

# A CLASSIFYING ALGEBRA FOR BOUNDARY CONDITIONS

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## Abstract

We introduce a finite-dimensional algebra that controls the possible boundary conditions of a conformal field theory. For theories that are obtained by modding out a  $\mathbb{Z}_2$  symmetry (corresponding to a so-called  $D_{\text{odd}}$ -type, or half-integer spin simple current, modular invariant), this classifying algebra contains the fusion algebra of the untwisted sector as a subalgebra. Proper treatment of fields in the twisted sector, so-called fixed points, leads to structures that are intriguingly close to the ones implied by modular invariance for conformal field theories on closed orientable surfaces.

CERN-TH/97-215  
August 1997

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# 1 Introduction

Recently, theories of open strings and conformal field theories on surfaces with boundaries have received considerable interest. As exemplified by the rôle played by D-branes in the description of non-perturbative aspects of string theory, it is a crucial task to obtain more insight into the possible boundary conditions for such theories. So far, however, most investigations have been limited to models based on free field theories or on orbifolds of such theories. In this paper we investigate the structure of boundary conditions in a general conformal field theory, including non-trivial modular invariants.

A conformal field theory typically admits several modular invariants. One always has the charge conjugation and the diagonal modular invariant. The possible boundary conditions in a theory with charge conjugation modular invariant have been explored in [1]. A first investigation of non-trivial modular invariants has been undertaken in [2,3] for WZW models based on  $\mathfrak{su}(2)$ . The goal of this letter is to extend this work to arbitrary rational conformal field theories with a specific type of non-diagonal modular invariant.

The type of modular invariant we will focus on generalizes the modular invariant of  $D_{\text{odd}}$ -type in the  $A$ - $D$ - $E$  classification of  $\mathfrak{su}(2)$  modular invariants, see equations (1) and (4) below. Quite generally, every non-trivial modular invariant can be obtained as follows: one first extends the chiral algebra and then superposes an automorphism of the fusion rules. The invariants (1) and (4) provide examples of non-trivial fusion rule automorphisms. The extension procedure is by now fairly well understood, at least in the case of extensions by so-called simple currents, and can be described entirely in terms of a chiral half of the theory. As a consequence, such extensions do not raise any problems in the construction of open string theories that were not already encountered for closed strings. In contrast, the implementation of fusion rule automorphisms is far from being understood. Our results provide new insight into this problem.

The further layout of this letter is as follows. In section 3 we discuss the reflection coefficients which characterize the various consistent boundary conditions and conclude that the construction of an open string theory requires a certain relation between the numbers of primary fields of various types. This non-trivial relation, given in (7) below, is established in section 4. These results enable us to introduce in section 5 a new finite-dimensional algebra, which in section 6 is shown to control the boundary conditions. In the same section we also study the implications for the annulus amplitude. In the last section we point out possible further consequences of our results.

## 2 The modular invariant

We analyze conformal field theories that are characterized by modular invariant combinations of characters that generalize the non-diagonal modular invariants

$$Z = \sum_{l=0}^{k/2} |\chi_{2l}|^2 + \sum_{l=0}^{k/2-1} \chi_{2l+1} \chi_{k-2l-1}^* \quad (1)$$

of the  $\mathfrak{su}(2)$  WZW theory, which exist at all levels  $k$  with  $k = 2 \pmod{4}$ . Here we have labelled the primary fields by their highest weight  $\Lambda$ , i.e. by twice their isospin. Notice that the primary fields come in two groups: when the  $\mathfrak{su}(2)$  representations carried by a field have integral isospin, then the representation is paired with itself, while a representation with half-integral isospin  $l$  gets paired with the representation of isospin  $k/2 - l$ . Note that the transition from  $l$  to  $k/2 - l$  corresponds to taking the fusion product with the primary field of highest weight  $k$ , which is a so-called *simple current*:  $\phi_k * \phi_\Lambda = \phi_{k-\Lambda}$ .

One can think of the modular invariant (1) as being obtained from the diagonal modular invariant by modding out a  $\mathbb{Z}_2$  symmetry; the first sum in (1) then constitutes the partition function of the untwisted sector, while the second sum is the contribution from the twisted sector. Note that the partition function of the twisted sector contains one term which superficially looks like an untwisted term, namely  $\chi_{k/2} \chi_{k/2}^*$ . The corresponding primary field has the property that it equals its fusion product with the simple current,  $\phi_k * \phi_{k/2} = \phi_{k/2}$ ; it is therefore termed a *fixed point*. In short, the primary fields in our example can be organized in *three* different types: we have  $N_0 = k/2 + 1$  left-right symmetric integer isospin fields,  $N_1 = k/2 - 1$  left-right asymmetric half-integer isospin fields that are not fixed points, and  $N_f = 1$  fixed point.

The situation summarized above has the following generalization. We consider a rational conformal field theory which contains a simple current, i.e. a primary field  $J$  whose fusion product with every primary field<sup>1</sup>  $\phi_\Lambda$  contains just a single primary field, which we denote by  $\phi_{J\Lambda} = J * \phi_\Lambda$ . Furthermore, we assume that the simple current  $J$  has order 2, i.e. satisfies  $J * J = \phi_0$ , and that its conformal weight is half-integral,  $\Delta_J \in \mathbb{Z} + 1/2$ . To each primary field  $\phi_\Lambda$  one associates its *monodromy charge*  $Q(\Lambda)$  with respect to  $J$ , which is the combination

$$Q(\Lambda) := \Delta_J + \Delta_\Lambda - \Delta_{J\Lambda} \pmod{\mathbb{Z}} \quad (2)$$

of conformal weights; the monodromy charge  $Q$  generalizes the conjugacy class – integral or half-integral isospin – of the  $\mathfrak{su}(2)$  example (1). An important property of  $Q$  is that it is conserved under operator products. It also appears in the relation [4]

$$S_{J\lambda, \mu} = e^{2\pi i Q(\mu)} S_{\lambda, \mu} \quad (3)$$

for the modular transformation matrix  $S$ .

It is known [4] that in the situation at hand the following non-diagonal combination of characters is modular invariant:

$$Z = \sum_{\Lambda: Q(\Lambda)=0} \chi_\Lambda \chi_{\Lambda^+}^* + \sum_{\Lambda: Q(\Lambda)=1/2} \chi_\Lambda \chi_{(J\Lambda)^+}^* \quad (4)$$

Here  $\lambda^+$  denotes the label of the field  $\phi_{\lambda^+} \equiv (\phi_\lambda)^+$  that is the charge conjugate of  $\phi_\lambda$ .<sup>2</sup>

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<sup>1</sup> The primary fields are now characterized by suitable labels  $\Lambda$ ; in the special case of WZW theories these correspond to integrable highest weights of the underlying affine Lie algebra. Also, we reserve the label 0 to stand for the identity (vacuum) primary field,  $\phi_0 \equiv \mathbf{1}$ .

<sup>2</sup> Let us remark that usually instead of (4) one considers the combination of characters where the charge conjugation is absent, which is modular invariant as well. (This type of modular invariant arises naturally in the conformal field theory description of type IIA compactifications of the superstring.) As will become clear below, in the open string context it is more natural to include the charge conjugation as in (4).

It follows from (2) that the primary fields  $\phi_\lambda$  which obey  $J * \phi_\lambda = \phi_\lambda$ , i.e. the fixed points, all have monodromy charge  $Q(\lambda) = \Delta_J \bmod \mathbb{Z} = 1/2$ . Also, we can again organize the primary fields in three different sets, the  $Q = 0$  fields, the  $Q = 1/2$  fields that are not fixed points, and the fixed points, with  $N_0$ ,  $N_1$  and  $N_f$  elements, respectively. Finally, we can again regard the invariant (4) as being obtained from the charge conjugation invariant by modding out the  $\mathbb{Z}_2$  symmetry that is induced by the simple current  $J$ . In this picture, the fixed points, even though left-right symmetric, all belong to the twisted sector; in the investigations below, this always must be kept in mind.

### 3 Boundary conditions and reflection coefficients

Let us now analyze such conformal field theories on surfaces with boundaries. Throughout this letter we assume that the boundary conditions are not only compatible with conformal invariance, but that they even preserve the full chiral symmetry of the theory. This condition effectively links left and right movers, and as a consequence a primary field in the bulk can survive in the presence of a boundary only if in the torus partition function it is paired with its charge conjugate. In the case of the modular invariant (4) this condition is fulfilled for the  $N_0$  fields with vanishing monodromy charge and for the  $N_f$  fixed points, so that we are left with  $N_0 + N_f$  bulk fields.

The investigation of conformal field theories in the presence of boundaries is based on the fact that every surface with boundaries admits a twofold cover that is orientable and does not have any boundaries. Under the lift to the covering surface points in the bulk have two pre-images, while for boundary points the lift is unique. It follows that when a bulk field  $\phi_{(\lambda, \lambda^+)}$  approaches a boundary, one effectively has to take the operator product of fields sitting at the two pre-images on the covering surface, and as a consequence it excites boundary fields  $\psi_\mu^{\alpha\alpha}$  [5]. On the (unit) disk, this is encoded in the expansion

$$\phi_{(\lambda, \lambda^+)}(re^{i\sigma}) \sim \sum_{\mu, \alpha} C_{(\lambda, \lambda^+), \mu}^\alpha (1 - r^2)^{-2\Delta_\lambda + \Delta_\mu} \psi_\mu^{\alpha\alpha}(e^{i\sigma}) \quad \text{for } r \rightarrow 1. \quad (5)$$

Here the possible boundary conditions are labelled by  $\alpha$ .

The constants  $C_{(\lambda, \lambda^+), \mu}^\alpha$  can be interpreted as reflection coefficients at a boundary  $\alpha$  with excitation of type  $\mu$ ; the determination of their explicit values is one of the necessary ingredients for formulating a conformal field theory on surfaces with boundaries. For all theories studied so far it was found that the reflection coefficients  $C_{(\lambda, \lambda^+), 0}^\alpha$  involving the identity boundary field form irreducible representations of some semisimple finite-dimensional algebra  $\tilde{\mathfrak{A}}$ . Accordingly we call  $\tilde{\mathfrak{A}}$  the *classifying algebra*. In the case of the charge conjugation modular invariant it was argued in [1] that this algebra  $\tilde{\mathfrak{A}}$  just coincides with the fusion rule algebra  $\mathfrak{A}$  of the theory, whose structure constants are the fusion rule coefficients  $\mathcal{N}_{\lambda, \mu}^\nu$ ; a distinguished basis of  $\mathfrak{A}$  is given by all bulk fields, so that one has the relation

$$C_{(\lambda, \lambda^+), 0}^\alpha C_{(\mu, \mu^+), 0}^\alpha = \sum_\nu \mathcal{N}_{\lambda, \mu}^\nu C_{(\nu, \nu^+), 0}^\alpha. \quad (6)$$

A similar structure was found in [3] for the  $\mathfrak{su}(2)$  WZW theory with the modular invariant (1). Again the possible boundary conditions can be related to the irreducible representations of a finite-dimensional semisimple associative algebra  $\tilde{\mathfrak{A}}$  which possesses a basis labelled by the allowed bulk fields; in particular the dimension of  $\tilde{\mathfrak{A}}$  is now  $N_0 + N_f$ . The structure constants of this algebra have been determined in [3] by using the explicit form of the duality (i.e., fusing and braiding) matrices of these models. (Besides the  $c < 1$  minimal models, the  $\mathfrak{su}(2)$  WZW models are actually the only conformal field theories for which explicit closed expressions for the duality matrices are known.) In this letter we will generalize the arguments of [1] to arrive at a general prescription for the classifying algebra  $\tilde{\mathfrak{A}}$  for *all* modular invariants of the type (4).

It was also observed in [3] that the number of boundary conditions, which equals the number of irreducible representations of  $\tilde{\mathfrak{A}}$ , is given by  $k/2 + 2 = \frac{1}{2}(N_0 + N_1) + 2N_f$ ; thus, provided one counts length-1 (i.e., fixed point) orbits with a multiplicity 2, the irreducible representations of  $\tilde{\mathfrak{A}}$  are in one-to-one correspondence with the orbits of  $J$ . Analogously we expect a total of  $\frac{1}{2}(N_0 + N_1) + 2N_f$  possible boundary conditions also in the general case (4). Since the number of irreducible representations of a semisimple algebra is equal to its dimension, this can hold only if the numbers of primary fields of different types satisfy the non-trivial relation

$$\frac{1}{2}(N_0 + N_1) + 2N_f = N_0 + N_f. \quad (7)$$

In the  $\mathfrak{su}(2)$  WZW case this identity is obviously valid. As a first step towards a formula for the reflection coefficients, we now show that the sum rule (7) holds in general.

## 4 The charge-zero subalgebra of the fusion algebra

We will establish the sum rule (7) by investigating the properties of a certain associative algebra  $\mathfrak{A}_0$ . We introduce  $\mathfrak{A}_0$  as the subalgebra of the fusion algebra  $\mathfrak{A}$  that is spanned by the  $N_0$  primary fields with vanishing monodromy charge  $Q$ . It is easily checked that  $\mathfrak{A}_0$  inherits from  $\mathfrak{A}$  the structure of a fusion algebra, so that in particular it is semisimple and its dimension  $N_0$  equals the number of its irreducible representations. We claim that this number is equal to the number of orbits of  $J$ , i.e. that

$$N_0 = \frac{1}{2}(N_0 + N_1) + N_f, \quad (8)$$

which is equivalent to the sum rule (7).

To derive (8), we first observe that the monodromy charge  $Q$  induces a  $\mathbb{Z}_2$  grading on the fusion algebra  $\mathfrak{A}$ . As a consequence, the representation matrices of  $Q = 0$  primary fields in the regular representation<sup>3</sup> of  $\mathfrak{A}$  are of block diagonal form; more precisely, they are built from two blocks, one of them being the representation matrix of the field in the regular representation of  $\mathfrak{A}_0$ . It follows that the characteristic polynomials of fusion matrices for such primary fields factorize and contain as a factor the characteristic polynomial in the regular representation of  $\mathfrak{A}_0$ . Since the roots of these polynomials are just the representation matrices of the (one-dimensional) irreducible representations, we conclude that every irreducible representation of the charge-zero algebra  $\mathfrak{A}_0$  is obtained by restricting an irreducible representation of  $\mathfrak{A}$ .

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<sup>3</sup> The representation matrices  $\mathcal{N}_\lambda$  in this representation are just the fusion matrices, i.e.  $(\mathcal{N}_\lambda)_\mu^\nu = \mathcal{N}_{\lambda,\mu}^\nu$ .

Now the irreducible representations  $\mathcal{R}_\alpha: \mathfrak{A} \rightarrow \mathbb{C}$  of  $\mathfrak{A}$ , the so-called generalized quantum dimensions, are in one-to-one correspondence to the primary fields; they are expressible through the modular matrix  $S$ :

$$\mathcal{R}_\alpha(\phi_\mu) = \frac{S_{\alpha,\mu}}{S_{0,\alpha}}. \quad (9)$$

As a consequence of the simple current relation (3), the restrictions of the  $\mathcal{R}_\alpha$  to the subalgebra  $\mathfrak{A}_0$  coincide whenever the labels  $\alpha$  belong to one and the same orbit of  $J$ . This implies that the dimension of  $\mathfrak{A}_0$  is smaller or equal to the number of orbits of  $J$ .

It remains to be shown that two irreducible representations of the fusion algebra  $\mathfrak{A}$  take the same value on all fields of vanishing monodromy charge *only* if they are related by the action of the simple current, since this implies that the dimension of  $\mathfrak{A}_0$  is larger or equal to the number of orbits. To this end, we use the fact that the modular matrix  $S$  is unitary, which when combined with (3) yields

$$\delta_{\alpha,\beta} = \sum_{\mu} S_{\alpha,\mu} S_{\mu,\beta}^* \quad \text{and} \quad \delta_{J\alpha,\beta} = \sum_{\mu} S_{J\alpha,\mu} S_{\mu,\beta}^* = \sum_{\mu} e^{2\pi i Q(\mu)} S_{\alpha,\mu} S_{\mu,\beta}^*, \quad (10)$$

from which we conclude that

$$\sum_{\mu: Q(\mu)=0} S_{\alpha,\mu} S_{\mu,\beta}^* = \frac{1}{2} (\delta_{\alpha,\beta} + \delta_{J\alpha,\beta}). \quad (11)$$

On the other hand, if two irreducible representations  $\mathcal{R}_\alpha$  and  $\mathcal{R}_\beta$  of the fusion algebra coincide (i.e. if  $S_{\alpha,\mu}/S_{0,\alpha} = S_{\beta,\mu}/S_{0,\beta}$ ) for all  $\mu$  with  $Q(\mu) = 0$ , then we have  $S_{\alpha,\mu} = \lambda S_{\beta,\mu}$  with  $\lambda \neq 0$  for all those  $\mu$ . As a consequence,

$$\frac{1}{2} (\delta_{\alpha,\beta} + \delta_{J\alpha,\beta}) = \sum_{\mu: Q(\mu)=0} S_{\alpha,\mu} S_{\mu,\beta}^* = \lambda \sum_{\mu: Q(\mu)=0} S_{\beta,\mu} S_{\mu,\beta}^* = \frac{1}{2} (1 + \delta_{J\beta,\beta}) \lambda, \quad (12)$$

which shows that  $\lambda = 1$  and that  $\alpha$  and  $\beta$  are on the same simple current orbit. This concludes our derivation of the relation (8), and hence of (7).

As a side remark, we mention the following generalization of the structure discovered above. We denote by  $\mathcal{U}$  any subgroup of the (abelian) group  $\mathcal{G}$  of all simple currents of a conformal field theory, and by  $\mathfrak{A}_0$  the sub-fusion algebra that is spanned by all fields whose monodromy charges with respect to all simple currents in  $\mathcal{U}$  vanish. Then the number of orbits of  $\mathcal{U}$  on  $\mathfrak{A}$  is just  $N_0$ , the dimension of  $\mathfrak{A}_0$ . To see this, we observe that by the same arguments as before the charge-zero algebra  $\mathfrak{A}_0$  is semisimple. Moreover, again one has a grading of  $\mathfrak{A}$  (this time over the group  $\mathcal{G}/\mathcal{U}$ ), leading to block diagonal fusion matrices for primary fields in the subspace  $\mathfrak{A}_0$ . The formula (3) still guarantees that the generalized quantum dimensions belonging to fields on one and the same orbit of  $\mathcal{U}$  give one and the same irreducible representation of  $\mathfrak{A}_0$ .

To generalize the relation (11) as well, we associate to any primary field  $\mu$  a function  $\Psi_\mu: \mathcal{G} \rightarrow \mathbb{C}$  by

$$\Psi_\mu(J) := \exp(2\pi i Q_J(\mu)). \quad (13)$$

From the relation  $Q_{J_1}(J_2\mu) = Q_{J_1}(J_2) + Q_{J_1}(\mu) \bmod \mathbb{Z}$  (which holds because the monodromy charge is additive under operator products) we learn that  $Q_{J_1}(\mu) + Q_{J_2}(\mu) = Q_{J_1 J_2}(\mu) \bmod \mathbb{Z}$ , which in turn implies that the function  $\Psi_\mu$  is a group character on  $\mathcal{G}$ . Summing the identity  $\delta_{J\alpha,\beta} = \sum_\mu S_{J\alpha,\mu} S_{\mu,\beta}^* = \sum_\mu \Psi_\mu(J) S_{\alpha,\mu} S_{\mu,\beta}^*$  over  $J \in \mathcal{G}$ , we therefore obtain

$$\frac{1}{|\mathcal{G}|} \sum_{J \in \mathcal{G}} \delta_{J\alpha,\beta} = \sum_{\mu: Q(\mu)=0} S_{\alpha,\mu} S_{\mu,\beta}^*. \quad (14)$$

This settles the generalization from  $\mathbb{Z}_2$  to an arbitrary subgroup  $\mathcal{U}$  of  $\mathcal{G}$ .

## 5 The classifying algebra

As the crucial ingredient which allows to obtain a formula for the boundary conditions, we now introduce a new  $\mathbb{Z}_2$  graded associative algebra  $\tilde{\mathfrak{A}}$  of dimension  $N_0 + N_f$  that contains the charge-zero algebra  $\mathfrak{A}_0$  as a subalgebra. We claim that this algebra  $\tilde{\mathfrak{A}}$  constitutes the classifying algebra for the case of the modular invariant (4). We define  $\tilde{\mathfrak{A}}$  as follows. A distinguished basis of  $\tilde{\mathfrak{A}}$  is labelled by all possible bulk fields, i.e. by the primary fields with vanishing monodromy charge and the fixed points.  $\mathfrak{A}_0$  is the subalgebra of  $\tilde{\mathfrak{A}}$  that corresponds to the unit element in the  $\mathbb{Z}_2$  grading; the description of the other structure constants  $\check{\mathcal{N}}_{\mu,\nu}^\lambda$  requires some preparation.

Recall that the fusion coefficients  $\mathcal{N}_{\lambda,\mu,\nu} = \mathcal{N}_{\lambda,\mu}^{\nu^+}$  count the (finite) dimension of the spaces of chiral blocks of the three-point functions on the sphere. They can be expressed in terms of the modular matrix  $S$  via the Verlinde formula

$$\mathcal{N}_{\lambda,\mu,\nu} = \sum_\rho \frac{S_{\lambda,\rho} S_{\mu,\rho} S_{\nu,\rho}}{S_{0,\rho}}. \quad (15)$$

The simple current relation (3) for the entries of  $S$  implies that  $\mathcal{N}_{\lambda,\mu,\nu} = \mathcal{N}_{J\lambda,J\mu,\nu}$ . In fact, the action of a simple current  $J$  can be naturally implemented on the spaces of chiral blocks, and the latter equality follows from the existence of an isomorphism  $\Theta_J$  between the respective spaces of blocks.

Now suppose that both  $\lambda = f$  and  $\mu = g$  are fixed points. In this case the isomorphism  $\Theta_J$  becomes an *endomorphism* of the space  $\mathcal{B}$  of chiral blocks, and one can compute its trace

$$\check{\mathcal{N}}_{f,g,\nu} := \text{tr}_{\mathcal{B}} \Theta_J. \quad (16)$$

Note that  $\check{\mathcal{N}}_{f,g,\nu}$  is an integer; in fact, this remains true for the general case of arbitrary simple current group  $\mathcal{U}$ , where the order of  $J$  is typically larger than 2. In addition, of course, the dimensions of the eigenspaces of  $\Theta_J$  are non-negative integers; since  $\text{tr}_{\mathcal{B}} \Theta_J = \check{\mathcal{N}}_{f,g,\nu}$  while  $\text{tr}_{\mathcal{B}} \text{id} = \mathcal{N}_{f,g,\nu}$ , this means that

$$\frac{1}{2} (\mathcal{N}_{f,g,\nu} \pm \check{\mathcal{N}}_{f,g,\nu}) \in \mathbb{Z}_{\geq 0}. \quad (17)$$

Similar traces have already appeared in the analysis of the so-called fixed point resolution in integer spin simple current modular invariants [6]. In fact, it is known [4, 6] that there is

some other conformal field theory, the so-called fixed point theory, whose primary fields are in one-to-one correspondence to the fixed points of the original theory (when there is only a single fixed point, as in the  $\mathfrak{su}(2)$  case (1), then the fixed point theory is trivial), and whose modular matrix  $\check{S}$  determines  $\check{N}$  via the formula

$$\check{N}_{f,g,\nu} = \sum_{h: Jh=h} \frac{\check{S}_{f,h}\check{S}_{g,h}S_{\nu,h}}{S_{0,h}}, \quad (18)$$

where the sum is over all fixed points.<sup>4</sup> In the case of a WZW model based on an affine Lie algebra  $\mathfrak{g}$ , the fixed point theory is governed by the so-called orbit Lie algebra  $\check{\mathfrak{g}}$  that is associated [8] to  $\mathfrak{g}$  and  $J$ , which in particular provides an explicit closed expression for  $\check{S}$ .

We are now ready to define the multiplication rules for the classifying algebra  $\check{\mathfrak{A}}$ : the product of  $Q=0$  fields is the ordinary fusion product, while the other non-vanishing structure constants are given by  $\check{N}_{f,g}^\lambda$  and  $\check{N}_{f,\lambda}^g$ .<sup>5</sup> That is,

$$\check{N}_{\lambda,\mu}^\nu = \begin{cases} \mathcal{N}_{\lambda,\mu}^\nu & \text{if } Q(\lambda) = Q(\mu) = Q(\nu) = 0, \\ \check{N}_{\lambda,\mu}^\nu & \text{if precisely one out of } \lambda, \mu, \nu \text{ has } Q = 0, \\ 0 & \text{else.} \end{cases} \quad (19)$$

Inspection shows that the classifying algebra  $\check{\mathfrak{A}}$  is commutative and associative, that it has  $\phi_0$  as a unit element, and that it has a conjugation which is still given by the evaluation of the product on  $\phi_0$ , i.e.  $\check{N}_{f,g}^0 = \delta_{f,g^+}$ . As a consequence,  $\check{\mathfrak{A}}$  is again a semisimple associative algebra. However, it is *not* a fusion algebra, because some of its structure constants are negative:

$$\check{N}_{f,g,J\lambda} = \sum_h \frac{\check{S}_{f,h}\check{S}_{g,h}S_{J\lambda,h}}{S_{0,h}} = -\check{N}_{f,g,\lambda}. \quad (20)$$

Also note that the algebra  $\check{\mathfrak{A}}$  is *not* a subalgebra of the original fusion algebra  $\mathfrak{A}$ .

Applying similar arguments as for  $\mathfrak{A}$  above, we can determine the  $N_0 + N_f = \frac{1}{2}(N_0 + N_1) + 2N_f$  irreducible representations  $\check{\mathcal{R}}_\alpha$  of  $\check{\mathfrak{A}}$ . They all restrict to irreducible representations of  $\mathfrak{A}_0$ , i.e.  $\check{\mathcal{R}}_\alpha(\phi_\mu) = S_{\alpha,\mu}/S_{0,\alpha}$  for  $Q(\mu)=0$ . If  $\alpha$  is on a full orbit of  $J$ , then this restriction is uniquely extended to the fixed points by zero. On the other hand, by direct calculation one checks that for irreducible representations corresponding to a fixed point  $h$ , the extension to fixed points is by  $\pm\check{S}_{h,f}/S_{0,h}$ , which accounts for two irreducible representations  $\check{\mathcal{R}}_{(h+)}$  and  $\check{\mathcal{R}}_{(h-)}$  each. That is,

$$\check{\mathcal{R}}_\alpha(\phi_f) = 0 \quad \text{for } Q(\alpha) = 0, \quad \check{\mathcal{R}}_{(h\pm)}(\phi_f) = \pm \frac{\check{S}_{h,f}}{S_{0,h}}. \quad (21)$$

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<sup>4</sup> It has also been conjectured [6] that  $\check{S}$  describes the modular properties of the one-point chiral blocks on the torus, where the insertion is the simple current  $J$ . Additional evidence for this relationship has been presented in [7].

<sup>5</sup> As usual, the indices of  $\check{N}$  are raised and lowered by complex conjugating the corresponding matrix element of  $S$  respectively  $\check{S}$  on the right hand side of (18).



## 6 The reflection coefficients

As a consequence of our claim that the algebra  $\tilde{\mathfrak{A}}$  introduced in the previous section governs the boundary conditions for a conformal field theory with modular invariant (4), the boundary coefficients  $C_{(\mu,\mu^+),0}^\alpha$  are given by the irreducible representation matrices of  $\tilde{\mathfrak{A}}$ . Hence our results about the representation theory of  $\tilde{\mathfrak{A}}$  tell us that there are two different types of boundary conditions: for length-two orbits  $\alpha$  of  $J$ , we obtain

$$B_\mu^\alpha \equiv C_{(\mu,\mu^+),0}^\alpha = \tilde{\mathcal{R}}_\alpha(\phi_\mu) = \begin{cases} \frac{S_{\alpha,\mu}}{S_{0,\alpha}} & \text{for } Q(\mu) = 0, \\ 0 & \text{for } J\mu = \mu \end{cases} \quad (22)$$

(because of (3) this does not depend on the choice of representative of the orbit  $\alpha$ ), while fixed point orbits yield two distinct sets of coefficients:

$$B_\mu^{(f\pm)} \equiv C_{(\mu,\mu^+),0}^{(f\pm)} = \tilde{\mathcal{R}}_{(f\pm)}(\phi_\mu) = \begin{cases} \frac{S_{f,\mu}}{S_{0,f}} & \text{for } Q(\mu) = 0, \\ \pm \frac{\check{S}_{f,\mu}}{S_{0,f}} & \text{for } J\mu = \mu. \end{cases} \quad (23)$$

In the case of  $\mathfrak{su}(2)$  this prescription reproduces the results of [3]. Also notice that the appearance of the modular matrix  $\check{S}$  of the fixed point theory is rather natural; indeed, fixed points belong to the twisted sector, and according to [6] the matrix  $\check{S}$  governs the modular transformations of that sector.

To provide more evidence for our prescription for the boundary coefficients, we study the annulus amplitude. The latter has the general form [9]

$$A_{ab}(t) = \sum_\mu \chi_\mu\left(\frac{2i}{t}\right) \left(\frac{S_{0,\mu}}{S_{0,0}}\right)^{-1} (B_\mu^a \tilde{C}_0^a)^* B_\mu^b \tilde{C}_0^b = \sum_\mu A_{ab}^\mu \chi_\mu\left(\frac{it}{2}\right). \quad (24)$$

Here  $a$  and  $b$  are the boundary conditions at the two boundaries of the annulus (i.e. each of them can take the values  $\alpha$  that label full orbits as well as the two values  $(f\pm)$  for each fixed point label  $f$ ), and  $t \in \mathbb{R}_{>0}$  is the standard modulus of the annulus (the modulus of its covering torus is then  $\tau = it/2$ ). The number  $\tilde{C}_0^a \equiv \langle \psi_0^{aa} \rangle$  is the normalization of the one-point function of the identity on a boundary of type  $a$ . The second equality in (24) is obtained by a modular transformation and gives the amplitude in the open string channel.

The natural value of  $\tilde{C}_0^a$  that generalizes the expressions for diagonal [9] and  $D_{\text{odd}}$ -type  $\mathfrak{su}(2)$  [3] theories reads

$$C_0^\alpha = \sqrt{2} S_{0,\alpha} \quad \text{for } Q(\alpha) = 0, \quad C_0^f = S_{0,f}/\sqrt{2} \quad \text{for } Jf = f. \quad (25)$$

Inserting the formulæ (22), (23) and (25) into the relation (24), we can determine the tensors  $A_{ab}^\mu$ ; we find

$$\begin{aligned} A_{\alpha\beta}^\mu &= \mathcal{N}_{\beta,\mu}^\alpha + \mathcal{N}_{\beta,\mu}^{J\alpha}, & A_{(f\pm)(g\pm)}^\mu &= \frac{1}{2} (\mathcal{N}_{f^+,g,\mu} + \check{\mathcal{N}}_{f^+,g,\mu}), \\ A_{\alpha(f\pm)}^\mu &= \mathcal{N}_{f,\mu}^\alpha, & A_{(f\pm)(g\mp)}^\mu &= \frac{1}{2} (\mathcal{N}_{f^+,g,\mu} - \check{\mathcal{N}}_{f^+,g,\mu}). \end{aligned} \quad (26)$$

We can now present evidence for our prescription (22) and (23). We first remark that the annulus amplitude can be regarded as the partition function for the boundary operators (before orientifold projection). For consistency it is therefore necessary that all coefficients  $A_{ab}^\mu$  in (26) are non-negative integers. Inspection shows that this highly non-trivial constraint is indeed satisfied for all values of  $a, b$  and  $\mu$ ; in the particular case of boundary conditions of fixed point type this is a consequence of the result (17), which in turn has its origin in the specific properties of the fixed point theories.

Further confirmation is provided by the following properties of the tensors  $A_{ab}^\mu$ , which we deduce from the formulæ (26). First, they obey the relation

$$\sum_{\mu} A_{ab}^{\mu} A_{cd}^{\mu+} = \sum_{\mu} A_{ac+}^{\mu} A_{b+d}^{\mu+} \quad (27)$$

for all choices of the boundary labels  $a, b, c, d$  (also note that  $A_{ba}^\mu = A_{a+b+}^\mu$ ). And second, considered as matrices in the boundary labels, they satisfy

$$A^\mu A^\nu = \sum_{\lambda} \mathcal{N}_{\mu,\nu}^{\lambda} A^\lambda. \quad (28)$$

(For diagonal theories, where  $A_{\alpha\beta}^\mu = \mathcal{N}_{\beta,\mu}^{\alpha}$ , these relations reduce to the statement that the fusion rules are associative and that the structure constants furnish a representation – the regular representation – of the fusion algebra.) The equality (28) can be interpreted as the assertion that the boundary conditions are complete. More specifically (compare equation (33) of [3]), it implies that the two distinct ways of factorizing a two-point function with bulk insertions lead to the same result.

It is known [3] that the relations (27) and (28) are highly restrictive, in particular when they are combined with the information about the spectrum that is contained in the torus partition function. The fact that our ansatz for the boundary coefficients reproduces these formulæ is therefore another strong indication that our prescription is correct.

As a final test, we study the boundaries along the lines followed in [1] for diagonal modular invariants. We first remark that when both boundaries of the annulus are in the  $\alpha = 0$  condition, then, in the terminology of [1], both the field  $\phi_0$  and  $\phi_J$  propagate in the bulk. This nicely fits with the observation of [3] that there is an effective enhancement of the boundary symmetry. Now just like in the case of integer spin simple current invariants, for fixed points such an extension can be performed in two inequivalent ways. Therefore for boundary conditions  $a = 0$  and  $b = (f\pm)$   $\phi_f$  propagates in the bulk, and it does so in two independent ways (as characters of the *non-extended* algebra, the associated characters are, however, identical,  $\chi_{(f+)} = \chi_{(f-)} = \chi_f$ ).

We now map the annulus to an infinitely long strip and consider the following configuration (compare figure 2 of [1]). We start with conditions of type  $\alpha = 0$  both on the left and on the right boundary of the strip; then we insert on the right boundary a boundary operator that switches to boundary conditions  $(f\pm)$  so that now  $\phi_f$  propagates in the bulk. Afterwards we insert a boundary operator on the left boundary that yields boundary condition  $(g\pm)$ .

This amounts to coupling the two fixed point primary fields, and we know that they can only couple to primary fields with vanishing monodromy charge. The latter are, however, ‘uncharged’ under the action of the simple current, and accordingly we get a restriction from the requirement that the couplings transform correctly under the simple current action. More specifically, if the couplings  $(f\pm)$  and  $(g\pm)$  are of like sign, then the coupling should be even; as we have seen in the discussion before equation (17), the number of such couplings is just  $\frac{1}{2}(\mathcal{N}_{f,g}^\mu + \check{\mathcal{N}}_{f,g}^\mu)$ . Similarly, the case of opposite signs yields  $\frac{1}{2}(\mathcal{N}_{f,g}^\mu - \check{\mathcal{N}}_{f,g}^\mu)$  couplings. Our argument thus reproduces precisely what we obtained in the second column of (26). It would be gratifying to corroborate this generalization of the rather heuristic arguments of [1] by an explicit calculation analogous to the one reported in [3]. The latter, however, relies on the explicit knowledge of the duality matrices, which are available for  $\mathfrak{su}(2)$ , but not for more complicated conformal field theories.

## 7 Conclusions

In this letter, we have determined the boundary conditions for conformal field theories with non-trivial modular invariants of ‘ $D_{\text{odd}}$ -type’ (4). We have shown that just as in the diagonal case they are controlled by a semisimple algebra, the classifying algebra  $\check{\mathfrak{A}}$ . The structure we discovered is closely related to the fusion algebra of another type of modular invariants, namely those of ‘ $D_{\text{even}}$ -type’ (also known as integer spin simple current extensions). In particular, it looks as if the boundary theory is extended by the *half*-integer spin simple current J.

This is indeed most remarkable, because in the case of extensions, modular invariance provides powerful consistency requirements. But for surfaces of Euler characteristic zero which are non-orientable or which have a boundary, there is no analogue of a modular group. In string theory it is usually argued that tadpole cancellation provides a substitute for such consistency conditions. Note, however, that for the investigations presented in this letter we did not have to assume that the conformal field theory is part of a string compactification (e.g., the central charge is not restricted), so that the conditions of tadpole cancellation cannot even be formulated. Still, it seems that already on a pure conformal field theory level there are similar powerful constraints; to unravel the underlying structure will be a promising task.

As a final and somewhat more speculative remark, we mention that the quantum dimensions, and hence also the space of possible boundary conditions, carry a natural action of the Galois group of a cyclotomic number field. When a conformal field theory admits a geometrical interpretation as a sigma model on some manifold  $\mathcal{M}$ , then a boundary condition frequently corresponds to a certain sheaf on  $\mathcal{M}$ . If those sheaves are the direct image of a line bundle over a spectral cover, one might speculate on a relation between the Galois group of the corresponding covering and the Galois action just mentioned.

### Acknowledgement:

We are grateful to A. Sagnotti, A.N. Schellekens and Ya.S. Stanev for helpful discussions.

## References

- [1] J.L. Cardy, *Boundary conditions, fusion rules and the Verlinde formula*, Nucl. Phys. B 324 (1989) 581
- [2] G. Pradisi, A. Sagnotti, and Ya.S. Stanev, *The open descendants of non-diagonal  $SU(2)$  WZW models*, Phys. Lett. B 356 (1995) 230
- [3] G. Pradisi, A. Sagnotti, and Ya.S. Stanev, *Completeness conditions for boundary operators in 2D conformal field theory*, Phys. Lett. B 381 (1996) 97
- [4] A.N. Schellekens and S. Yankielowicz, *Simple currents, modular invariants, and fixed points*, Int. J. Mod. Phys. A 5 (1990) 2903
- [5] D.C. Lewellen, *Sewing constraints for conformal field theories on surfaces with boundaries*, Nucl. Phys. B 372 (1992) 654
- [6] J. Fuchs, A.N. Schellekens, and C. Schweigert, *A matrix  $S$  for all simple current extensions*, Nucl. Phys. B 473 (1996) 323
- [7] P. Bantay, *Simple current extensions and mapping class group representations*, preprint hep-th/9611124
- [8] J. Fuchs, A.N. Schellekens, and C. Schweigert, *From Dynkin diagram symmetries to fixed point structures*, Commun. Math. Phys. 180 (1996) 39
- [9] J.L. Cardy and D.C. Lewellen, *Bulk and boundary operators in conformal field theory*, Phys. Lett. B 259 (1991) 274