \left(\frac{\bar{v}}{2} \right) \bar{\theta}_{[1]}^{[1]} \left(\frac{\bar{v}}{2} \right) \bar{\theta}_{[1+G]}^{[1+H]} \left(\frac{\bar{v}}{2} \right) \bar{\theta}_{[1-G]}^{[1-H]} \left(\frac{\bar{v}}{2} \right) . \end{aligned} \quad (2.38)$$

To evaluate the various derivative terms for obtaining B_{2n} , we also use the properties that among theta functions, θ_1 and its even derivatives with respect to v are odd under $v \rightarrow -v$ and vanish at $v = 0$. The odd derivatives of the remaining theta functions, as well as that of $\xi(v)$, are also odd under $v \rightarrow -v$ and vanish at $v = 0$. These properties simplify our calculations significantly. For example, the RHS of (2.38) immediately implies $B_2 = 0$, as expected. The helicity supertrace B_4 , which counts the number of short multiplets of $N = 3$ string theory,

$$B_4 = (Q + \bar{Q})^4 Z(v, \bar{v})|_{v=\bar{v}=0}, \quad (2.39)$$

has a non-zero contribution only from the term $6 Q^2 \bar{Q}^2 Z(v, \bar{v})|_{v=\bar{v}=0}$ in the above expression. There are two sectors that can give a non-zero result, namely, the two “ $N = 4$ ” sectors:

- (i) $\vec{h} = (h, g) = \vec{0}$, $\vec{H} = (H, G) \neq \vec{0}$ and
- (ii) $\vec{h} = \vec{H} \neq \vec{0}$.

In the first “ $N = 4$ ” sector the (1,2)-complex plane remains untwisted; the left- and the right-moving currents $J_1 = \omega_1 y_1 + i\omega_2 y_2$, $\bar{J}_1 = \bar{\omega}_1 \bar{y}_1 + i\bar{\omega}_2 \bar{y}_2$ remain untwisted while the remaining ones $J_2 = \omega_3 y_3 + i\omega_4 y_4$, $J_3 = \omega_5 y_5 + i\omega_6 y_6$, $\bar{J}_2 = \bar{\omega}_3 \bar{y}_3 + i\bar{\omega}_4 \bar{y}_4$, $\bar{J}_3 = \bar{\omega}_5 \bar{y}_5 + i\bar{\omega}_6 \bar{y}_6$ are twisted. In the second “ $N = 4$ ” sector the untwisted planes are the (3,4)-left and the (1,2)-right with untwisted currents the J_2 left and \bar{J}_1 right. The two “ $N = 4$ ” sectors give identical contributions. For the first case we find:

$$\begin{aligned} B_4^{\vec{h}=\vec{0}} &= \frac{3}{4\eta^6 \bar{\eta}^6} \frac{1}{2} \sum_{H,G} |\theta_{[1-G]}^{[1-H]} \theta_{[1+G]}^{[1+H]}|^2 \\ &\quad \times \frac{1}{2} \sum_{(\gamma,\delta)} |\theta_{[\delta]}^{[\gamma]}|^4 |\theta_{[\delta]}^{[\gamma]} \theta_{[\delta-G]}^{[\gamma-H]}|^2 \\ &\quad \times \frac{1}{2} \sum_{(\gamma',\delta')} |\theta_{[\delta']}^{[\gamma']} \theta_{[\delta'+G]}^{[\gamma'+H]}|^2 . \end{aligned} \quad (2.40)$$

This expression is further simplified by using identities involving theta functions [3]. We then get

$$B_4^{\vec{h}=\vec{0}} = 12 \frac{1}{2} \sum_{\gamma,\delta} |\theta_{[\delta]}^{[\gamma]}|^4 \equiv 12 \Gamma_{2,2}^{[0]}|_{T=U=i}, \quad (2.41)$$

where in the final step we have written the helicity trace as a lattice contribution of signature (2, 2) [3]; notice that both T and U moduli are fixed at their self-dual

points ($T = U = i$). Adding the contributions of the two $N = 4$ sectors ($\vec{h} = (h, g) = \vec{0}$, $\vec{H} = (H, G) \neq \vec{0}$ and $\vec{h} = \vec{H}$, $\vec{H} \neq \vec{0}$), we have

$$\begin{aligned} B_4^{\text{total}} &= B_4^{\vec{h}=\vec{0}, \vec{H}\neq\vec{0}} + B_4^{\vec{h}=\vec{H}\neq\vec{0}} \\ &= 12\Gamma_{2,2}[0] |_{T=U=i} + 12\Gamma'_{2,2}[0] |_{T=U=i} = 24 \frac{1}{2} \sum_{\gamma,\delta} |\theta[\gamma] |^4. \end{aligned} \quad (2.42)$$

At the massless level, eq. (2.42) implies $B_4(\text{massless}) = 24$, which matches with the combined contributions from the supergravity sector and 11 vector multiplets, written earlier in eq. (2.30); $B_4(\text{massless}) = 15/2 + 11 \times 3/2 = 24$.

The helicity supertrace B_6 can also be computed in a similar way. In this case an analysis of the terms in the generating function $Z^{\text{string}}(v, \bar{v})$:

$$B_6 = (Q + \bar{Q})^6 Z^{\text{string}}(v, \bar{v}) |_{(v=\bar{v}=0)}, \quad (2.43)$$

shows that there are contributions from various sectors:

- **The two “ $N = 4$ ” sectors,**

$$(i) \quad B_6^{\vec{h}=\vec{0}, \vec{H}\neq\vec{0}} \quad \text{and} \quad (ii) \quad B_6^{\vec{h}=\vec{H}\neq\vec{0}},$$

which give a non-vanishing contributions from the terms $15 Q^2 \bar{Q}^2 (Q^2 + \bar{Q}^2)$ in eq. (2.43).

- **The “ $N = 6$ ” sector,**

$$(iii) \quad B_6^{\vec{h}\neq\vec{0}, \vec{H}=\vec{0}},$$

which gives a non-vanishing contribution from the term $15 Q^2 \bar{Q}^4$ in eq. (2.43).

The extra derivative $(Q^2 + \bar{Q}^2)$ on $B_4(v, \bar{v}) \equiv Q^2 \bar{Q}^2 Z^{\text{string}}(v, \bar{v})$ in the two $N = 4$ sectors give rise to a multiplicative factor $\left(4 + \chi[G] + \bar{\chi}[G]\right)$ where $\chi[G]$ are defined as:

$$\chi[G] \equiv \frac{12}{i\pi} \partial_\tau \log \frac{\theta_{[1+G]}^{1+H}}{\eta} = \frac{1}{2} \sum_{\gamma,\delta} \theta^4[\gamma] \left[e^{i\pi(H+G\gamma)} - e^{i\pi(G+H\delta)} \right]. \quad (2.44)$$

We will give our result for B_6 in terms of the above functions $\chi[G]$ and in terms of the “shifted” lattice $\Gamma_{2,2}[G] |_{T=U=i}$:

$$\Gamma_{2,2}[G] |_{T=U=i} = \frac{1}{2} \sum_{\gamma,\delta} |\theta_{[\delta+G]}^{[\gamma+H]}|^4 e^{i\pi [\delta H + \gamma G + GH]}. \quad (2.45)$$

In terms of $\chi[G]$ and $\Gamma_{2,2}[G]$ the contribution of the two $N = 4$ sectors is:

$$B_6^{\vec{h}=\vec{0}, \vec{H}\neq\vec{0}} = B_6^{\vec{h}=\vec{H}\neq\vec{0}} = \quad (2.46)$$

$$\begin{aligned}
&= \frac{15}{2} \sum_{(H,G) \neq (0,0)} \left(1 + \frac{\chi[G^H] + \bar{\chi}[G^H]}{4} \right) \left(\Gamma_{2,2[0]} + \Gamma_{2,2[G^H]} \right) |_{T=U=i} \\
&= 30 \Gamma_{2,2[0]} |_{T=U=i} + \frac{15}{2} \sum_{(H,G) \neq (0,0)} \frac{\chi[G^H] + \bar{\chi}[G^H]}{4} \Gamma_{2,2[G^H]} |_{T=U=i}. \tag{2.47}
\end{aligned}$$

The final equality in the above equation follows from the identities:

$$\sum_{(H,G) \neq (0,0)} \chi[G^H] = 0, \quad \sum_{(H,G) \neq (0,0)} \Gamma_{2,2[G^H]} = \Gamma_{2,2[0]} |_{T=U=i}. \tag{2.48}$$

Finally the contribution from the $N = 6$ sector is:

$$B_6^{\vec{h} \neq \vec{0}, \vec{H} = \vec{0}} = \frac{45}{4} \sum_{(h,g) \neq (0,0)} \frac{\bar{\chi}[g^h]}{2} \Gamma_{2,2[g^h]} |_{T_3 U_3}, \tag{2.49}$$

where T_3, U_3 are the moduli of the third complex plane. In the fermionic construction the moduli (T_3, U_3) are fixed to their self dual points $T_3 = U_3 = i$. In the above expression we have extended the validity of the model for arbitrary T_3, U_3 moduli:

$$\begin{aligned}
\Gamma_{2,2[g^h]} |_{T U} &= \sum_{m_i, n_i} \exp \left[i\pi\tau |P_L(h)|^2 - i\pi\bar{\tau} |P_R(h)|^2 + i\pi g m_1 \right] \\
|P_L(h)|^2 &= \frac{|m_1 U + (n_1 + \frac{h}{2}) T - m_2 + n_2 T U|^2}{2\text{Im}T \text{Im}U}, \\
|P_L(h)|^2 - |P_R(h)|^2 &= 2m_1 \left(n_1 + \frac{h}{2} \right) + 2m_2 n_2. \tag{2.50}
\end{aligned}$$

Note that $\bar{\chi}[g^h]$ in the $N = 6$ sector originates from the contribution of the 1st and 2nd right-moving complex planes; these two planes are “untwisted” on the right and “twisted” on the left. The combination of the left-twisted and right-untwisted is proportional to $\bar{\chi}[g^h]$. In the $N = 4$ sectors $\chi[g^h]$ and $\bar{\chi}[g^h]$ have a different origin; they appear because of the extra derivative operation $Q^2 + \bar{Q}^2$ on $B_4(v, \bar{v})$.

The total B_6^{total} is the sum of all contributions:

$$B_6^{\text{total}} = B_6^{\vec{h} = \vec{0}, \vec{H} \neq \vec{0}} + B_6^{\vec{h} = \vec{H} \neq \vec{0}} + B_6^{\vec{h} \neq \vec{0}, \vec{H} = \vec{0}}. \tag{2.51}$$

In the infrared limit $\text{Im} \tau \rightarrow \infty$ only the massless states give a non-zero contribution. In this limit

$$\begin{aligned}
\sum_{(h,g) \neq 0} \chi[g^h] \Gamma[g^h] |_{T,U} &\rightarrow 2, & \Gamma_{2,2[0]} |_{T,U} &\rightarrow 1, \\
B_6^{\vec{h} = \vec{0}, \vec{H} \neq \vec{0}} &\rightarrow \frac{75}{2}, & B_6^{\vec{h} = \vec{H} \neq \vec{0}} &\rightarrow \frac{75}{2}, & B_6^{\vec{h} \neq \vec{0}, \vec{H} = \vec{0}} &= \frac{45}{4}. \tag{2.52}
\end{aligned}$$

Then $B_6^{\text{total}}(\text{Im} \tau \rightarrow \infty) = 345/4$, which corresponds to the contribution of the massless fields of the $N = 3$ supergravity together with the contribution of 11 $N = 3$

massless vector multiplets; $B_6(\text{massless}) = 525/8 + 11 \times 15/8 = 345/4$. Furthermore, the contribution of the $N = 6$ sector matches (up to a factor of 2), the contribution of the massless fields of the $N = 6$ supergravity:

$$B_6(N = 6, \text{ sugra}) = 2B_6^{\vec{h} \neq \vec{0}, \vec{H} = \vec{0}} = \frac{45}{2}. \quad (2.53)$$

The factor of 2 is due to the extra projection (H, G) in $N = 3$ theory. In $N = 6$ supergravity the massless sector is *uniquely* determined by the supergravity multiplet.

Both B_4^{total} and B_6^{total} are then consistent with an $N = 3$ supersymmetric structure with 11 vector multiplets. The moduli space of the scalars form the Kähler manifold

$$\frac{SU(3, 3+8)}{U(1) \times SU(3) \times SU(3+8)} \quad \text{with Kahler potential :} \quad (2.54)$$

$$K = -\log \det \left[i(T_{i\bar{j}} - \bar{T}_{i\bar{j}}) - W_{i\bar{k}} \bar{W}_{\bar{k}j} \right]. \quad (2.55)$$

In our case the moduli $T_{i\bar{j}}$ correspond to the following type II fields:

- the type II dilaton: $S = T_{11}$,
- the moduli of the 3rd complex plane:

$$T_3 = T_{22} \quad \text{and} \quad U_3 = T_{33} ,$$

- the Wilson lines corresponding to the marginal deformations of the currents associated to the $(3_L, 2_R)$ -complex plane:

$$\left[(J_3)_L \quad \text{and/or} \quad (\bar{J}_3)_L \right] \times \left[(J_2)_R \quad \text{and/or} \quad (\bar{J}_2)_R \right] \rightarrow Y_1, \quad iY_2:$$

$$T_{23} = Y_1 + iY_2, \quad T_{32} = Y_1 - iY_2 ,$$

- the untwisted R–R scalars: $\rightarrow T_{12}, T_{21}, T_{13}, T_{31}$,
- the twisted R–R scalars: $\rightarrow W_{1,\bar{k}}$,
- the twisted scalars: $\rightarrow W_{2,\bar{k}}, W_{3,\bar{k}}$.

The perturbative string spectrum varies with the values of T_3, U_3 moduli and with the values of the two Wilson lines Y_1, Y_2 . On the other hand the perturbative spectrum does not depend on the values of all other moduli. As we will see in the next chapter, using the heterotic–type II and type II (4,0)–type II (2,2) non-perturbative \mathcal{U} -duality map, we find that the heterotic dilaton is frozen at the fixed non-perturbative value $S_H = i$. In type II theory S_H is mapped to the frozen $T_0 = i$ moduli of the first complex plane. Indeed, all $N = 3$ perturbative BPS states are selected by the B_4 helicity supertrace, which, as we have shown, does not display any dependence on the perturbative moduli; it depends only on the radii of the complex planes $[(1_L, 1_R) + (2_L, 1_R)]$; the values of these radii *are fixed at special points* in such a way that the left-right-asymmetric projection defined by \vec{h} is compatible with

modular invariance. Only in B_6 is there a non-trivial dependence on the perturbative moduli; it comes from the $N = 6$ sector in which the 3rd complex plane is untwisted. In the example we gave above the moduli dependence of B_6 is through the “shifted” lattice $\Gamma_{2,2}^{[h]} |_{T_3} U_3$. Our result can be easily generalized to include the presence of non-zero Wilson lines Y_1 and Y_2 . In order to do that we must perform the following replacement in B_6^{total} in eq. (2.49):

$$\frac{1}{2} \sum_{(h,g) \neq (0,0)} \bar{\chi}_{[g]}^{[h]} \Gamma_{2,2}^{[h]} \rightarrow \frac{1}{2} \sum_{(h,g) \neq (0,0)} \sum_{\epsilon, \xi} \bar{\rho}_{[\xi]}^{[\epsilon]} \Gamma_{2,4}^{[h, \epsilon]} |_{T,U,Y_i}, \quad (2.56)$$

where the shifted $\Gamma_{2,4}^{[h, \epsilon]}$ lattice and the function $\rho[\epsilon, \xi]$ are defined as:

$$\rho_{[g, \xi]}^{[h, \epsilon]} = \frac{1}{2} \sum_{\epsilon, \xi} \theta^2 \left[\begin{smallmatrix} 1-\epsilon \\ 1-\xi \end{smallmatrix} \right] e^{i\pi\epsilon} \left[e^{i\pi\xi h} - e^{i\pi\epsilon g} \right] \quad (2.57)$$

$$\Gamma_{2,4}^{[h, \epsilon]} = \sum_{m_i, n_i} \exp \left[i\pi\tau |P_L|^2 - i\pi\bar{\tau} \sum_1^4 (P^I)_R^2 + i\pi(gm_1 + \xi(Q_1 - Q_2)) \right], \quad (2.58)$$

with

$$|P_L|^2 = \frac{|m_1 U + (n_1 + \frac{h}{2})T - m_2 + n_2(TU - \frac{1}{2}\vec{Y}\vec{Y}) + (Q_1 + \frac{\epsilon}{2})Y_1 + (Q_2 - \frac{\epsilon}{2})Y_2|^2}{2\text{Im}T \text{Im}U - \text{Im}\vec{Y} \text{Im}\vec{Y}}$$

and

$$\begin{aligned} |P_L|^2 - \sum_1^4 (P^I)_R^2 - (Q_1 + \frac{\epsilon}{2})^2 - (Q_2 - \frac{\epsilon}{2})^2 &= \\ &= 2m_1(n_1 + \frac{h}{2}) + 2m_2n_2 - Q_1^2 - Q_2^2 - \frac{\epsilon^2}{2}. \end{aligned} \quad (2.59)$$

We have therefore presented the explicit form of the helicity supertrace formulae for an $N = 3$ string construction. Our study has been restricted to a particular $N = 3$ model with 11 vector multiplets. However similar results can be derived for $N = 3$ models with a lower number of gauge fields. We give below $N = 3$ models with 7, 4, 3 and 1 gauge fields.

The $SU(3,3+4)$ model

In the fermionic construction one uses the basis vectors F, S, \bar{S} , and $b_7^H = b_{11}^H$, as previously [see eqs. (2.1) and (2.2)]. The vector b_7^H implies the symmetric projection that is defined by \vec{H} [see eqs. (2.1) and (2.2)], while the additional vector

$$b_7^h = [\psi_\mu^L, y_{1,2,3}^L, \omega_4^L \chi_{5,6}^L, y_{5,6}^L, \omega_{5,6}^L \mid y_5^R, \omega_5^R, y_1^R, \omega_1^R] \quad (2.60)$$

defines the asymmetric projection denoted by \vec{h} .

As in the rank 11 model, there are two $N = 4$ sectors and one $N = 6$ sector, which give non-zero contributions to B_4 and B_6 supertraces.

For B_4 we have the following results:

$$\begin{aligned}
B_4^{\vec{h}=\vec{0}, \vec{H}\neq\vec{0}} &= 12 \frac{1}{2} \sum_{\gamma,\delta} |\theta[\gamma]_\delta|^4 = 12 \Gamma_{2,2}[0] |_{T=U=i} \\
B_4^{\vec{h}=\vec{H}\neq\vec{0}} &= 6 \frac{1}{2} \sum_{\gamma,\delta} \sum_{(H,G)\neq(0,0)} |\theta[\gamma]_\delta|^2 |\theta[\delta+G]^{\gamma+H}|^2 \\
&= 12 \Gamma_{2,2}[0] |_{T=i,U=2i} - 6 \Gamma_{2,2}[0] |_{T=U=i}.
\end{aligned} \tag{2.61}$$

At the massless level, $B_4^{\text{total}} = B_4^{\vec{h}=\vec{0}} + B_4^{\vec{h}=\vec{H}\neq\vec{0}} \rightarrow 12 + 6 = 18$, which matches the combined contributions from the supergravity sector and 7 vector multiplets, $B_4(\text{massless}) = 15/2 + 7 \times 3/2 = 18$.

For B_6 we find:

$$\begin{aligned}
B_6^{\vec{h}=\vec{0}, \vec{H}\neq\vec{0}} &= 30 \Gamma_{2,2}[0] |_{T=U=i} + \frac{15}{2} \sum_{(H,G)\neq(0,0)} \frac{\chi[G]^H + \bar{\chi}[G]^H}{4} \Gamma_{2,2}[G]^H |_{T=U=i}, \\
B_6^{\vec{h}=\vec{H}\neq\vec{0}} &= \frac{15}{2} \sum_{(H,G)\neq(0,0)} \frac{4 + \chi[G]^H + \bar{\chi}[G]^H}{4} \sum_{\gamma,\delta} |\theta[\gamma]_\delta|^2, |\theta[\delta+G]^{\gamma+H}|^2, \\
B_6^{\vec{h}\neq\vec{0}, \vec{H}=\vec{0}} &= \frac{45}{4} \sum_{(h,g)\neq(0,0)} \frac{\bar{\chi}[g]^h}{2} \Gamma_{2,2}[g]^h |_{T_3 U_3}.
\end{aligned} \tag{2.62}$$

Some comments are in order:

- In the limit $\text{Im } \tau \rightarrow \infty$, $B_6^{\vec{h}=\vec{0}, \vec{H}\neq\vec{0}} \rightarrow 75/2$, $B_6^{\vec{h}=\vec{H}\neq\vec{0}} \rightarrow 30$ and $B_6^{\vec{h}\neq\vec{0}, \vec{H}=\vec{0}} \rightarrow 45/4$. Then $B_6^{\text{total}}(\text{Im } \tau \rightarrow \infty) = 315/4$ corresponds to the contribution of the massless fields of the $N = 3$ supergravity together with the contribution of 7 $N = 3$ massless vector multiplets; $B_6(\text{massless}) = 525/8 + 7 \times 15/8 = 315/4$ as expected.

- $B_6^{\vec{h}\neq\vec{0}, \vec{H}=\vec{0}}$ is identical to the rank 11 model due to the universal behaviour of the $N = 6$ sector.

- The T -moduli of the first and second complex planes is always fixed at the self-dual point $T = i$. The U moduli of the same planes may take several discrete values.

- In the rank 7 model the $H = h = 1$ twisted sector is massive.

The $SU(3, 3)$ model

In this model there are no extra vector multiplets from the twisted sectors. In the fermionic construction one uses the basis vectors F, S, \bar{S} as before, plus two extra basis vectors defining the symmetric projection (\vec{H}):

$$b_3^H = [\psi_\mu^L, \chi_{1,2}^L, y_{3,\dots,6}^L, y_1^L, \omega_1^L \mid \psi_\mu^R, \chi_{1,2}^R, y_{3,\dots,6}^R, y_1^R, \omega_1^R],$$

and the asymmetric projection \vec{h} :

$$b_3^h = [\psi_\mu^L, y_{1,2,3}^L, \omega_4^L \chi_{5,6}^L, y_5^L, \omega_5^L \mid y_5^R, \omega_5^R]. \quad (2.63)$$

Here also there are two $N = 4$ sectors and one $N = 6$ sector. Their contributions to B_4 and B_6 supertraces are as follows.

- For B_4 we find:

$$\begin{aligned} B_4^{\vec{h}=\vec{0}, \vec{H} \neq \vec{0}} &= B_4^{\vec{h}=\vec{H} \neq \vec{0}} = 6 \frac{1}{2} \sum_{\gamma, \delta} \sum_{(H,G) \neq (0,0)} |\theta_{[\delta}^{[\gamma]}|^2 |\theta_{\delta+G}^{[\gamma+H]}|^2 \\ &= 12 \Gamma_{2,2}^{[0]} |_{T=i, U=2i} - 6 \Gamma_{2,2}^{[0]} |_{T=U=i} . \end{aligned} \quad (2.64)$$

At the massless level, $B_4^{\text{total}} = B_4^{\vec{h}=\vec{0}} + B_4^{\vec{h}=\vec{H} \neq \vec{0}} \rightarrow 6 + 6 = 12$, which matches the combined contributions from the supergravity sector and three vector multiplets, $B_4(\text{massless}) = 15/2 + 3 \times 3/2 = 12$.

- For B_6 we find:

$$\begin{aligned} B_6^{\vec{h}=\vec{0}, \vec{H} \neq \vec{0}} &= B_6^{\vec{h}=\vec{H} \neq \vec{0}} = \frac{15}{2} \sum_{(H,G) \neq (0,0)} \frac{4 + \chi_{[G]}^{[H]} + \bar{\chi}_{[G]}^{[H]}}{4} \sum_{\gamma, \delta} |\theta_{[\delta}^{[\gamma]}|^2 |\theta_{\delta+G}^{[\gamma+H]}|^2 \\ B_6^{\vec{h} \neq \vec{0}, \vec{H}=\vec{0}} &= \frac{45}{4} \sum_{(h,g) \neq (0,0)} \frac{\bar{\chi}_{[g]}^{[h]}}{2} \Gamma_{2,2}^{[h]} |_{T_3 U_3} \end{aligned} \quad (2.65)$$

At the massless level, $B_6^{\text{total}} = B_4^{\vec{h}=\vec{0}} + B_6^{\vec{h}=\vec{H} \neq \vec{0}} + B_6^{\vec{h} \neq \vec{0}, \vec{H}=\vec{0}} \rightarrow 30 + 30 + 45/4 = 285/4$, which matches the combined contributions from the supergravity sector and three vector multiplets, $B_6(\text{massless}) = 525/8 + 3 \times 15/8 = 285/4$.

Here again the contribution of the $N = 6$ sector is the same as in the previous models, thanks to the uniqueness of the $N = 6$ sector. Notice also that the T moduli of the first and second complex plane are fixed to their self-dual value.

The SU(3,1) model

This model is somewhat different from the previous ones in the sense that it is defined by three asymmetric projections, of which two are acting on the left-moving gravitinos h_1, h_2 while the other is acting on the right-moving ones h_3 . All projections are freely acting and thus there is no extra massless states coming from the twisted sectors. The h_1 acts on the 2nd and 3rd left moving complex planes, h_2 acts on the 1st and 3rd left-moving complex planes and h_3 acts on the 2nd and 3rd right-moving complex planes. In the fermionic construction one uses the basis vectors F, S, \bar{S} as before, as well as the three asymmetric basis vectors, which define the asymmetric projections h_i :

$$b^{h_1} = [\psi_\mu^L, \chi_{1,2}^L, y_{3,4,5,6}^L, y_1^L, \omega_1^L \mid y_1^R, \omega_1^R],$$

$$\begin{aligned}
b^{h_2} &= [\psi_\mu^L, y_{1,2}^L, \chi_{3,4}^L, \omega_{5,6}^L, y_3^L, \omega_3^L | y_3^R, \omega_3^R], \\
b^{h_3} &= [\psi_\mu^R, \chi_{1,2}^R, y_{3,4,5,6}^R, y_2^R, \omega_2^R | y_2^L, \omega_2^L].
\end{aligned} \tag{2.66}$$

In this model there are three $N = 4$ and four $N = 6$ sectors, which give non-zero contributions to the B_4 and B_6 supertraces. Namely, the three $N = 4$ sectors are:

$$\begin{aligned}
1) \vec{h}_1 &= \vec{0}, \vec{h}_2 = \vec{h}_3 = \vec{H} \neq \vec{0}, \\
2) \vec{h}_2 &= \vec{0}, \vec{h}_1 = \vec{h}_3 = \vec{H} \neq \vec{0}, \\
3) \vec{h}_1 + \vec{h}_2 &= \vec{0}, \vec{h}_1 = \vec{h}_2 = \vec{h}_3 = \vec{H} \neq \vec{0}.
\end{aligned}$$

Whereas the four $N = 6$ sectors are:

$$\begin{aligned}
1) \vec{h}_1 &= \vec{h} \neq \vec{0}, \vec{h}_2 = \vec{h}_3 = \vec{0} \\
2) \vec{h}_2 &= \vec{h} \neq \vec{0}, \vec{h}_1 = \vec{h}_3 = \vec{0} \\
3) \vec{h}_3 &= \vec{h} \neq \vec{0}, \vec{h}_2 = \vec{h}_1 = \vec{0} \\
4) \vec{h}_1 &= \vec{h}_2 = \vec{h} \neq \vec{0}, \vec{h}_3 = \vec{0}
\end{aligned}$$

All $N = 4$ sectors give equal contributions to B_4 :

$$\begin{aligned}
B_4^1 &= B_4^2 = B_4^3 = 3 \frac{1}{2} \sum_{\gamma, \delta} \sum_{(H,G) \neq (0,0)} |\theta[\gamma]|^2 |\theta[\delta+H]|^2 \\
&= 6 \Gamma_{2,2}^{[0]} |_{T=i, U=2i} - 3 \Gamma_{2,2}^{[0]} |_{T=U=i}.
\end{aligned} \tag{2.67}$$

In the limit $\text{Im } \tau \rightarrow \infty$, $B_4^{\text{total}} = B_4^1 + B_4^2 + B_4^3 \rightarrow 3 \times 3 = 9$, which corresponds to the contribution of the massless fields of the $N = 3$ supergravity together with the contribution of one $N = 3$ massless vector multiplet; $B_4(\text{massless}) = 15/2 + 1 \times 3/2 = 9$.

The B_6 receives contributions from the three $N = 4$ sectors as well as from the four $N = 6$ sectors. We find:

$$\begin{aligned}
B_6^{N=4,(1)} &= B_6^{N=4,(2)} = B_6^{N=4,(3)} = \frac{15}{4} \sum_{(H,G) \neq (0,0)} \frac{4 + \chi[G] + \bar{\chi}[G]}{4} \sum_{\gamma, \delta} |\theta[\gamma]|^2 |\theta[\delta+H]|^2 \\
B_6^{N=6,(1)} &= B_6^{N=6,(2)} = B_6^{N=6,(3)} = \frac{45}{8} \sum_{(h,g) \neq (0,0)} \frac{\bar{\chi}[g]}{2} \Gamma_{2,2}^{[g]} |_{T=U=i} \\
B_6^{N=6,(4)} &= \frac{45}{8} \sum_{(h,g) \neq (0,0)} \frac{\chi[g]}{2} \Gamma_{2,2}^{[g]} |_{T=U=i}.
\end{aligned} \tag{2.68}$$

In the limit $\text{Im } \tau \rightarrow \infty$, $B_6^{\text{total}} = B_6^{N=4,(1)} + B_6^{N=4,(2)} + B_6^{N=4,(3)} + B_6^{N=6,(1)} + B_6^{N=6,(2)} + B_6^{N=6,(3)} + B_6^{N=6,(4)} \rightarrow 3 \times 15 + 4 \times 45/8 = 135/2$, which corresponds to the contribution of the massless fields of the $N = 3$ supergravity together with the

contribution of one $N = 3$ massless vector multiplet; $B_6(\text{massless}) = 525/8 + 1 \times 15/8 = 135/2$.

In the rank 1 model all perturbative moduli are fixed. There is no marginal deformation in this model and all T and U moduli are fixed in all complex planes; in particular the T moduli are fixed to their self-dual values $T = i$.

In the next section we extend these results and write down the non-perturbative BPS formula for $N = 3$ string theory. These results can be used for computing the one-loop corrections to the R^4 and R^6 terms in string theory as well as for verifying its duality with the heterotic string theory presented in the next section.

3 Non-perturbative BPS States

3.1 Mapping to (3, 0) Models

In this section the projections, used for (2,1)-superstring constructions in type II models, are mapped either to the (3,0)-heterotic or (3,0) type II theories. The knowledge of this mapping defines at the non-perturbative level a specific Z_2 projection reducing the supersymmetries from $N = 4$ to $N = 3$ either in (4,0)-heterotic or in (4,0)-type II theories. As a result, the $N = 3$ non perturbative BPS mass formula is obtained from the one which is valid in $N = 4$ theories via the non-perturbative Z_2 truncation defined by the string-string duality maps.

First we consider the $SU(3, 11)$ model defined in previous section. To derive the non-perturbative BPS mass formula it is more convenient to use the asymmetric orbifold language. In this language the b_{11}^H and b_{11}^h act by twisting (asymmetrically) some of the internal coordinates. Namely,

$$\begin{aligned}
 (i) \quad b_{11}^H : \quad & X_i^{L,R} \rightarrow -X_i^{L,R}, (i = 5, \dots, 8), \\
 (ii) \quad b_{11}^h : \quad & (X_{3,4,5,6}^L) \rightarrow -(X_{3,4,5,6}^L) \\
 & X_7^{L,R} \rightarrow X_7^{L,R} + \pi
 \end{aligned} \tag{3.1}$$

In order to keep the world-sheet supercurrent invariant, the orbifold projections also act on the world-sheet fermions. These actions, in this case, are identical to the ones specified above for the bosons. It is also evident that b_{11}^h acts freely whereas the action of b_{11}^H has 16 fixed points.

In section 2, several other models with different numbers of gauge fields have been constructed. As all of them have lower-rank gauge sector, the corresponding BPS formula is obtained by setting to zero some of the charges in the above model.

The projection b_{11}^H gives a (2, 2) supersymmetric model in four dimensions. This model also has a space-time interpretation directly in six dimensions and is in fact

a special case of the type II compactification on K3. In four dimensions, it has twelve gauge fields, and associated scalars, from the untwisted sector. Out of these, only four associated with the momentum and winding modes of T^2 , specified by the compactified dimensions $X_{3,4}$, are from the NS–NS sector and eight from the R–R sector. In addition, there are sixteen $N = 4$ vector multiplets from the twisted sectors as well. These are located at the points $X_i = (0, \pi)$ along directions $i = 5, \dots, 8$.

The projection b_{11}^h applies an asymmetric twist to break another 1/2 supersymmetry from the left-moving sector. Moreover, it also applies a half-shift (on a lattice vector) in the X_7 direction. As a result, there are no extra massless states due to this projection. At the massless level, b_{11}^h projects out half of the vectors from both the untwisted and the twisted sectors. The numbers of vectors in the untwisted (NS–NS as well as R–R) sectors decrease, because of the twist part of b_{11}^h , since they act as a Z_2 which permutes, in pairs, the eight vectors in the R–R sector. In the NS–NS sector two vectors are even and the other two are odd under this Z_2 . The shift part of b_{11}^h , namely $X^7 \rightarrow X^7 + \pi$ permutes the sixteen fixed points of the first projection (b_{11}^H) in eight pairs. Its action on the vectors, through the appropriate twist operators associated with the fixed points, is by a similar permutation. The shift does not have any effect on the transformation of the vectors in the untwisted sector. Since the model constructed by a twist b_{11}^H is dual by standard string-string duality to the heterotic string, the projection b_{11}^h can be mapped to the heterotic side.

However, before going to the heterotic dual, we use the string duality which is valid among pairs of type II string constructions, and map the projection b_{11}^h to a model possessing $(4, 0)$ supersymmetry. The duality between type II string models with $(4, 0)$ and $(2, 2)$ supersymmetries has been discussed earlier [10], [18]; There, it was shown that such type II dual pairs can be constructed, and they are related through an element of the $SO(5, 5)$ U-duality group in six dimensions [10]. In addition, the relationship between the two models in four dimensions also involves an interchange between the S and T moduli fields associated with the T^2 specified by $X_{3,4}$.

We now obtain the mapping of the orbifold element b_{11}^h into the $(4, 0)$ side in the type II theory. Later on, this will be extended to the heterotic string theory, through the inclusion of the twisted sector states in the type II models. The transformation b_{11}^h , up to a shift, is represented by a six-dimensional $O(4, 4)$ transformation, in the choice of metric:

$$\bar{L} = \begin{pmatrix} -I_4 & \\ & I_4 \end{pmatrix}, \quad (3.2)$$

as

$$\bar{\Omega} = \begin{pmatrix} -I_2 & & \\ & I_2 & \\ & & I_4 \end{pmatrix}. \quad (3.3)$$

By complexifying the six internal coordinates as:

$$Z_1 = X_3 + iX_4, Z_2 = X_5 + iX_6, Z_3 = X_7 + iX_8, \quad (3.4)$$

in the notations of [10], $\bar{\Omega}$ can also be alternatively represented as:

$$\bar{\Omega} = (\pi, 0; 0, 0), \quad (3.5)$$

where the entries in the right-hand side of (3.5) denote the rotation in the planes represented by Z_2 and Z_3 in the left- and the right-moving sectors, respectively; b_{11}^h also has a part that acts as an $O(2, 2)$ transformation:

$$\Omega_D = \begin{pmatrix} -I_2 & \\ & I_2 \end{pmatrix}, \quad (3.6)$$

in the planes defined by Z_1 in the left- and right-moving sectors, for a choice of the $O(2, 2)$ metric:

$$L_D = \begin{pmatrix} -I_2 & \\ & I_2 \end{pmatrix}. \quad (3.7)$$

The subscripts in eqs. (3.6) and (3.7) denote the choice of a diagonal metric for $O(2, 2)$. For later convenience, we will also use an off-diagonal metric:

$$L = \begin{pmatrix} & I_2 \\ I_2 & \end{pmatrix}. \quad (3.8)$$

These are related by a map:

$$L_D = \eta L \eta^T, \quad (3.9)$$

with

$$\eta = \frac{1}{\sqrt{2}} \begin{pmatrix} -I_2 & I_2 \\ I_2 & I_2 \end{pmatrix}. \quad (3.10)$$

It has been shown in [10] that the mapping of an $O(4, 4)$ element to the $(4, 0)$ side involves the use of the triality between the $SO(4, 4)$ representations, such that $\bar{\Omega}$ transforms to

$$\tilde{\tilde{\Omega}} = (\pi/2, -\pi/2; \pi/2, -\pi/2). \quad (3.11)$$

In the matrix notation, this is written explicitly in a block diagonal form as

$$\tilde{\tilde{\Omega}} = \text{diag} (i\sigma_2, -i\sigma_2; i\sigma_2, -i\sigma_2) \quad (3.12)$$

with σ_2 a Pauli matrix.

In order to map $O(2, 2)$ transformation to the $(4, 0)$ side, we use the metric L in eq. (3.8). We also use the fact that an $O(2, 2)$ transformation Ω can be identified with two $SL(2)$ transformations Λ_T and Λ_U thanks to the equivalence:

$$O(2, 2) \equiv SL(2)_T \times SL(2)_U. \quad (3.13)$$

In particular, when the $SL(2, Z)$ transformations for the moduli T and U are given as:

$$\Lambda_T = \begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix} \in SL(2, Z)_T, \quad \Lambda_U = \begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix} \in SL(2, Z)_U, \quad (3.14)$$

the $O(2, 2)$ transformation is parametrized as [10]

$$\Omega = \begin{pmatrix} p_1 p_2 & p_1 q_2 & -q_1 q_2 & q_1 p_2 \\ p_1 r_2 & p_1 s_2 & -q_1 s_2 & q_1 r_2 \\ -r_1 r_2 & -r_1 s_2 & s_1 s_2 & -s_1 r_2 \\ r_1 p_2 & r_1 q_2 & -s_1 q_2 & s_1 p_2 \end{pmatrix}. \quad (3.15)$$

The transformation to the diagonal metric of the form described in eq. (3.7) is through a map:

$$\Omega_D = \eta \Omega \eta^T. \quad (3.16)$$

It can then be verified that the $O(2, 2)$ transformation (3.6), associated with the action of the projection element in the two-dimensional space $X_{3,4}$ can be identified with the $SL(2)_T$ and $SL(2)_U$ elements:

$$\Lambda_T = i\sigma_2, \quad \Lambda_U = -i\sigma_2. \quad (3.17)$$

For completeness, we also mention that the $N = 3$ type II construction being completely perturbative in nature, the S -duality element associated with b_{11}^h is trivial:

$$\Lambda_S = I_2. \quad (3.18)$$

Now an $S \leftrightarrow T$ interchange, together with $U \rightarrow U$, implies that the projection elements: Λ_T , Λ_U and Λ_S transform in the $(4, 0)$ side to

$$\tilde{\Lambda}_T = I_2, \quad \tilde{\Lambda}_S = i\sigma_2, \quad \text{and} \quad \tilde{\Lambda}_U = -i\sigma_2. \quad (3.19)$$

As a result the projection element in the $(4, 0)$ side, $b_{11}^{\tilde{h}}$, now has a non-trivial S -duality action. The action of $\tilde{\Lambda}_T$ and $\tilde{\Lambda}_U$ is further combined into an $O(2, 2)$ projection; for the diagonal choice of the metric L , this now has the form:

$$\tilde{\Omega}_D = \begin{pmatrix} -i\sigma_2 & \\ & -i\sigma_2 \end{pmatrix}. \quad (3.20)$$

The difference in the form of the $O(4, 4)$ part of the projection elements b_{11}^h and $b_{11}^{\tilde{h}}$ in the $(2, 2)$ and the $(4, 0)$ side can be understood from the fact that the gauge-field sectors on the two sides are related through an interchange of the NS-NS gauge fields with the R-R ones. This is essentially a change from the vector to the spinor representation of $SO(4, 4)$ and leads to a form of $b_{11}^{\tilde{h}}$ seen above. On the other hand, the remaining action of the projections in the two sides has its origin in the $S \leftrightarrow T$

interchange, which has also been used in showing the $SL(2, Z)$ duality of the heterotic string starting from the string/string-duality conjecture in six dimensions.

To promote the above mapping to the full heterotic string theory, one has to analyse the action of b_{11}^h on the twisted sector states in the type II side with $(2, 2)$ supersymmetry and use a mapping of the massless fields from the type II (on $K3$) to the heterotic string (on T^4). This map transforms the various 6-dimensional fields as [19]:

$$\begin{aligned} \phi' &= -\phi, & G'_{\mu\nu} &= e^{-\phi} G_{\mu\nu}, \\ M' &= M, & A'^a_{\mu} &= A^a_{\mu}, \\ \sqrt{-G} e^{-\phi} H^{\mu\nu\rho} &= \frac{1}{6} \epsilon^{\mu\nu\rho\sigma\tau\epsilon} H'_{\sigma\tau\epsilon}, \end{aligned} \tag{3.21}$$

where prime and unprime variables denote the fields in the type II side, compactified on $K3$ and the heterotic side, compactified on T^4 .

Now, in the type II side with $(2, 2)$ supersymmetry, the action of b_{11}^h on the sixteen vectors from the twisted sectors of b_{11}^H can be written as an $O(16)$ matrix in a block diagonal form:

$$\tilde{\Omega}_T = \text{diag}(i\sigma_2, i\sigma_2, \dots, i\sigma_2). \tag{3.22}$$

This is because the shift part of the projection b_{11}^h transforms the $X_7 = 0$ fixed points to that of $X_7 = \pi$ and vice versa. The mapping of these transformations to the heterotic side is then achieved through eq. (3.21). The relative minus sign in the transformation of the twisted sectors, within a pair, can be explained from the fact that, although the twist fields associated with the vertex operators of these gauge fields have identical weights at the fixed points $X^7 = 0$ and $X^7 = \pi$, the $U(1)$ vacuum charges are opposite and give a relative minus sign in the transformations.

Since the projection b_{11}^h acts freely, the twisted sector states, from this particular projection, in the $(2, 2)$ side are heavy. This also holds in the $(4, 0)$ side, where the projection b_{11}^h introduces non-zero R–R gauge field flux at the new fixed points and makes them heavy.

We have therefore identified, through the type II/heterotic map, the appropriate transformations in the heterotic side, which will give an $N = 3$ supersymmetric model. We notice that, unlike in the type II side, in the heterotic case, the projection acts non-perturbatively. However, this map does define a consistent model and allows us to write down the expression for the masses of the BPS states. We would once again like to mention the similarity between this $N = 3$ model and the orbifold limit of certain F -theory constructions. In the F -theory context, the orbifold limit is identified with the degeneration of the fibre through appropriate Weierstrass equations, and in this limit the moduli on the base remain constant, modulo the monodromies around certain points, which are identified with the fixed points of the orbifold group. A Z_4 projection, which forces the 10-dimensional type IIB axion-dilaton moduli to be fixed to its value at the self-dual point, was also identified in the F -theory context [12]. It

may be of interest to examine whether such a similarity between the two cases also leads to an understanding of certain non-perturbative aspects in our case.

We conclude this subsection by summarizing the action of the $N = 3$ projection $b_{11}^{\tilde{h}}$ in the heterotic side. This is given by eqs. (3.19), (3.20) on the coordinates $X_{3,4}$, by $\tilde{\Omega}$ in eq. (3.12) on the coordinates $X_{5,6,7,8}$, and by $\tilde{\Omega}_T$ in eq. (3.22) on the extra sixteen right-moving coordinates of the heterotic string. Since these projections act as an exchange of internal coordinates, they reduce the rank of the gauge group by half. In the next subsection we will see the same phenomena from a different point of view. The S -duality projection $\tilde{\Lambda}_S$ will be shown to preserve only half of an $N = 4$ vector multiplet. These exchange operations then combine two such half-multiplets to construct a self-conjugate vector multiplet of $N = 3$.

3.2 Massless States

After having identified the projection in the heterotic side, we now obtain the spectrum of the heterotic string theory, when the above projection is applied. We show that there are three surviving supersymmetries, and at the massless level, we get a correct spectrum for the $N = 3$ string theory. In an $N = 4$ theory, the supersymmetry algebra and representations are specified by a group structure, $SO(2) \times SU(4) \times U(1)_S$, where $SO(2)$ is the little group of the Lorentz group and $U(4) \equiv SU(4) \times U(1)_S$ is the R -symmetry for the $N = 4$ theory. Gravitinos transform as $\mathbf{4}$ and the $U(1)$ gauge fields of the supergravity multiplet transform as a $\mathbf{6}$ of $SU(4)$. In addition, $N = 4$ supergravity multiplet also has two scalars, which are neutral under $SU(4)$. In heterotic string theory, the $SU(4)$ symmetry can be identified with the $SO(6)_L$ subgroup of the $SO(6, 22)$ T -duality group, which originates from its left-moving sector. $U(1)_S$ is the maximal compact subgroup of the S -duality group: $SL(2, R)$.

For the projection $b_{11}^{\tilde{h}}$, the $SU(4)$ element is given by the diagonal $U(1)$ subgroup of $SO(6)_L$: $\Omega_L \equiv \text{diag}(-i\sigma_2, i\sigma_2, -i\sigma_2)$. In this case, we also have, in the right-moving sector, a projection given by the diagonal $U(1)$ element of $SO(22)_R$, given by Ω_T in eq. (3.22) together with Ω_R , which acts on the right-moving part in the 6-dimensional internal space and is identical to Ω_L . In addition, as mentioned before the $U(1)_S$ element is specified by the matrix $\tilde{\Lambda}_S$ given in eq. (3.19).

This projection breaks $SU(4) \times U(1)_S$ to its subgroup $SU(3) \times U(1)_V$, where the $U(1)_V$ that remains unbroken is a combination of the $U(1)_S$ and $U(1)_I$, with $U(1)_I$ as the diagonal $U(1)$ subgroup of $SU(4)$: $SU(3) \times U(1)_I \subset SU(4)$. The four supercharges and their CPT conjugates transform, under $SU(4) \times U(1)_S$ as:

$$Q_{1/2} = \mathbf{4}_{(1/2,1)} + \bar{\mathbf{4}}_{(-1/2,-1)}, \quad (3.23)$$

where the first entry in the bracket shows the helicity of the state. By decomposing the supercharges in the representation of $SU(3) \times U(1)_I$, the three supersymmetry generators that survive the projection are: $\mathbf{3}_{[1/2,-1/3,1]}$ and its complex conjugate.

In the following, the entries in the curly brackets “ (\cdot, \cdot) ” denote helicity and $U(1)_S$ charges while the quantities in the square brackets “ $[\cdot, \cdot, \cdot]$ ” denote helicity, $U(1)_I$ and $U(1)_S$ charges. The supercharge transforming as a singlet of $SU(3)$ is projected out. As a result we are left with $N = 3$ supersymmetry.

The states that survive the above projection also belong to a representation of the residual supersymmetry. The $N = 4$ gravity multiplet is constituted out of helicity ± 2 supermultiplets of the $N = 4$ supersymmetry, which are complex conjugates of each other. They have the transformation property: $\mathbf{1}_{(-2,0)} + \mathbf{4}_{(-3/2,1)} + \mathbf{6}_{(-1,2)} + \bar{\mathbf{4}}_{(-1/2,3)} + \mathbf{1}_{(0,4)}$, together with the complex-conjugate representation. The $N = 3$ projection then selects out of the above, the $\mathbf{1}_{[2,0,0]}$ and $\bar{\mathbf{3}}_{[-1,-2/3,2]}$ states among the bosonic ones, due to the decomposition of $SU(4) \rightarrow SU(3) \times U(1)_I$. These, together with their CPT conjugates, give us $g_{\mu\nu}$ and three A_μ 's, which is the correct spectrum for the $N = 3$ supergravity sector.

The vector multiplet of $N = 4$ supersymmetry is self-conjugate under CPT, whereas the $N = 3$ one is constructed out of two different multiplets containing helicity -1 and $+1$ states and are conjugate to each other. The $N = 3$ projection mentioned above does not leave the complete vector multiplet of the $N = 4$ theory invariant. Instead, it projects out some of the states and leaves only an $N = 3$ multiplet, of either $+1$ or -1 helicity invariance. However, as seen before, the $N = 3$ projection also permutes the internal indices of the heterotic string theory in the right-moving sector. As a result, the complete $N = 3$ vector multiplet is a linear combination of two half-vectors from $N = 4$. The final spectrum is CPT-invariant. But the rank of the gauge group is reduced by a factor $1/2$.

3.3 Mass Formula

After identifying the correct $N = 3$ massless spectra from the projection of the heterotic strings, we now proceed to write down the BPS mass formula in this theory. In this context, the starting point is the $N = 4$ BPS formula, which can be written in terms of six “electric” and six “magnetic” charges, associated with the supergravity sector, as [3]:

$$M_{BPS}^2 = \frac{[(P_m + SQ_m)(P^m + \bar{S}Q^m)]}{4 \text{Im } S} + \frac{1}{2} \sqrt{(P_m P^m)(Q_m Q^m) - (P_m Q^m)^2}, \quad (3.24)$$

where the contractions of the indices are defined with respect to the internal metric on T^6 , namely G^{mn} . The quantities P_m and Q_m are defined in terms of the integer valued electric charges $(\alpha^L, \alpha^R, \alpha^I)$ and magnetic charges $(\tilde{\alpha}^L, \tilde{\alpha}^R, \tilde{\alpha}^I)$ of the heterotic string theory as:

$$\begin{aligned} P &= \alpha^L + (G + B + C)\alpha^R + A\alpha^I \\ Q &= \tilde{\alpha}^L + (G + B + C)\tilde{\alpha}^R + A\tilde{\alpha}^I. \end{aligned} \quad (3.25)$$

Here $C = \frac{1}{2}AA^T$, G and B are the moduli fields associated with the internal T^6 , and the A 's are the Wilson-line moduli.

The square-root factor in the BPS formula is proportional to the square of the difference in the two $N = 4$ central charges. This term vanishes for the states preserving 1/2 supersymmetry. Such states belong to the “short” multiplets of the $N = 4$. The non-zero contribution comes from the states belonging to the intermediate multiplets of $N = 4$ and preserve the 1/4 of the supersymmetry. The BPS states in the perturbative construction of the heterotic string theory are examples of “short” multiplets.

To obtain the BPS formula for the $N = 3$ case, using the Z_2 projection mentioned before, we must set at the beginning first the “dilaton-axion” moduli of the $N = 4$ theory to the value $S = i$. This is due to an observation that we made earlier, namely a transformation by $\tilde{\Lambda}_S$ in eq. (3.19) is a symmetry of the $N = 4$ theory, for a given coupling, only for this value of the S field. In other words, the $N = 3$ projection transforms the strong coupling to the weak one, and can therefore be applied consistently only for its fixed value at the self-dual point. Furthermore, since the gauge fields, associated with the charges mentioned above, transform under $SU(4) \times U(1)_S$ as $\mathbf{6}_{(-1,2)}$ and $\mathbf{6}_{(1,-2)}$, only those belonging to the $SU(3) \times U(1)_V$ representations, $\mathbf{3}_{[1,2/3,-2]}$ and $\bar{\mathbf{3}}_{[-1,-2/3,2]}$, survive the $N = 3$ projection. To select the appropriate combinations that remain invariant under this projection, we define:

$$\Pi_m = (P_m + iQ_m), \quad \bar{\Pi}_{\bar{m}} = (P_m - iQ_m). \quad (3.26)$$

We also use the complexifications of the coordinates introduced in eq. (3.4) and note that out of the original twelve charges, those existing after the projection are six charges Π_m and $\bar{\Pi}_{\bar{m}}$. With this projection, both the terms in the $N = 4$ mass formula give identical contributions, and one gets

$$M_{BPS}^2 = \frac{1}{2} \Pi_m G^{m\bar{m}} \bar{\Pi}_{\bar{m}}. \quad (3.27)$$

In obtaining the $N = 3$ mass formula from $N = 4$ in eq. (3.27), we have also used the fact that the internal space for $N = 3$ is parametrized by a Kähler metric. This can be observed independently of the action of $b_{11}^{\tilde{}}$ on the moduli fields. In particular, only the components $G_{m\bar{m}}$, $B_{m\bar{m}}$, A_m^{I+} and $A_{\bar{m}}^{I-}$, with I^{\pm} being the complexifications of the sixteen right-moving coordinates, survive the $N = 3$ projection. The mass formula (3.27) is also the unique quadratic invariant of the charges, the latter transforming as $\mathbf{3}$ and $\bar{\mathbf{3}}$ under the residual $U(3)$ symmetry.

We now rewrite the BPS mass formula (3.27) in terms of the physical charges associated with the gauge fields in the theory. This is also given by a projection over the charges Π_m , defined in terms of the physical charges of $N = 4$ theory, using eqs. (3.25) and (3.26). The final expression has a form similar to that of eq. (3.25):

$$\Pi_m = \alpha^L_m + (G + B + C)_{m\bar{m}} \alpha_{\bar{m}}^R + A_m^{I+} \alpha_{I+}. \quad (3.28)$$

The mass formula (3.27) can now be rewritten in an $SU(3, n)$ -invariant form³:

$$M_{BPS}^2 = q^\dagger \cdot (M + L) \cdot q, \quad (3.29)$$

with q denoting the column vector:

$$q = \begin{pmatrix} \alpha_m^L \\ \alpha_m^R \\ \alpha^{I^\pm} \end{pmatrix}. \quad (3.30)$$

The $SU(3, n)$ matrix M has the standard expression in terms of the Wilson-line moduli $A_{z_i}^{I^\pm}$ and the Hermitian and anti-Hermitian matrices G and B , respectively:

$$\begin{pmatrix} G^{-1} & G^{-1}(B+C) & G^{-1}A \\ (-B+C)G^{-1} & (G-B+C)G^{-1}(G+B+C) & (G-B+C)G^{-1}A \\ A^\dagger G^{-1} & A^\dagger G^{-1}(G+B+C) & \mathbf{I}_8 + A^\dagger G^{-1}A \end{pmatrix}, \quad (3.31)$$

with $C = \frac{1}{2}AA^T$ and the $SU(3, n)$ metric L having the form:

$$L = \begin{pmatrix} 0 & I_3 & 0 \\ I_3 & 0 & 0 \\ 0 & 0 & -I_8 \end{pmatrix}. \quad (3.32)$$

It can be verified that the matrix M is Hermitian and satisfies the $SU(3, n)$ property: $M^\dagger LM = L$.

In this section we have presented the $SU(3, n)$ -invariant BPS mass formula for the $N = 3$ string theories in four dimensions. There can be many applications of these results, including black-hole physics. The entropy formula for the $N = 3$ case has already been presented in the literature [13]. Our results can be used to obtain these expressions from a microscopic description through an appropriate truncation of the type II or heterotic string models.

4 $N = 3$ String Effective Action

The projections applied to the $N = 4$ theory, in the previous section, can also be used for writing down the $N = 3$ effective action. By restricting this to the bosonic sector, the $N = 4$ effective action, at a generic point \hat{M} in the moduli space of the heterotic string has a form:

$$\begin{aligned} \hat{S} = \frac{1}{32\pi} \int \sqrt{-g} & \left[R - \frac{1}{2(\lambda)^2} g^{\mu\nu} \partial_\mu \lambda \partial_\nu \bar{\lambda} - \lambda_2 F_{\mu\nu}^{(a)} (\hat{L} \hat{M} \hat{L}) F^{(b) \mu\nu} + \right. \\ & \left. \lambda_1 F_{\mu\nu}^{(a)} \hat{L}_{ab} \tilde{F}^{(b) \mu\nu} + \frac{1}{8} g^{\mu\nu} \text{Tr}(\partial_\mu \hat{M} \hat{L} \partial_\nu \hat{M} \hat{L}) \right], \end{aligned} \quad (4.1)$$

³In the present case $n = 11$. But the general structure of the invariant expressions is preserved for other values of n as well.

where ($a = 1, \dots, 28$) and \hat{L} is an $SO(6, 22)$ metric:

$$\hat{L} = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix}. \quad (4.2)$$

By taking into account the fact that the axion-dilaton moduli of the heterotic string are fixed to the self-dual point, the $N = 3$ projection then leads to an action of the form:

$$S = \frac{1}{32\pi} \int d^4x \sqrt{-g} \left[R - F_{\mu\nu}^+ (LML) F^{-\mu\nu} + \frac{1}{8} g^{\mu\nu} \text{Tr}(\partial_\mu ML \partial_\nu ML) \right], \quad (4.3)$$

where M denotes the $SU(3, n)$ moduli (3.31), L is the $SU(3, n)$ metric (3.32) and $F^\pm = F \pm i\tilde{F}$. In obtaining action (4.3) from (4.1), we expanded various terms of the $N = 4$ action and then recombine them after collecting the invariants.

It is observed that the action (4.3) is manifestly invariant under the $U(3, n)$ symmetry. The manifest invariance of the action is due to the fact that the $N = 3$ projection on the heterotic strings leaves only perturbative moduli in the spectrum. The above reduction of $N = 4$ effective action to $N = 3$ can also be seen at the level of the equations of motion, which for the $N = 3$ case can be written, starting from the $N = 4$ one, as:

$$\begin{aligned} R_{\mu\nu} &= 2F_{\mu\nu}^+ (LML) F_\nu^\rho - \frac{1}{2} g_{\mu\nu} F_{\rho\sigma}^+ (LML) F^{-\rho\sigma} \\ D_\mu (MLF^{+\mu\nu}) &= 0 \end{aligned} \quad (4.4)$$

It will be of interest to study the solutions of these equations of motion, in order to obtain the classical background configurations that are consistent with $N = 3$ supersymmetry. Among them, those that preserve 1/2 of the supersymmetry, such as extremal black-hole configurations, are of particular interest. They will provide the examples of the $N = 3$ BPS states found in the previous sections.

5 Conclusions

We have presented an explicit expression for the $N = 3$ BPS formula. It was also shown how the known $N = 3$ models are incorporated in this picture. The BPS states associated with the perturbative spectrum of the known $N = 3$ string theory were also described. It will be of interest to further study compactification of the $N = 3$ effective action to 2- and 3-dimensional space-times. In particular, it is expected that the effective action in three dimensions will possess an $SU(4, n+1)$ symmetry. The coset structure $SU(4, n+1)/SU(4) \times SU(n+1) \times U(1)$ can be seen by a direct counting of the matter degrees of freedom, which in the bosonic sector contains only scalars, in three dimensions. Our results, through a duality between the type II and the heterotic

sides, also indicate that type II string models with $N = 3$ supersymmetry and non-Abelian gauge symmetries can be obtained at special points of the moduli space. It may be interesting to examine whether some of these symmetry enhancements in the type II case can take place at special values of the perturbative type II moduli as well. One will then be able to study them using conformal field-theory techniques.

It will also be of interest to directly construct $N = 3$ orientifold models in four dimensions, whose open string sectors, with Dirichlet boundary conditions, can be interpreted as BPS states appearing in the mass formulae derived earlier. This turns out to be a difficult exercise due to the asymmetric nature of the $N = 3$ construction on the one hand and to the requirement of the left-right symmetry for the orientifolding operation on the other. In order to achieve the desired results, the orientifolding operation must therefore be combined with appropriate operation on the internal space. Although the orientifolding operation seems difficult in four dimensional models, it becomes much simpler in two dimensions where models preserving $3/8$ supersymmetry can be easily constructed.

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