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An SL(2, Z) Multiplet of Black Holes in $D = 4$ Type II Superstring Theory

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ABSTRACT

It is well-known that the conjectured SL(2, Z) invariance of type IIB string theory in ten dimensions also persists in lower dimensions when the theory is compactified on tori. By making use of this recent observation, we construct an infinite family of magnetically charged black hole solutions of type II superstring theory in four space-time dimensions. These solutions are characterized by two relatively prime integers corresponding to the magnetic charges associated with the two gauge fields (from NS-NS and R-R sectors) of the theory and form an SL(2, Z) multiplet. In the extremal limit these solutions are stable as they are prevented from decaying into black holes of lower masses by a 'mass gap' equation.

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Black hole solutions in string theory [1] provide a very interesting arena to address some of the long-standing issues involving thermodynamics of black holes, their evaporation and information loss paradox[2]. It is well-known that string theories admit a rich variety of static spherically symmetric as well as rotating black hole solutions in various dimensions. For example, spherically symmetric black hole solutions in string theory having purely magnetic, purely electric and both charges (dyonic black holes) have been constructed before [3–6]. Furthermore, rotating black hole solutions containing electric, magnetic and both charges have been discussed in refs.[7,8]. Since black holes have profound conceptual implications in our understanding of the nature of general relativity in the quantum domain and since string theory is believed to lead to a finite, consistent theory of quantum gravity, it is very important to construct various kinds of black hole solutions in string theory and study their properties. As black holes are intrinsically non-perturbative, it is in general difficult to study their properties in the perturbative framework of string theory. However, there has been a spectacular advancement in our understanding of the non-perturbative behavior of string theory in recent times. Subsequently, Strominger and Vafa [9] constructed a special class of black hole solutions in type II string theory in $D = 5$ and reproduced the Bekenstein-Hawking area entropy relation through a D-brane description [10] of such black holes and by counting the number of microstates in this framework. Black holes in that case saturate the BPS condition in the extremal limit and carry an electric as well as an axionic charge. Further developments along this line could be found in refs.[†][11,12].

Recently, we have shown [13,14] that the low energy effective action of type IIB string theory has a manifest SL(2, R) invariance in lower dimensions when compactified on tori as a consequence of the corresponding symmetry in ten dimensions [15–18]. This symmetry is non-perturbative as it transforms the string coupling constant in a non-trivial way. A discrete subgroup of this SL(2, R) group has been conjectured to be an exact symmetry of the quantum type IIB string theory. A strong evidence in favor of this conjecture has been given in ref.[19], when we showed that there exist SL(2, Z) multiplets of macroscopic string-like solutions in type II string theories in $D < 10$. The tensions and the charges of these BPS saturated string-like solutions have been shown to be given by SL(2, Z) covariant expressions.

In this paper, we will construct another class of black hole solution in $D = 4$ type II

[†]These contain only a partial list.

string theory. By making use of this SL(2, R) invariance of the lower dimensional type II string theory, we construct an SL(2, Z) multiplet of black hole solutions in $D = 4$. First, we construct a magnetically charged black hole solution, similar to the one obtained by Garfinkle, Horowitz and Strominger (GHS) [4]. This solution arises due to the presence of an Abelian gauge field in the NS-NS sector of the theory in four dimensions as a consequence of compactification of the corresponding antisymmetric tensor field in the ten dimensional action. Next, we implement the SL(2, R) transformations so that the resulting solution carries both NS-NS and R-R charges[†]. The two Abelian gauge fields correspond to the dimensionally reduced antisymmetric tensor fields coming from the NS-NS and R-R sectors of type II string theory. As the magnetic charges are quantized, the final solution will, therefore, be characterized by the two integers corresponding to the magnetic charges associated with the two gauge fields of NS-NS and R-R sectors. In the extremal case, we will show that both the charges and the masses of such black holes are given by SL(2, Z) covariant expressions. Since in the extremal limit the magnetically charged black holes are BPS saturated, the SL(2, Z) covariant results give a strong evidence in favor of the conjectured SL(2, Z) invariance of the quantum theory. We mention in passing that type II string theory in four dimensions has been conjectured to possess a much bigger non-compact global symmetry group $E_{7(7)}(Z)$ [20,16], known as the U-duality group [16], which contains both the S-duality [21] group SL(2, Z) and the T-duality [22] group O(6, 6; Z) as the subgroup. But we will restrict ourselves only to a part of this bigger symmetry group, namely, the S-duality group. Then we will show that these extremal black holes are stable when they are characterized by two relatively prime integers. In that case, as common to BPS saturated states, the masses of the black holes satisfy a triangle inequality which prevents the black holes to decay into black holes of lower masses.

Let us recapitulate how the four dimensional magnetically charged black hole solution of GHS [4] arises from four dimensional effective action.

The complete low energy four dimensional effective action of interest to us is

$$S = \int d^4x\sqrt{-G}e^{-2\phi}\Big(R + 4\partial_\mu\phi\partial^\mu\phi + \frac{1}{8}\mathrm{tr}\,\partial_\mu\mathbf{M}^{-1}\partial^\mu\mathbf{M} - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{4}\mathcal{F}^T_{\mu\nu}\mathbf{M}^{-1}\mathcal{F}^{\mu\nu}\Big) \qquad (1)$$

[†]In the recent discussions on the microscopic origin of the Bekenstein-Hawking area entropy relation [9,11,12], black holes with R-R charges have been considered and this is crucial to have D-brane description of the black holes.

where $G = (\det G_{\mu\nu})$, $G_{\mu\nu}$ being the four dimensional metric in the string frame, ϕ is the dilaton field in $D = 4$, R is the scalar curvature corresponding to the metric $G_{\mu\nu}$. This four dimensional action is of generic form which can be obtained through toroidal compactification on T^6 of a ten dimensional string effective action. For example, if we start from the ten dimensional heterotic string, the matrix $\mathbf{M}$ which contains the scalar fields, parametrizes the coset, $\frac{O(22,6)}{O(22)\times O(6)}$ and $\mathcal{F}_{\mu\nu}$ corresponds to 28 Abelian gauge field strengths [23]. On the other hand if we start from ten dimensional action of type II theories, then the reduced action (1) can be identified with the one that is obtained by dimensional reductions of the NS-NS sector and now there will be only 12 gauge fields (6 from the metric and 6 from antisymmetric tensor) and $\mathbf{M}$ will contain scalars parametrizing the coset $\frac{O(6,6)}{O(6)\times O(6)}$. The superscript '$T$' denotes the transpose of a matrix. Definitions of the field strengths are

$$\begin{aligned}
\mathcal{F}_{\mu\nu} &= \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu \\
H_{\mu\nu\rho} &= \partial_\mu B_{\nu\rho} + \mathcal{A}_\mu^T \eta \mathcal{F}_{\nu\rho} + \text{cyc. in } \mu\nu\rho
\end{aligned} \quad (2)$$

where $\mathcal{A}_\mu$ is a 28 dimensional vector field containing the 28 gauge fields coming from the dimensional reduction of the ten dimensional metric, antisymmetric tensor field and $U(1)^{16}$ gauge fields in the case of heterotic string.

In order to obtain magnetically charged black hole solution, we choose $\mathbf{M}$ to be constant and put $H_{\mu\nu\rho} = 0$ and set all the gauge fields except one (denoted as $A_\mu^{(1)}$) to zero, then the action (1) reduces in the Einstein frame to,

$$\bar{S} = \int d^4x \sqrt{-g} \left(R - 2\partial_\mu \phi \partial^\mu \phi - \frac{1}{4} e^{-2\phi} F_{\mu\nu}^{(1)} F^{(1)\,\mu\nu} \right) \quad (3)$$

where the Einstein metric is related to the string metric by $g_{\mu\nu} = e^{-2\phi} G_{\mu\nu}$. R now denotes the scalar curvature with respect to the Einstein metric $g_{\mu\nu}$. Note that this action is precisely the one considered by GHS [4]. In their case, the gauge field $A_\mu^{(1)}$ came from one of the $U(1)^{16}$ gauge fields in ten dimensions, whereas, we choose $A_\mu^{(1)}$ to come from the dimensional reduction of the ten dimensional antisymmetric tensor field.

Our conventions and notations are as follows: the signature of the tangent space Lorentz metric is $(-, +, +, \ldots)$. The covariant derivative, connection and the Riemann curvature tensor are:

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda{}_{\mu\nu} V_\lambda$$

$$
\begin{aligned}
\Gamma^{\mu}{}_{\nu\lambda} &= \frac{1}{2} g^{\mu\rho}\left(\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\nu\lambda}\right) \\
R^{\mu}{}_{\nu\lambda\rho} &= \partial_\lambda \Gamma^{\mu}{}_{\nu\rho} - \partial_\rho \Gamma^{\mu}{}_{\nu\lambda} + \Gamma^{\mu}{}_{\lambda\sigma}\Gamma^{\sigma}{}_{\nu\rho} - \Gamma^{\mu}{}_{\rho\sigma}\Gamma^{\sigma}{}_{\nu\lambda}
\end{aligned} \tag{4}
$$

where V_μ is any vector. The scalar curvature is given as $R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^{\rho}{}_{\mu\rho\nu}$.

The equations of motion derived from the action (3) are as given by

$$
\nabla_\mu \left(e^{-2\phi} F^{(1)\,\mu\nu}\right) = 0 \tag{5}
$$

$$
\nabla^2 \phi + \frac{1}{8} e^{-2\phi} F^{(1)}_{\mu\nu} F^{(1)\,\mu\nu} = 0 \tag{6}
$$

$$
R_{\mu\nu} = 2\partial_\mu \phi \partial_\nu \phi + \frac{1}{2} e^{-2\phi} F^{(1)}_{\mu\lambda} F^{(1)}_{\nu}{}^{\lambda} - \frac{1}{8} g_{\mu\nu} e^{-2\phi} F^{(1)}_{\lambda\rho} F^{(1)\,\lambda\rho} \tag{7}
$$

A static spherically symmetric black hole solution of these equations can be obtained from the following ansatz of the space-time metric

$$
\begin{aligned}
ds^2 &= -f^2 dt^2 + f^{-2} dr^2 + R^2 d\Omega \\
&= -f^2 dt^2 + f^{-2} dr^2 + R^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)
\end{aligned} \tag{8}
$$

and the Maxwell field

$$
F^{(1)}_{23} = Q \sin\theta \tag{9}
$$

We denote the coordinates t, r, θ and φ by 0, 1, 2 and 3 respectively. f, R and ϕ are functions of the radial coordinate r only. Asymptotically, as $r \to \infty$, $f \to 1$, $R \to r$ and $\phi \to \phi_0$, otherwise the functions are arbitrary. We also note from (9) that since $F^{(1)}_{23}$ is the only non-zero component of the Maxwell field, it is magnetic and Q is the corresponding charge defined as $Q \equiv \frac{1}{4\pi}\int F^{(1)}$. It can be easily checked from (8) and (9) that $F^{(1)}_{\mu\nu} F^{(1)\,\mu\nu} = 2Q^2/R^4$. Also, the only non-zero components of the connection we find from (4) and (8) are

$$
\begin{aligned}
\Gamma^{0}{}_{01} &= \frac{f'}{f} \\
\Gamma^{1}{}_{00} &= f^3 f', \quad \Gamma^{1}{}_{11} = -\frac{f'}{f}, \quad \Gamma^{1}{}_{22} = -f^2 R R', \quad \Gamma^{1}{}_{33} = -f^2 R R' \sin^2\theta \\
\Gamma^{2}{}_{33} &= -\sin\theta\cos\theta, \quad \Gamma^{2}{}_{12} = \frac{R'}{R} \\
\Gamma^{3}{}_{13} &= \frac{R'}{R} \quad \text{and} \quad \Gamma^{3}{}_{23} = \cot\theta
\end{aligned} \tag{10}
$$

Here ‘prime’ denotes the derivative with respect to the radial coordinate r. Note that Eq.(5) is automatically satisfied using above information and then from Eq.(7) we arrive at the relations,

$$R_{00} = \frac{f^2Q^2}{4R^4}e^{-2\phi} = -f^2\nabla^2\phi \tag{11}$$

$$R_{22} = \frac{Q^2}{4R^2}e^{-2\phi} = -R^2\nabla^2\phi \tag{12}$$

where we have made use of Eq.(6) to write the last expressions in (11) and (12). By comparing (11) and (12) we have,

$$R_{00} = \frac{f^2}{R^2}R_{22} \tag{13}$$

Furthermore, R_{00} and R_{22} can also be expressed in terms of the functions f and R, appearing in the metric (8), through the definition of Ricci tensor (4) as follows,

$$R_{00} = f^2\left(ff'' + (f')^2 + 2ff'\frac{R'}{R}\right) \tag{14}$$

$$R_{22} = 1 - \left(f^2RR'\right)' \tag{15}$$

where we have made use of (10). Substituting (14) and (15) in (13), we obtain an equation involving the two unknown functions f and R as,

$$\left(f^2R^2\right)'' = 2 \tag{16}$$

Now using (11), (14) and the expression for $\nabla^2\phi = f^2\phi'' + 2ff'\phi' + 2f^2\frac{R'}{R}\phi'$, we obtain another equation involving f, R and the dilaton ϕ as,

$$\left(\frac{f^2R^2X'}{X}\right)' = 0 \tag{17}$$

where, $X \equiv f^2e^{2\phi}$. Eqs.(16) and (17) can now be easily solved if we impose the asymptotic limit as $r \to \infty$, $f \to 1$ and $R \to r$. The solution is:

$$\begin{aligned} f^2(r) &= \left(1 + \frac{a}{r}\right) \\ R^2(r) &= r^2\left(1 + \frac{b}{r}\right) \\ e^{-2\phi} &= e^{-2\phi_0}\left(1 + \frac{b}{r}\right) \end{aligned} \tag{18}$$

with $ab = \frac{1}{2}Q^2e^{-2\phi_0}$. Here a, b are integration constants and to find ab, we have used Eq.(6). The integration constant 'a' can be identified from the weak field limit† as $-2M$,

†We have set the Newton's constant $G = 1$.

where M is the mass of the black hole and therefore $b = -\frac{Q^2}{4M}e^{-2\phi_0}$, Q being the magnetic charge of the black hole defined earlier. The background field configurations, therefore, take the following form:

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(1-\frac{Q^2}{4Mr}e^{-2\phi_0}\right)d\Omega && (19)\\
e^{-2\phi} &= e^{-2\phi_0}\left(1-\frac{Q^2}{4Mr}e^{-2\phi_0}\right) = e^{-2\phi_0}\frac{R^2}{r^2} = e^{-2\phi_0}\frac{(1-f^2)^2R^2}{4M^2} && (20)\\
F^{(1)}_{23} &= Q\sin\theta && (21)
\end{aligned}
$$

The solution may be written in a more symmetric fashion by introducing the dilaton charge $\mathcal{D} \equiv \frac{1}{4\pi}\int d^2\Sigma^\mu \nabla_\mu \phi = \frac{R^2(r)}{4\pi}\int d\Omega \phi' = -\frac{Q^2}{8M}e^{-2\phi_0}$ as,

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(1-\frac{2|\mathcal{D}|}{r}\right)d\Omega\\
e^{-2\phi} &= e^{-2\phi_0}\left(1-\frac{2|\mathcal{D}|}{r}\right) && (22)
\end{aligned}
$$

Now with the coordinate transformation of the form:

$$
\begin{aligned}
\rho^2 &= r^2\left(1-\frac{2|\mathcal{D}|}{r}\right), \quad \text{for} \quad r \geq 2|\mathcal{D}|\\
&= -r^2\left(1-\frac{2|\mathcal{D}|}{r}\right), \quad \text{for} \quad r \leq 2|\mathcal{D}| && (23)
\end{aligned}
$$

which implies, $r = |\mathcal{D}| + \sqrt{\mathcal{D}^2+\rho^2}$, for $r \geq 2|\mathcal{D}|$, $r = |\mathcal{D}| + \sqrt{\mathcal{D}^2-\rho^2}$, for $|\mathcal{D}| \leq r \leq 2|\mathcal{D}|$ and $r = |\mathcal{D}| - \sqrt{\mathcal{D}^2-\rho^2}$, for $r \leq |\mathcal{D}|$, it can be easily checked that (22) represents a black hole with an event horizon located at

$$\rho = 2\left[M\left(M-|\mathcal{D}|\right)\right]^{\frac{1}{2}} \tag{24}$$

for $M > |\mathcal{D}|$. However, for $M < |\mathcal{D}|$, there is no event horizon and consequently the space-time singularity at $r = 0$ is directly observable representing a "naked" singularity. Thus the transition between the black hole and the "naked" singularity occurs at $M = |\mathcal{D}| = \frac{Q^2}{8M}e^{-2\phi_0}$ or $Q^2 = 8M^2e^{2\phi_0}$. The transition point is known as the extremal limit. Note from (22) that at the extremal limit the area of the event horizon vanishes causing the surface to be singular. Thus the black hole solution of GHS with magnetic charge was obtained for a special background configuration of four dimensional heterotic string theory and the one we presented exactly coincides with GHS; however, the four dimensional action is the NS-NS sector of type II theory as remarked earlier. Now, we proceed to discuss compactification of type IIB theory to four dimensions with relevant massless fields in both NS-NS and R-R sector.

Let us recall that the massless spectrum of the type IIB string theory in the bosonic sector contains a graviton, a dilaton and an antisymmetric tensor field as NS-NS sector states, whereas, in the R-R sector it contains another scalar, another antisymmetric tensor field and a four-form gauge field whose field-strength is self-dual. It is well known that a covariant action for self dual five index antisymmetric tensor fields in ten dimensions does not exist [24] and we set this field strength to zero, since this field is of no relevance to us in what follows. Therefore, a consistent, covariant action can be written [17] from which the type IIB supergravity equations of motion can be derived. We have studied the dimensional reduction of this action on a $(10-D)$ dimensional torus in ref.[13,14]. When $D=4$, the corresponding four dimensional type IIB string effective action in the Einstein frame takes the following form:

$$\begin{aligned}\bar{S}_{\rm II} \;=\; \int d^4x\sqrt{-g}\Big[R &+ \frac{1}{4}{\rm tr}\,\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1} + \frac{1}{8}\partial_\mu\log\bar{\Delta}\partial^\mu\log\bar{\Delta} + \frac{1}{4}\partial_\mu g_{mn}\partial^\mu g^{mn} \\ &-\frac{1}{4}g_{mn}F^{(3)\,m}_{\mu\nu}F^{(3)\,\mu\nu,\,n} - \frac{1}{4}(\bar{\Delta})^{1/2}g^{mp}g^{nq}\partial_\mu\mathcal{B}^T_{mn}\mathcal{M}\partial^\mu\mathcal{B}_{pq} \\ &-\frac{1}{4}(\bar{\Delta})^{1/2}g^{mp}\mathcal{H}^T_{\mu\nu\,m}\mathcal{M}\mathcal{H}^{\mu\nu}{}_{p} - \frac{1}{12}(\bar{\Delta})^{1/2}\mathcal{H}^T_{\mu\nu\lambda}\mathcal{M}\mathcal{H}^{\mu\nu\lambda}\Big]\end{aligned} \tag{25}$$

Here $g = (\det g_{\mu\nu})$, where $g_{\mu\nu}$ is the four dimensional Einstein metric and R is the scalar curvature associated with $g_{\mu\nu}$. $\mathcal{M}$ is an SL(2, R) matrix defined as

$$\mathcal{M} \equiv \begin{pmatrix} \chi^2 + e^{-2\tilde{\phi}} & \chi \\ \chi & 1 \end{pmatrix} e^{\tilde{\phi}} \tag{26}$$

where χ is the R-R scalar and $\tilde{\phi} = \phi + \frac{1}{2}\log\Delta$, ϕ being the NS-NS scalar, the four dimensional dilaton and $\Delta^2 = (\det G_{mn})$, G_{mn} being the scalars coming from the dimensional reduction of the ten dimensional string metric. $g_{mn} = e^{-2\phi}G_{mn}$ and $(\bar{\Delta})^2 = (\det g_{mn})$. $F^{(3)\,m}_{\mu\nu} = \partial_\mu A^{(3)\,m}_\nu - \partial_\nu A^{(3)\,m}_\mu$, where $A^{(3)\,m}_\mu$ is the gauge field resulting from the dimensional reduction of the string metric. $\mathcal{B}_{mn} \equiv \begin{pmatrix} B^{(1)}_{mn} \\ B^{(2)}_{mn}\end{pmatrix}$, where $B^{(i)}_{mn}$, for $i=1,2$ are the moduli coming from the dimensional reduction of the NS-NS and R-R antisymmetric tensor fields. $\mathcal{H}_{\mu\nu\,m} \equiv \begin{pmatrix} H^{(1)}_{\mu\nu\,m} \\ H^{(2)}_{\mu\nu\,m}\end{pmatrix}$, where $H^{(i)}_{\mu\nu\,m} = F^{(i)}_{\mu\nu\,m} - B^{(i)}_{mn}F^{(3)\,n}_{\mu\nu}$ and $F^{(i)}_{\mu\nu\,m} = \partial_\mu A^{(i)}_{\nu\,m} - \partial_\nu A^{(i)}_{\mu\,m}$, with $A^{(i)}_{\mu\,m}$ being the gauge fields resulting from the dimensional reduction of the NS-NS and R-R sector antisymmetric tensor fields. Finally, $\mathcal{H}_{\mu\nu\lambda} \equiv \begin{pmatrix} H^{(1)}_{\mu\nu\lambda} \\ H^{(2)}_{\mu\nu\lambda}\end{pmatrix}$, $H^{(i)}_{\mu\nu\lambda} = \Big(\partial_\mu B^{(i)}_{\nu\lambda} - F^{(3)\,m}_{\mu\nu}A^{(i)}_{\lambda\,m} + \text{cyc. in } \mu\nu\lambda\Big)$. The action (25) can be easily seen to be invariant under the following global SL(2, R) transformation [13,14]:

$$\begin{array}{rcl} \mathcal{M} & \rightarrow & \Lambda \mathcal{M} \Lambda^T, \qquad \mathcal{B}_{mn} \ \rightarrow \ (\Lambda^{-1})^T \mathcal{B}_{mn} \\ \left(\begin{array}{c} A^{(1)}_{\mu\, m} \\ A^{(2)}_{\mu\, m} \end{array} \right) \equiv \mathcal{A}_{\mu\, m} & \rightarrow & (\Lambda^{-1})^T \mathcal{A}_{\mu\, m}, \qquad \left(\begin{array}{c} B^{(1)}_{\mu\nu} \\ B^{(2)}_{\mu\nu} \end{array} \right) \equiv \mathcal{B}_{\mu\nu} \ \rightarrow \ (\Lambda^{-1})^T \mathcal{B}_{\mu\nu} \\ g_{\mu\nu} & \rightarrow & g_{\mu\nu}, \qquad g_{mn} \ \rightarrow \ g_{mn}, \qquad \text{and} \quad A^{(3)\, m}_{\mu} \ \rightarrow \ A^{(3)\, m}_{\mu} \end{array} \tag{27}$$

where $\Lambda = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right)$, is the SL(2, R) transformation matrix and a, b, c and d are constants satisfying $ad - bc = 1$.

We shall consider a truncated action, rather than the full action (25). Let us, from now on, set $H^{(i)}_{\mu\nu\lambda} = 0$, $A^{(3)\, m}_{\mu} = 0$, $G_{mn} = \delta_{mn}$, $\Delta = 1$, $B^{(i)}_{mn} = 0$ and all the components of $A^{(1)}_{\mu\, m}$ and $A^{(2)}_{\mu\, m}$ to zero except one (we call the non-zero components of the gauge fields as $A^{(1)}_{\mu}$ and $A^{(2)}_{\mu}$ with the corresponding field-strength $F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu$), then the action (25) reduces to:

$$\begin{array}{l} \int d^4x \sqrt{-g} \Big[R + \frac{1}{4} \mathrm{tr}\, \partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1} + \frac{1}{8} \partial_\mu \log \bar{\Delta} \partial^\mu \log \bar{\Delta} \\ \qquad + \frac{1}{4} \partial_\mu g_{mn} \partial^\mu g^{mn} - \frac{1}{4} (\bar{\Delta})^{\frac{1}{6}} \mathcal{F}^T_{\mu\nu} \mathcal{M} \mathcal{F}^{\mu\nu} \Big] \end{array} \tag{28}$$

Here $\mathcal{M}$ is as given in (26) with $\tilde{\phi}$ replaced by ϕ since we have set $\Delta = 1$. $\mathcal{F}_{\mu\nu} \equiv \left(\begin{array}{c} F^{(1)}_{\mu\nu} \\ F^{(2)}_{\mu\nu} \end{array} \right)$. The action (28) is invariant under the global SL(2, R) transformation:

$$\begin{array}{l} \mathcal{M} \ \rightarrow \ \Lambda \mathcal{M} \Lambda^T, \qquad \left(\begin{array}{c} A^{(1)}_{\mu} \\ A^{(2)}_{\mu} \end{array} \right) \equiv \mathcal{A}_\mu \ \rightarrow \ (\Lambda^{-1})^T \mathcal{A}_\mu \\ g_{\mu\nu} \ \rightarrow \ g_{\mu\nu} \qquad \text{and} \qquad g_{mn} \ \rightarrow \ g_{mn} \end{array} \tag{29}$$

Note that type IIB string effective action (28) reduces precisely to the action (3) considered by GHS, when the R-R fields are set to zero. We would like to exploit this SL(2, R) invariance to rotate the magnetically charged black hole solution of GHS to a more general black hole solution of type II string theory in $D = 4$. In order to describe the complete black hole solution, we have to specify the asymptotic values of the dilaton ϕ and R-R scalar χ. Under the transformation (29), the complex scalar field $\lambda = \chi + i e^{-\phi}$ and the gauge field $A^{(i)}_\mu$ transform as follows,

$$\lambda \ \rightarrow \ \frac{a\lambda + b}{c\lambda + d} \tag{30}$$

$$\begin{array}{rcl} A^{(1)}_3 & \rightarrow & d A^{(1)}_3 - c A^{(2)}_3 \\ A^{(2)}_3 & \rightarrow & -b A^{(1)}_3 + a A^{(2)}_3 \end{array} \tag{31}$$

Note from (9) that the only non-zero component of the field-strength $F^{(1)}_{\mu\nu}$ is $F^{(1)}_{23} = Q \sin\theta$ and so, upto the gauge transformation the only non-zero component of the gauge field $A^{(1)}_{\mu}$ is $A^{(1)}_3 = -Q\cos\theta$. Let us first construct the black hole solution for the simplest choice of $\lambda_0 = i$ (i.e. for $\phi_0 = \chi_0 = 0$), where the subscript 'zero' represent the asymptotic value of scalars in $\mathcal{M}$ and this black hole carries only NS-NS charge. Here Q can be argued to be quantized in some basic units. Although the actions (25) and (28) are invariant under SL(2,R) transformations, in the quantized theory, the remnant, robust symmetry is expected to be SL(2,Z) and elements of Λ are integers satisfying the constraint $\det \Lambda = 1$. Thus starting from a black hole with a given Q and $\lambda_0 = i$, we can obtain a black hole which carries both type of charges. The relevant transformation matrix has the following form:

$$\Lambda = \frac{1}{\sqrt{q_1^2 + q_2^2}} \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix} \tag{32}$$

where q_1 and q_2 are two integers and still $\lambda_0 = i$. Note here that we have used the fact that the charges $\begin{pmatrix} Q^{(1)} \\ Q^{(2)} \end{pmatrix}$ associated with the gauge fields $A^{(1)}_\mu$ and $A^{(2)}_\mu$ transform as $\Lambda \begin{pmatrix} Q^{(1)} \\ Q^{(2)} \end{pmatrix}$ eventhough the gauge fields themselves transform as given in (29). This is in contrary to the usual Maxwell theory where both the gauge fields and the charges should transform in the same way. This difference can be understood by looking at the gauge field kinetic term in our action (28). The equation of motion in this case has the form

$$\nabla_\mu \left(\mathcal{M}\mathcal{F}^{\mu\nu}\right) = \mathcal{J}^\nu \tag{33}$$

It is clear from (33) that the charges would transform contragradiently with respect to the gauge fields under the SL(2, Z) transformation [18] as happened in our case. Once we have derived the form of Λ, we can easily calculate the gauge field components and the complex scalar from (31) and (30) as follows:

$$A^{(i)}_3 = -q_i Q\cos\theta \tag{34}$$

$$\begin{aligned} \lambda &= \frac{i q_1 R\left(1-f^2\right) - 2q_2 M}{i q_2 R\left(1-f^2\right) + 2q_1 M} \\ &= \frac{q_1 q_2 \left[R^2\left(1-f^2\right)^2 - 4M^2\right] + 2iMR\left(1-f^2\right)\left(q_1^2+q_2^2\right)}{4q_1^2M^2 + q_2^2R^2\left(1-f^2\right)^2} \end{aligned} \tag{35}$$

We note from (35) that asymptotically $R\left(1-f^2\right) \to 2M$ and therfore $\lambda \to i$ as $r \to \infty$.

Let us now generalize our construction for an arbitrary vacuum modulus λ_0. In this case we replace the charge Q with an arbitrary value $\alpha_{(q_1,q_2)} = \Delta^{1/2}_{(q_1,q_2)} Q$. $\Delta^{1/2}_{(q_1,q_2)}$ will be determined later. We take the SL(2, R) transformation matrix as

$$\begin{aligned}\Lambda &= \Lambda_1\Lambda_2 = \begin{pmatrix} e^{-\phi_0/2} & \chi_0 e^{\phi_0/2} \\ 0 & e^{\phi_0/2}\end{pmatrix}\begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix} \\ &= \begin{pmatrix} e^{-\phi_0}\cos\alpha + \chi_0\sin\alpha & -e^{-\phi_0}\sin\alpha + \chi_0\cos\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix} e^{\phi_0/2} \end{aligned} \tag{36}$$

Here Λ_2 is the most general SL(2, R) matrix which preserves the vacuum modulus $\lambda_0 = i$ and Λ_1 is the SL(2, R) matrix which transforms it to an arbitrary value $\lambda = \lambda_0$. α is an arbitrary parameter which will be fixed from the charge quantization condition. Now since we have Λ, we can find the magnetic charges associated with the gauge fields $A_3^{(1)}$ and $A_3^{(2)}$ as,

$$\begin{aligned} Q^{(1)} &= \left(e^{-\phi_0/2}\cos\alpha + \chi_0 e^{\phi_0/2}\sin\alpha\right)\Delta^{1/2}_{(q_1,q_2)} Q \\ Q^{(2)} &= e^{\phi_0/2}\sin\alpha\Delta^{1/2}_{(q_1,q_2)} Q \end{aligned} \tag{37}$$

Using the charge quantization condition, we get from (37),

$$\begin{aligned} \sin\alpha &= e^{-\phi_0/2}\Delta^{-1/2}_{(q_1,q_2)} q_2 \\ \cos\alpha &= e^{\phi_0/2}\left(q_1 - q_2\chi_0\right)\Delta^{-1/2}_{(q_1,q_2)} \end{aligned} \tag{38}$$

where q_1, q_2 are integers. $\Delta_{(q_1,q_2)}$ can be evaluated if we use $\sin^2\alpha + \cos^2\alpha = 1$, as follows,

$$\begin{aligned} \Delta_{(q_1,q_2)} &= e^{-\phi_0} q_2^2 + \left(q_1 - q_2\chi_0\right)^2 e^{\phi_0} \\ &= (q_1, q_2)\mathcal{M}_0^{-1}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \end{aligned} \tag{39}$$

where $\mathcal{M}_0 = \begin{pmatrix} \chi_0^2 + e^{-2\phi_0} & \chi_0 \\ \chi_0 & 1 \end{pmatrix} e^{\phi_0}$. We note that the expression for $\Delta_{(q_1,q_2)}$ is SL(2, Z) invariant and therfore the charges of the black holes $\alpha_{(q_1,q_2)} = \Delta^{1/2}_{(q_1,q_2)} Q$ are also given by SL(2, Z) covariant expressions. With the Λ in (36), we write below the transformed gauge fields,

$$\begin{aligned} A_3^{(1)} &= -e^{-\phi_0}\left(q_1 - q_2\chi_0\right) Q\cos\theta \\ A_3^{(2)} &= -e^{-\phi_0}\left(q_2|\lambda|^2 - q_1\chi_0\right) Q\cos\theta \end{aligned} \tag{40}$$

which can be written compactly as,

$$\begin{pmatrix} A_3^{(1)} \\ A_3^{(2)} \end{pmatrix} = -\mathcal{M}_0^{-1}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} Q\cos\theta \tag{41}$$

Also, the value of the transformed complex scalar field is:

$$\begin{aligned} \lambda &= \frac{iq_1 e^{-\phi_0} R\left(1-f^2\right) + 2M\left(q_1\chi_0 - q_2|\lambda_0|^2\right)}{iq_2 e^{-\phi_0} R\left(1-f^2\right) + 2M\left(q_1 - q_2\chi_0\right)} \\ &= \frac{4M^2 e^{-\phi_0}\chi_0 \Delta_{(q_1,q_2)} + q_1 q_2 e^{-2\phi_0}\left(R^2\left(1-f^2\right)^2 - 4M^2\right) + i2MR\left(1-f^2\right)e^{-2\phi_0}\Delta_{(q_1,q_2)}}{4M^2\left(q_1 - q_2\chi_0\right)^2 + q_2^2 e^{-2\phi_0} R^2\left(1-f^2\right)^2} \end{aligned} \tag{42}$$

It can be checked that asymptotically as $r \to \infty$, $R(1-f^2) \to 2M$ and so, $\lambda \to \lambda_0$. From (42) we obtain

$$e^{-2\phi} = \frac{4M^2 R^2\left(1-f^2\right)^2 e^{-4\phi_0}\Delta^2_{(q_1,q_2)}}{\left[4M^2\left(q_1 - q_2\chi_0\right)^2 + q_2^2 e^{-2\phi_0} R^2\left(1-f^2\right)^2\right]^2} \tag{43}$$

So, starting from the magnetically charged black hole solution of GHS, we have constructed an SL(2, Z) multiplet of magnetically charged black hole solution of type II string theory given by the field configurations (19) and (40–43). Note that in (19) Q is now replaced by $\Delta^{1/2}_{(q_1,q_2)}Q$. The black hole solutions in this case are characterized by two integers corresponding to the magnetic charges associated with the two gauge fields of the NS-NS and R-R sectors. Since the canonical metric does not transform under the SL(2, R) transformation the transition between the black hole and the "naked" singularity occurs at the same point ($Q^2 = 8M^2 e^{-2\phi_0}$) as discussed earlier. However, we can no longer express the metric nicely in terms of the new dilaton charge as was done in Eq.(22). Therefore, we note that in the extremal case, mass of the black hole should also be given by $M_{(q_1,q_2)} = \Delta^{1/2}_{(q_1,q_2)}M$ and the black holes will be BPS saturated.

Since

$$M_{(q_1,q_2)} = \sqrt{e^{-\phi_0}q_2^2 + \left(q_1 - q_2\chi_0\right)^2 e^{\phi_0}}\,M \tag{44}$$

masses satisfy the following relation when $\chi = 0$,

$$\begin{aligned} &\left(M_{(q_1,q_2)} + M_{(p_1,p_2)}\right)^2 - M^2_{(q_1+p_1,q_2+p_2)} \\ = \;& 2M^2\left\{\left[\left(p_1q_1e^{\phi_0} + p_2q_2e^{-\phi_0}\right)^2 + \left(p_1q_2 - p_2q_1\right)^2\right]^{1/2} - \left(p_1q_1e^{\phi_0} + p_2q_2e^{-\phi_0}\right)\right\} \\ \geq \;& 0 \end{aligned} \tag{45}$$

As M is a real, positive number (45) gives a triangle inequality among the masses which we write below:

$$M_{(q_1,q_2)} + M_{(p_1,p_2)} \geq M_{(q_1+p_1,q_2+p_2)} \tag{46}$$

The equality holds when $p_1q_2 = p_2q_1$ i.e. when $p_1 = nq_1$ and $p_2 = nq_2$, with n being an integer. Therefore, when q_1, q_2 are relatively prime integers, the inequality prevents the black holes to decay into black holes of lower masses. So, because of the 'mass gap' relation (46), the extremal black holes are stable. Furthermore, note that since charges also satisfy the same relation (44) like the masses, when q_1 and q_2 are relatively prime the charge conservation can not be satisfied if the black holes decay [25]. Finally, we note from (19) that even in the case of type II black hole solution the area of the event horizon vanishes in the extremal limit, like what happened for the heterotic string case of GHS, causing the surface to be singular.

To summarize, we first argued that the magnetically charged black hole solution GHS derived in the context of $D = 4$ heterotic string theory can also be interpreted as black hole solution of $D = 4$ type IIB theory such that the gauge field appears due to compactification of the NS-NS antisymmetric field (of $D = 10$ action) with all R-R fields set to zero. It has been demonstrated that the low energy effective action of type IIB string theory compactified on torus possesses an SL(2, Z) invariance if the $D = 10$ theory is endowed with the same symmetry. By exploiting this symmetry of type IIB string theory, we have constructed an infinite family of magnetically charged black hole solutions in $D = 4$. Black hole solutions in string theory having electric, magnetic and both charges associated with the gauge fields originating from the dimensional reduction of the various heterotic string states as well as the NS-NS sector states of type II string theory have been constructed before. The solutions we have constructed in this paper are characterized by two integers corresponding to the charges associated with both NS-NS sector and R-R sector gauge fields. We have shown that in the extremal limit, when these two integers (q_1, q_2) are relatively prime, the black holes are stable as they are prevented from decaying due to the inequality (46). In this context, we are tempted to interprete the BPS saturated (q_1, q_2) black holes as marginal bound states of q_1 NS-NS black holes and q_2 R-R black holes analogous to the arguments due to Witten [26] for the string solutions of Schwarz [18]. The class of black hole solutions obtained here have zero area of event horizon. It is well known that in order to construct $D = 4$ extremal black holes, with nonzero area, one must have four nonzero charges (corresponding to the number of 1-D-branes, the number of 5-D-branes, the number of solitonic 5-branes or Kaluza-Klein monopoles and the Kaluza-Klein charges) as has been discussed by several authors [12] in the context of intersecting D-brane approach [27]. In our case, the solutions are characterized by only two charges. However, the computation of entropy for the type of black holes, presented here, can be carried out by using the concept of "stretched horizon" [28]. It will also be interesting to construct other classes of black hole solutions in type IIB string theory by implementing the electric-magnetic duality transformations, for example, the dyonic black holes for which the area is known not to vanish.

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