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NORMAL FORM VIA TRACKING OR BEAM DATA

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Normal Form is a powerful tool to analyse the nonlinear content of a complicated system like an accelerator with its many thousands of high order multipolar errors. This technique needs as input a mapping from the initial to the final coordinates. Unfortunately, a map of an accelerator is not known beforehand but has to be determined for a complete lattice of the accelerator including all calculated and/or measured error tables of the guiding and focussing magnet elements. It is however possible to obtain turn-by-turn position data of a kicked beam for many turns. In various experiments at existing accelerators it has been shown that these data are equivalent to tracking data produced by simulation programs. In this report we will demonstrate how tracking data can be used to determine, with excellent precision, the coefficients of the generating function. The well tested Normal Form tools can then be used to construct the underlying Hamiltonian and the map. The essential tool is the recently developed frequency analysis which allows for a very precise determination of the tunes using beam or tracking data.

Keywords: Single beam dynamics; Normal form; Frequency

1 INTRODUCTION

Since many years perturbation theory[1] and more recently the Normal Form[2–5] techniques have been used to understand nonlinear motion of single particles in hadron accelerators. This has proven to be very useful in the design phase of an accelerator. When it comes to existing machines these sophisticated tools have been rarely in use up to now. In part this is due to the complexity of the theory but also due to the fact that a nonlinear model of the accelerator cannot be predicted easily. Checking such a model experimentally[6,7] may prove even more difficult.

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One well documented attempt to overcome this problem has been made by Bengtsson.[8] In the framework of the first order perturbation theory he has studied how the real spectra from tracking or experimental turn-by-turn data can be related to resonances. This study has stopped short of a complete solution. An important prerequisite to his analysis was a tune measurement technique superior to the standard FFT.[9] Similar attempts were performed in the field of celestial mechanics.[10]

Recently, new techniques were developed,[11,12] allowing an even more precise determination of the tunes. It seems therefore appropriate to review the link between experimental data and theoretical models. The frequency map analysis by Laskar[11] can be used not only to derive the tune, but also to find spectral lines in descending order of magnitude. It has already been shown how these spectra can be applied to remove from a sequence of tracking data unwanted regular complexity. Moreover, this method has been successfully used to correct resonances excited by sextupoles.[13]

In Section 3 we will demonstrate how the generating function in resonance basis, the coefficients of which are equivalent to the resonance driving terms, can be derived approximately from this high precision spectrum of tracking data. In Section 4 a prescription will be given to get the terms of the generating function order-by-order. As an example we will show how well resonance driving terms in the Hamiltonian can be derived from tracking data for a very complicated accelerator model. Lastly in Section 5 we will discuss the limits of the method.

2 SPECTRUM FROM TRACKING OR BEAM DATA

The time series of tracking (or experimental) data is analysed with the algorithms for the precise measurement of the betatron tune. They provide the main frequencies (i.e. the tunes) and the corresponding Fourier coefficients. By subtracting from the time series these tune lines, using the proper amplitudes and phases, we obtain a new signal of equal length which can be reanalysed in the same way. This iterative procedure provides the set of frequencies contained in the betatron motion.

In case the motion is regular it can be decomposed into a series of spectral lines[14] the frequencies of which are the fundamental tunes ν_x and ν_y and their linear combinations. For the horizontal particle coordinates (x, p_x) transformed into the linearly normalised Courant–Snyder variables[15] $(\hat{x}, \hat{p}_x)$, we get after N turns

$$\hat{x}(N) - \mathrm{i}\hat{p}_x(N) = \sum_{j=1}^{\infty} a_j \mathrm{e}^{\mathrm{i}[2\pi(m_j\nu_x + n_j\nu_y)N + \psi_j]}, \quad m_j, n_j \in \mathbf{Z}, \tag{1}$$

where the a_j and ψ_j are amplitude and phase of the corresponding spectral line.

3 GENERATING FUNCTION VERSUS DATA SPECTRUM

In the following we will consider the four dimensional case of coupled horizontal and vertical motion. We consider this to be minimal as the coupling resonances are mandatory to understand the dynamics of a real accelerator. On the other hand we did not consider the full six dimensional case so as to avoid the formalism to become too complicated. We like to stress, however, that the formalism as well as all numerical tools can be fully applied to the general case of six dimensions provided 6D particle data are available.

The 4D particle coordinates $\mathbf{x}' = (x, p_x, y, p_y)$, after one turn in the machine, are related to the initial coordinates $\mathbf{x}$ by the mapping

$$\mathbf{x}' = \mathbf{M}(\mathbf{x}). \tag{2}$$

The perturbative theory for the maps gives a powerful tool to parametrise the nonlinear contents of the particle motion in terms of resonances and detuning terms. The initial map $\mathbf{M}$ is reduced to a simpler map $\mathbf{U}$ by means of a symplectic change of coordinates following the scheme

$$\begin{array}{ccc} \mathbf{x} & \xrightarrow{\mathbf{M}} & \mathbf{x}' \\ \mathbf{\Phi}^{-1}\downarrow & & \downarrow\mathbf{\Phi}^{-1} \\ \zeta & \xrightarrow[\mathbf{U}]{} & \zeta' \end{array} \tag{3}$$

The goal of the transformation $\mathbf{\Phi}^{-1}$ is to perform a change of variable towards the action–angle variables leaving in the transformed map $\mathbf{U}$ only action dependent terms (detuning terms), the so-called non-resonant Normal Form. In the case of the resonant Normal Form angle dependent terms are left in the Hamiltonian. The Normal Form transformation is accomplished via an order-by-order procedure in the perturbative parameter which in our case is the distance from the origin of the phase space. The transformation $\mathbf{\Phi}$ and the map $\mathbf{U}$ may be expressed as Lie operators with generating function F and H respectively:

$$\mathbf{\Phi} = \mathrm{e}^{:F:}, \quad \mathbf{\Phi}^{-1} = \mathrm{e}^{-:F:}, \quad \mathbf{U} = \mathrm{e}^{:H:}. \tag{4}$$

In the lowest order, the linear case, the action angle variables $(J_x, \phi_x, J_y, \phi_y)$ are simply related to the Courant–Snyder variables $(\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y)$ by the formula (z stands for x or y)

$$\begin{aligned} \hat{z} &= \sqrt{2J_z}\cos(\phi_z + \phi_{z0}); \\ \hat{p}_z &= -\sqrt{2J_z}\sin(\phi_z + \phi_{z0}), \end{aligned} \tag{5}$$

where ϕ_{z0} is the initial phase. It is convenient to express the linearly normalised variables in the so-called resonance basis $\mathbf{h} = (h_x^+, h_x^-, h_y^+, h_y^-)$ defined by the relations:

$$h_z^{\pm} = \hat{z} \pm \mathrm{i}\hat{p}_z = \sqrt{2J_z}\mathrm{e}^{\mp\mathrm{i}(\phi_z + \phi_{z0})}. \tag{6}$$

The transformation to the new set of canonical coordinates $\zeta = (\zeta_x^+, \zeta_x^-, \zeta_y^+, \zeta_y^-)$ which brings the map into the Normal Form is expressed as a Lie series

$$\zeta = \mathrm{e}^{-:F_\mathrm{r}:}\mathbf{h}, \tag{7}$$

where

$$\zeta_z^{\pm} = \sqrt{2I_z}\mathrm{e}^{\mp\mathrm{i}(\psi_z + \psi_{z0})} \tag{8}$$

and $(I_x, \psi_x, I_y, \psi_y)$ are the nonlinear action–angle variables. In the resonance basis, F_r can be written as a sum of homogeneous polynomials of the variables ζ as

$$F_\mathrm{r} = \sum_{jklm} f_{jklm} \zeta_x^{+j} \zeta_x^{-k} \zeta_y^{+l} \zeta_y^{-m}. \tag{9}$$

Introducing the $\zeta_z^{\pm}$ of Eq. (8), we arrive at:

$$F_{\mathrm{r}} = \sum_{jklm} f_{jklm} (2I_x)^{(j+k)/2} (2I_y)^{(l+m)/2} \mathrm{e}^{-\mathrm{i}[(j-k)(\psi_x+\psi_{x_0})+(l-m)(\psi_y+\psi_{y_0})]}. \quad (10)$$

The transformation from the new action–angle variables to the linearly normalised variables is given by

$$\mathbf{h} = \mathrm{e}^{:F_{\mathrm{r}}:}\zeta = \zeta + [F_{\mathrm{r}}, \zeta] + \frac{1}{2}[F_{\mathrm{r}}, [F_{\mathrm{r}}, \zeta]] + \cdots \quad (11)$$

where $[F_{\mathrm{r}}, \zeta]$ denotes the Poisson bracket of F_{r} and ζ. To the first order the transformation to the h_x^- reads

$$h_x^- \approx \zeta_x^- + [F_{\mathrm{r}}, \zeta_x^-] = \zeta_x^- - 2\mathrm{i} \sum_{jklm} j f_{jklm} \zeta_x^{+^{j-1}} \zeta_x^{-^k} \zeta_y^{+^l} \zeta_y^{-^m}, \quad (12)$$

where we have used the property of the Poisson bracket

$$[\zeta_x^{+^j}, \zeta_x^-] = -2\mathrm{i}\, j \zeta_x^{+^{j-1}}. \quad (13)$$

The evolution of the variable in Normal Form after N turns is given by

$$\zeta_x^-(N) = \sqrt{2I_x}\mathrm{e}^{\mathrm{i}(2\pi\nu_x N + \psi_{x_0})}, \quad (14)$$

where ν_x is the horizontal tune of the particle including the amplitude dependent detuning and ψ_{x_0} is the horizontal initial phase. We can therefore obtain the evolution of the linearly normalised horizontal variable in the form

$$h_x^-(N) = \sqrt{2I_x}\mathrm{e}^{\mathrm{i}(2\pi\nu_x N + \psi_{x_0})} - 2\mathrm{i} \sum_{jklm} j f_{jklm} (2I_x)^{(j+k-1)/2} \times (2I_y)^{(l+m)/2} \mathrm{e}^{\mathrm{i}[(1-j+k)(2\pi\nu_x N+\psi_{x_0})+(m-l)(2\pi\nu_y N+\psi_{y_0})]}. \quad (15)$$

This expression is equivalent to the spectral decomposition of Eq. (1). The motion appears as a superposition of spectral lines given by the tune (first term) and the contribution from the resonant terms (second term).

The expression in Eq. (15) can be compared with the generating function (Eq. (10)) in order to determine the coefficients f_{jklm} term-by-term. To this aim we rewrite the Eq. (15) as

$$h_x^-(N) = \sum_{jklm} HSL_{jklm} \mathrm{e}^{2\pi\mathrm{i}[(1-j+k)\nu_x+(m-l)\nu_y]N}, \quad (16)$$

where the complex Fourier coefficient of the horizontal spectral line is indicated by HSL_{jklm} with amplitude $|HSL_{jklm}|$ and phase $PHSL_{jklm}$. Analogously VSL_{jklm} will indicate the complex Fourier coefficient of the vertical motion. The following table compares the amplitudes and phases of the spectral lines with those of the generating function coefficients. Note that the $-\pi/2$ in the phase is due to the multiplicative factor $-i$ in front of the second term of Eq. (15). The ϕ_{jklm} is the phase of the complex generating function term f_{jklm}.

It has to be mentioned that the approximation in Eq. (12) which leads to Table I, is only valid for the resonances due to sextupoles in first order of their strengths. For each additional multipole order, one has to add one Poisson bracket which in the table will lead to corresponding higher order contribution of the $|f_{jklm}|$ to the spectral lines. In the next section we discuss how to overcome this problem.

It is also possible to relate the spectral lines directly with the driving terms. We use the well known relation[16] between the Hamiltonian coefficients h_{jklm} and generating function coefficients f_{jklm}:

$$f_{jklm} = \frac{h_{jklm}}{1 - \mathrm{e}^{\mathrm{i}\{2\pi[(j-k)\nu_x + (l-m)\nu_y]\}}}, \tag{17}$$

with ψ_{jklm} being the phase of the driving term h_{jklm}, we obtain Table II.

TABLE I Relation between spectral lines and coefficients of the generating function

	Generating function coefficient	*Spectral line*
Amplitude	$\lvert f_{jklm}\rvert$	$\lvert HSL_{jklm}\rvert = 2 \cdot j \cdot (2I_x)^{(j+k-1)/2}(2I_y)^{(l+m)/2}\lvert f_{jklm}\rvert$
		$\lvert VSL_{jklm}\rvert = 2 \cdot l \cdot (2I_x)^{(j+k)/2}(2I_y)^{(l+m-1)/2}\lvert f_{jklm}\rvert$
Phase	ϕ_{jklm}	$PHSL_{jklm} = \phi_{jklm} + \psi_{x_0} - \pi/2$
		$PVSL_{jklm} = \phi_{jklm} + \psi_{y_0} - \pi/2$

TABLE II Relation between Horizontal Spectral Lines HSL_{jklm} and coefficients of the resonant Hamiltonian (the vertical case is equivalent)

	Driving term	*Horizontal Spectral Line* (HSL_{jklm})
Amplitude	$\lvert h_{jklm}\rvert$	$(2I_x)^{(j+k-1)/2}(2I_y)^{(l+m)/2}(2 \cdot j \cdot \lvert h_{jklm}\rvert/(2\sin(\lvert\hat{\phi}\rvert)))$
Phase	ψ_{jklm}	$\psi_{jklm} + \psi_{x_0} - \pi/2 + \mathrm{sign}(\hat{\phi})(\pi/2 - \lvert\hat{\phi}\rvert)$, where $\hat{\phi} = \pi[(j-k)\nu_x + (l-m)\nu_y]$

4 APPLICATION MANUAL

We describe here how to construct the generating function and/or map from the data.

- Sufficient amount of raw data are needed from tracking or beam position monitors, i.e. a minimum of 1,000–10,000 turns seems adequate. For the beam data a horizontal and vertical pickup at the same longitudinal position is required.
- In principle, the knowledge of one beam position monitor per plane is sufficient to derive all terms of the generating function (see below). However it is preferable to have a second pair of pickups available at a phase advance of 90° with respect to the first pair. Such a second pickup will allow to have both beam positions and momenta in normalised coordinates. This is true for pickups in a FODO lattice with a 90° phase advance per cell since the β function at the pickups are equal. If this condition is not exactly fulfilled some preprocessing of the beam data is needed.[17]
- Each spectral line of Eq. (15) defined by $(1-j+k)\nu_x+(m-l)\nu_y$ is driven by Hamiltonian coefficients h_{jklm} which is responsible for the excitation of the $(j-k, l-m)$ resonance. For example the third order resonance $(3, 0)$ which means $j-k=3$ and $l-m=0$ drives the spectral line $(-2, 0)$, i.e. $1-j+k=-2$ and $l-m=0$. This also implies that several driving terms h_{jklm} can contribute to the same spectral line.
- In the case of experimental data[17] the kick strength has to be varied until the signal is detectable but not disturbed by higher order contributions. The terms in the Hamiltonian of order $(n=j+k+l+m)$ correspond to lines which depend on the amplitudes to the power $(n-1)$. The corresponding resonances, however, can also be driven by Hamiltonian terms of order $(n+2, n+4, \ldots)$. The lines generated by these subresonances depend on the amplitudes to the power $(n+1, n+3, \ldots)$, e.g. the $(3, 0)$ resonance is driven by sextupoles but also by decapoles and the corresponding lines depend on the amplitude to the order $(n=2)$ and $(n=4)$ respectively.
- As mentioned above, Eq. (11) allows to derive the sextupole related terms considering the first Poisson bracket only. The details are given in Section 4.1. When octupoles are examined, the second Poisson bracket is also needed because terms second order in the sextupoles

strengths will contribute to the spectral lines with the same amplitude dependence as the octupoles. Two approaches seem appropriate: either one calculates these second order Poisson brackets with the knowledge of the previously determined third order terms or one takes out the third order terms from the raw tracking data. The latter implies that after this third order transformation the frequency analysis has to be redone for the transformed data. As a result Eq. (12) can be applied to calculate the octupoles terms using the first Poisson bracket only and following the scheme described in Section 4.2. This procedure can be extended to higher order if needed.

- Once the generating function is constructed from the line spectra the DaLie package[18] allows the construction of the generating function and the Hamiltonian in resonance and Cartesian basis. It also allows to determine all terms in the map order-by-order.
- For the map construction there remains one last point which concerns the detuning terms. The first line of the spectra is always the precise tune at the chosen kick amplitude, i.e. the contributions to all orders are included. One can proceed in two ways: either the map is constructed without the detuning terms so that it only contains the information concerning resonances, or the detuning terms are derived from a fit from tunes versus kick amplitudes.

4.1 Third Order Contributions to the Line Spectra

According to the theory outlined in Section 3 the resonances due to normal sextupoles and the corresponding line spectra can be related as shown in Tables III and IV. The lines expected from the x^3 term are given in Table III while from the xy^2 term we obtain Table IV.

The third order resonance coefficients have a one-to-one correspondence with the lines. This is not the case for the subresonance $(1, 0)$ (the $(0, 0)$ term in the line spectrum) which has contributions from h_{2100}

TABLE III Resonances and spectral lines due to the normal sextupole term x^3

D.T.	*Resonance*	$j+k$	j	*Lines hor.*	$l+m$	l	*Lines ver.*
h_{3000}	$(3,0)$	3	3	$(-2,0)$	0	0	—
h_{2100}	$(1,0)$	3	2	$(0,0)$	0	0	—
h_{1200}	$(1,0)$	3	1	$(2,0)$	0	0	—
h_{0300}	$(3,0)$	3	0	—	0	0	—

TABLE IV Resonances and spectral lines due to the normal sextupole term xy^2

D.T.	*Resonance*	$j+k$	j	*Lines hor.*	$l+m$	l	*Lines ver.*
h_{1020}	$(1,2)$	1	1	$(0,-2)$	2	2	$(1,-1)$
h_{1011}	$(1,0)$	1	1	$(0,0)$	2	1	$(1,1)$
h_{1002}	$(1,-2)$	1	1	$(0,2)$	2	0	—
h_{0120}	$(1,-2)$	1	0	—	2	2	$(-1,-1)$
h_{0111}	$(1,0)$	1	0	—	2	1	$(-1,1)$
h_{0102}	$(1,2)$	1	0	—	2	0	—

and h_{1011}. The term h_{2100} is the complex conjugate of h_{1200} which leaves h_{1011} as the only unknown term. This term can however be calculated by considering that the $(0,0)$ horizontal spectral line has two complex contributions (HSL^*_{1200}, HSL_{1011}), the amplitude and phase of which are found in Table II. Therefore we get

$$HSL_{1011} = HSL_{\text{total}} - 2 \cdot HSL^*_{1200}, \tag{18}$$

note that the factor 2 is the index j of the h_{2100} in Table III.

The lines generated by the skew driving terms are not mixed with the previous ones and can therefore be calculated in an analogous way. The complete set of third order driving terms can therefore be found by using only horizontal spectral lines. The additional informations contained in the vertical lines can however be used to determine all Hamiltonian lines in the case that only one pair of position monitors is available.[17]

4.2 Fourth Order Contributions to the Line Spectra

The driving terms and the corresponding resonances excited by a normal octupole in the first order perturbative theory can be determined (see Tables V–VII) in the same way as described in the above Section 4.1. However, one first has to consider the quadratic sextupole contributions as described above.

The lines expected from the x^4 term are given in Table V while from the x^2y^2 term we get Table VI and lastly there are the y^4 terms in Table VII.

As in the case of the sextupoles, all Hamiltonian terms can be determined in a direct way or by combining lines.

TABLE V Resonances and spectral lines due to the normal octupole term x^4

D.T.	*Resonance*	$j+k$	j	*Lines hor.*	$l+m$	l	*Lines ver.*
h_{4000}	(4, 0)	4	4	(−3, 0)	0	0	—
h_{3100}	(2, 0)	4	3	(−1, 0)	0	0	—
h_{1300}	(2, 0)	4	1	(3, 0)	0	0	—
h_{0400}	(4, 0)	4	0	—	0	0	—

TABLE VI Resonances and spectral lines due to the normal octupole term x^2y^2

D.T.	*Resonance*	$j+k$	j	*Lines hor.*	$l+m$	l	*Lines ver.*
h_{2020}	(2, 2)	2	2	(−1, −2)	2	2	(2, −1)
h_{2011}	(2, 0)	2	2	(−1, 0)	2	1	(2, 1)
h_{2002}	(2, −2)	2	2	(−1, 2)	2	0	—
h_{1120}	(0, 2)	2	1	(1, 2)	2	2	(0, −1)
h_{1102}	(0, 2)	2	1	(1, −2)	2	0	—
h_{0220}	(2, −2)	2	0	—	2	2	(−2, −1)
h_{0211}	(2, 0)	2	0	—	2	1	(−2, 1)
h_{0202}	(2, 2)	2	0	—	2	0	—

TABLE VII Resonances and spectral lines due to the normal octupole term y^4

D.T.	*Resonance*	$j+k$	k	*Lines hor.*	$l+m$	l	*Lines ver.*
h_{0040}	(0, 4)	0	0	—	4	4	(0, −3)
h_{0031}	(0, 2)	0	0	—	4	3	(0, −1)
h_{0013}	(0, 2)	0	0	—	4	1	(0, 3)
h_{0004}	(0, 4)	0	0	—	4	0	—

The contributions to the spectral lines coming from the skew driving terms are completely separated from the contributions of the normal driving term given in the previous tables. Furthermore, we find that none of the horizontal spectral lines mix with the lines already generated by the third order terms.

4.3 Application to the LHC Tracking Data

We are now in the position to apply the method to the well studied LHC lattice version 4.[19] A set of 60 realisations of the random multipolar errors are included in the dipoles and quadrupoles. For each realisation we generated a set of 10^4 tracking data starting with a small initial amplitude of 1σ. In this region of phase space the amplitude dependence of the lines is to a very good approximation quadratic for the lines generated by third order terms and cubic for the lines generated by fourth order terms. We conclude therefore that higher order contributions are not relevant at 1σ. Moreover, knowing that the sextupole contributions, which are largest in the main dipoles, are quasi-locally corrected we have neglected those contributions to the octupole resonances that are quadratic in the strength of the sextupoles. In parallel we have calculated the maps and with DaLie the resonant Hamiltonian in resonance basis. Figure 1 shows that the

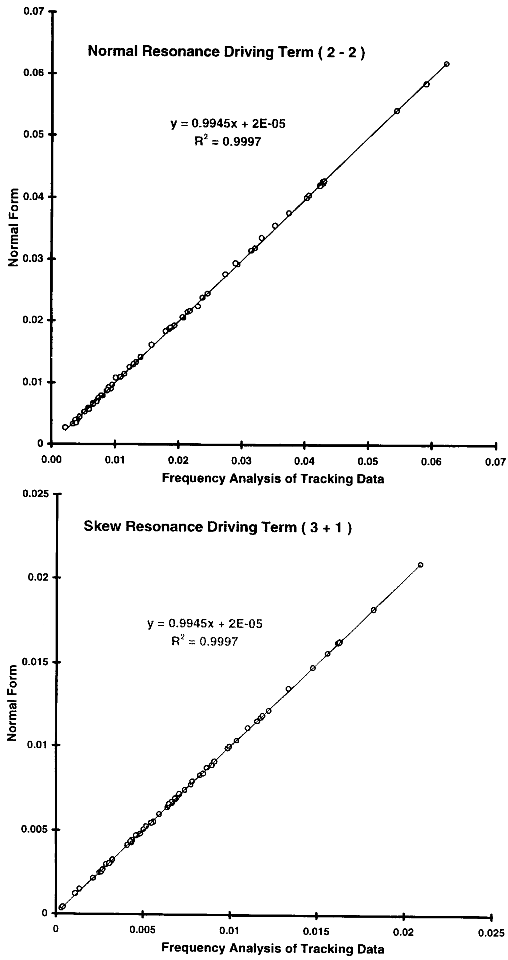

FIGURE 1 Hamiltonian term from Normal Form and from tracking data for 60 seeds of the LHC Lattice Version 4.

Hamiltonian terms of the regular resonance $(2, -2)$ and the skew resonance $(3, 1)$ can be predicted with excellent precision from the line spectra.

5 LIMITS OF THE METHOD

It is well known that the applied techniques have their limits:

- The algorithms used for the high precision frequency analysis rely on the hypothesis that the time series is a quasi-periodic signal of length T with frequencies separated by a distance larger than $1/T$. This implies that, in general, regular motion can always be treated by choosing large enough sample lengths. However, in the chaotic regime the method will break down since the above condition is not fulfilled. Of course in the chaotic regime Normal Form is also not applicable.
- Both Normal Form and spectral analysis break down in the vicinity of resonances even before reaching the chaotic regime.
- The first order approximation applied in this method can only be justified for small initial amplitudes or kick strengths. This is in conflict with obtaining sufficiently large signal strength.
- The length of the data samples of experimental data is given by the unavoidable decoherence of the oscillations of the kicked beam. Given the high precision of tune determination even with a very small number of turns this limitation may not be very serious.
- In presence of noise the high precision frequency analysis will lose some of its precision.[20] It remains to be seen if spectra can be derived from experimental data with high enough quality to predict the third and fourth order resonances.

6 CONCLUSIONS

We have shown that, in first approximation, the spectral lines derived with frequency map analysis from tracking data can be related to the terms of the generating function in resonance basis. An example for a complex accelerator structure shows that Hamiltonian terms can indeed be derived from tracking data with excellent precision and under realistic conditions. In principle the method is not limited in order and a complete Normal Form toolkit is available, so that tracking data can be easily transfered into a Hamiltonian or a map.

If the method works under realistic conditions for an existing accelerator it may become an important diagnostic tool to evaluate and compensate offending resonances. An experiment is therefore very desirable to test the applicability of this method.

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