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# The semi–classical approach to the exclusive electron scattering

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**Abstract**

The semiclassical approach, successfully applied in the past to the inelastic, inclusive electron scattering off nuclei, is extended to the treatment of exclusive processes. The final states interaction is accounted for in the mean field approximation, respecting the Pauli principle. The impact on the exclusive cross section of the shape of the potential binding the nucleons into the nucleus and of the distortion of the outgoing nucleon wave are explored. The exclusive scattering is found to be quite sensitive to the mean field final states interaction, unlike the inclusive one. Indeed we verify that the latter is not affected, as implied by unitarity, by the distortion of the outgoing nucleon wave except for the effect of relativity, which is modest in the range of momenta up to about 500 MeV/c. Furthermore, depending upon the correlations between the directions of the outgoing and of the initial nucleon, the exclusive cross–section turns out to be remarkably sensitive to the shape of the potential binding the nucleons. These correlations also critically affect the domain in the missing energy– missing momentum plane where the exclusive process occurs.

# 1 Introduction

The plane wave impulse approximation (PWIA) has been a framework extensively employed in analyzing the exclusive (or semi–inclusive) processes of inelastic scattering of electrons off nuclei, like, e.g., the $(e, e'p)$ one[1]. The advantage of such an approach lies of course in its simplicity : indeed, in the PWIA, the final nucleon is described by a plane wave and is not antisymmetrized with the daughter $(A-1)$ nucleus. Accordingly in PWIA one deals with the diagonal component of the spectral function $S(\vec{p}, E_p)$ only, which is the easiest to calculate.

In this connection the Fermi gas (FG) model, where translational invariance forces the outgoing nucleon wavefunction to be a plane wave anyway, appears naturally related, in the non–Pauli blocked domain, to the PWIA, its spectral function being intrinsically diagonal[2]: at the same time, however, the FG spectral function cannot be considered as a realistic one, as it stems from a uniform distribution of particles.

The chief flaw of PWIA is of course the neglect of final state interactions (FSI). These play an important role also in the FG, where in fact they can be more easily treated, although the infinite volume of the system appears hardly suitable for describing exclusive processes. Yet finite size (surface) and binding effects can be easily inserted into the FG model semi–classically. Indeed a semi–classical formalism has been developed[3, 4], which starting from the FG satisfactorily accounts for the impact of the finite size and of the binding energy of the nucleus in the "inclusive" response functions, but for their low energy side, where the semi–classical approach interpolates the quantum mechanical cross sections without reproducing their rapid variation with the energy.

The aim of this paper is to extend the semi–classical method to deal with exclusive processes, going beyond the PWIA by accounting for the FSI. Of course the problem of the FSI in the exclusive processes has already been widely addressed[5]: not, however, in the semi–classical framework. In this paper we shall show that the latter not only allows an adequate treatment of the inclusive processes, but also of the exclusive (or semi–inclusive) ones, yielding an off–diagonal spectral function $S(\vec{p}, \vec{p}\,'E)$ and a projection operator $\hat{\rho}_N$ (to be later defined) properly describing (the first) the dynamics of a nucleon inside the finite nucleus and (the second) the distortion the outgoing nucleon suffers in crossing the nuclear surface, at least in a mean field framework. Indeed this is the scheme we shall adhere to, for simplicity, in the present work: however the extension of the semi–classical method to encompass nucleon–nucleon (NN) dynamical correlations appears to be entirely feasible, both for the spectral function and for the distortion of the final nucleon wave. We actually intend to carry it out since the data unambiguously point to the existence of a substantial exclusive cross–section in kinematical regions hardly compatible with a pure mean field description of the $(e, e' p)$ process[7].

As it is well–known the semi–classical method expresses the physical observables in powers of the Planck's constant $\hbar$. It turns out that the leading term of the expansion already accounts, in the mean field approximation, for the basic elements of the exclusive physics, namely the nuclear confinement and the FSI. This important finding allows then to address several questions related to the physics of the exclusive (and inclusive) processes, whose answer would otherwise be much harder to get in a fully quantum mechanical many–body scheme.

In order to appreciate the impact on the exclusive process of the FSI, we describe the distortion of the outgoing nucleon wave in two opposite, schematic models, the so–called eikonal and uniform approximations: in the first the outgoing nucleon is not deflected from the direction of the initial momentum, while in the latter the final nucleon is isotropically emitted from the nucleus. For the same purpose we also felt it useful to evaluate, within the semi–classical approach, the exclusive cross–sections in the PWIA and to compare it with the results which include FSI.

For the purpose of testing the sensitivity of the semi–classical exclusive cross–section to the shape of the shell model potential binding the nucleons in the nucleus, we employ both an harmonic oscillator and a Woods–Saxon potential well. We remind that a unified approach, where hole and particle states are treated on equal footing, has been developed within a complex shell model potential which extrapolates the mean field from positive toward negative energies[8, 9]. Also in the present treatment particle and holes states are affected by the same mean field, although we consider here, for simplicity, only real potential wells.

Indeed in the exclusive $(e, e' p)$ cross–section the outgoing nucleon wave can keep track of the original bound state, again depending upon the distortion mechanism. On the contrary, different mean fields are known to play a minor role in the inclusive cross–section, which turns out to be almost unaffected by the shape of the potential.

We obtained the latter by integrating the exclusive cross–section in the appropriate domain of the missing energy – missing momentum plane: due to the approximate treatment of the distortion mechanism, it is not "a priori" guaranteed that the inclusive cross–sections are, as indeed they must in the mean field approximation exploited here, insensitive to the distortion of the outgoing nucleon.

Although the main focus of this work is centered on the effects of FSI, special attention has been also devoted to the spectral function, which is a key ingredient of the exclusive cross–section: direct calculation of the hole spectral density have been performed long ago with variational methods in few–nucleon systems[10, 11] and later on in nuclear matter[12, 13] in the frame of the correlated basis theory. Spectroscopic factors have been recently evaluated in a relativistic shell model approach[14], both in medium and heavy nuclei. The effect of a relativistic optical potential in exclusive processes is also investigated, to account for the distortion of the ejected nucleon[15]. While the semi–classical approach admittedly fails in reproducing specific quantum mechanical effects, we will show that in the spectral function it retains some important features which are lost in the framework of a Fermi gas description, still keeping a deal of simplicity.

We summarize now the organization of the paper: in Section 2 the general expressions for the inclusive and exclusive cross–section in the one–photon exchange approximation are shortly revisited. Furthermore the off–diagonal spectral function and the distortion operator are introduced. In Section 3 we derive and discuss the semi–classical expression for the exclusive cross–section, both in PWIA and DWIA. In Section 4 the diagonal part of the semi–classical spectral function is obtained in the mean field framework. Analytic expressions of this quantity for a few one–body potential wells are provided in Appendix A. In Section 5 we deduce the mean field expression for the distortion operator and discuss it in the context of the two rather extreme models (eikonal and uniform approximation) referred to above. In Section 6 we calculate, for both models, the exclusive cross–sections and, in Section 7, the inclusive ones, by integrating the former over appropriate regions of the missing energy–missing momentum plane. Finally in Section 8 we present and discuss our numerical results, while Section 9 illustrates the merits of the semi–classical approach and its possible extensions.

## 2 The cross–sections

The inclusive cross-section for the scattering of an electron, with initial and final four-momenta $k$ and $k'$ respectively, out of a nucleus, initially in its ground state $|A>$ and then excited into "any" final state $|X>$, reads

$$\frac{d^2\sigma}{d\Omega_e d\epsilon'} = 2\alpha^2 \frac{1}{Q_\mu^4}\,\frac{k'}{k}\eta^{\mu\nu}\sum_X <A|\hat{J}_\mu^\dagger(\vec{q})|\,X><X|\,\hat{J}_\nu(\vec{q})|A>\delta(E_X - E_A - \omega) \quad (2.1)$$

(all the states are normalized to one in a large box of volume V). In the above $Q_\mu^2 = \omega^2 - \vec{q}^{\,2}$ is the space–like four momentum transferred from the electron to the nucleus and

$$\eta^{\mu\nu} = k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu} k\cdot k' \quad (2.2)$$

is the well–known symmetric leptonic tensor of rank two. The diagram describing the inclusive process is displayed in Fig. 1.

In the present work we confine ourselves, for sake both of simplicity and of illustration, to consider one-body current only, disregarding meson exchange currents (MEC). Likewise correlations among nucleons beyond the mean field will be dealt with in future research. In any case the matrix elements of the nucleon's one–body current $\hat{J}_\nu$ entering into (2.1) read (the symbols are self–explanatory)

$$< X|\hat{J}_\nu(\vec{q})|A >= \sum_s \int \frac{d\vec{p}}{(2\pi)^3} \int \frac{d\vec{p}'}{(2\pi)^3} < X|\hat{a}^\dagger(\vec{p}\,', s')\, \hat{a}(\vec{p}, s)|A >< \vec{p}\,', s'|j_\nu(\vec{q})|\vec{p}, s > , \tag{2.3}$$

the standard annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ being normalized according to the anticommutation rule

$$\{\hat{a}\ (\vec{p}, s),\ \hat{a}^\dagger(\vec{p}\,', s')\} = (2\pi)^3\delta(\vec{p} - \vec{p}\,')\delta_{ss'}\,. \tag{2.4}$$

In the above

$$< \vec{p}\,', s'|j_\nu(\vec{q})|\vec{p}, s >= (2\pi)^3\delta(\vec{p}' - \vec{p} - \vec{q})j_\nu^{s's}(\vec{p} + \vec{q}, \vec{p}) \tag{2.5}$$

and

$$j_\nu^{s's}(\vec{p} + \vec{q}, \vec{p}) = \bar{u}(\vec{p} + \vec{q}, s')\Big[F_1(Q_\mu^2)\gamma_\nu + F_2(Q_\mu^2)i\frac{\kappa}{2M_N}\sigma_{\mu\nu}Q^\mu\Big]u(\vec{p}, s) \tag{2.6}$$

where $F_1$ and $F_2$ are the Dirac and Pauli nucleon's form factors, $\kappa = 1.79$ for protons and $\kappa = -1.91$ for neutrons. Finally the spinor normalization is $u^\dagger u = 1$.

Clearly the insertion of (2.5) into (2.3) leads to

$$< X|\hat{J}_\nu(\vec{q})|A >= \sum_s \int \frac{d\vec{p}}{(2\pi)^3} < X|\hat{a}^\dagger(\vec{p} + \vec{q}, s')\ \hat{a}\ (\vec{p}, s)|A > j_\nu^{s's}(\vec{p} + \vec{q}, \vec{p}). \tag{2.7}$$

Now in the PWIA framework the final nuclear state is factorized as follows ($V$ is the normalization volume)

$$|X >= |n > \otimes \frac{1}{\sqrt{V}}|\tilde{\vec{p}}_N, s_N > \tag{2.8}$$

where $|n>$ represents an excited state, normalized to one, of the residual $(A - 1)$ nucleus and $|\tilde{\vec{p}}_N, s_N>$ is just a plane wave, normalized according with the anticommutators (2.4). In such a scheme, to the matrix element (2.7) only a single nucleon with a given momentum $\vec{p}$ will contribute, the rest of the nucleus behaving just as a spectator.

In the present approach instead the wave of the outgoing nucleon, $|\tilde{\vec{p}}_N, s_N>$, is "distorted" by the interaction with the residual nucleus and no longer is a momentum eigenstate; as a consequence the nuclear matrix element (2.7) will be expressed through an integral (sum) over all possible initial nucleons' momenta (spin) and will read:

$$< X|\hat{J}_\nu(\vec{q})|A > = \frac{1}{\sqrt{V}} \sum_s \int \frac{d\vec{p}}{(2\pi)^3} < \tilde{\vec{p}}_N|\vec{p} + \vec{q} > j_\nu^{s_N s}(\vec{p} + \vec{q}, \vec{p}) < n \mid \hat{a}\ (\vec{p}, s)|A > \tag{2.9}$$

providing the spin of the outgoing nucleon, which will no longer be explicitly indicated in the state vectors, is unaffected by the distortion.

In Fig. 2 the exclusive process is diagrammatically displayed : the bubble on the final nucleon leg should be ignored in the PWIA scheme, whereas in the framework of the present paper it is meant to embody the FSI.

Using (2.9) we can now compute the cross–section for the exclusive process. In the laboratory frame, where the nucleus is at rest (hence $E_A = M_A$), we obtain

$$\begin{aligned}\frac{d^4\sigma}{d\Omega_e d\epsilon' d\vec{p}_N} &= \frac{2\alpha^2}{(2\pi)^3}\frac{1}{Q_\mu^4}\frac{k'}{k}\,\eta_{\mu\nu}\int\frac{d\vec{p}}{(2\pi)^3}\int\frac{d\vec{p}\,'}{(2\pi)^3} \\ &\times\sum_{s_N,s',s} j^\mu_{s's_N}\,(\vec{p}\,',\vec{p}\,'+\vec{q}) < \vec{p}\,'+\vec{q}\,|\,\widetilde{\vec{p}}_N >< \widetilde{\vec{p}}_N\,|\,\vec{p}+\vec{q} > j^\nu_{s_N\,s}\,(\vec{p}+\vec{q},\vec{p}) \\ &\times\sum_n < A|\hat{a}^\dagger\,(\vec{p}\,',s')|n > \delta[E^n_{A-1}-(\omega+M_A-E_N)] < n|\hat{a}(\vec{p},s)|A >\end{aligned} \tag{2.10}$$

the sum extending over the whole spectrum of eigenfunctions of the residual (A-1) nucleus, with eigenvalues $E^n_{A-1}$, while $E_N$ is the energy of the outgoing nucleon.

Concerning the electromagnetic vertices they will be taken, according to (2.6), with the nucleon on the mass–shell (which is the case for the FG, but clearly not for a finite nucleus). Of course this assumption can be corrected by employing, *e.g.*, the CC1 prescription of De Forest [6] to move the nucleon off the energy shell preserving gauge invariance.

Let us now transform the $\delta$–function appearing in (2.10). For this purpose it is customary to introduce the positive "nuclear separation energy"

$$E_S = M_N + M_{A-1} - M_A = M_N - \mu, \tag{2.11}$$

$\mu$ being the nuclear chemical potential, and the positive "excitation energy"

$$\mathcal{E}_n \;=\; E^n_{A-1} - E^0_{A-1} \;\cong\; E^n_{A-1} - M_{A-1} - E_{\rm rec} \tag{2.12}$$

of the residual (A-1) nucleus recoiling with momentum $\vec{q}-\vec{p}_N$ (the "missing momentum") and energy

$$E_{\rm rec} \;\cong\; \frac{(\vec{q}-\vec{p}_N)^2}{2\;M_{A-1}}. \tag{2.13}$$

The above formula is valid in the non–relativistic approximation, which is adequate for the recoil energy. Accordingly we can write

$$\delta[E^n_{A-1}-(\omega+M_A-E_N)] \;=\; \delta[\mathcal{E}_n-(\omega-T_N-E_S-E_{\rm rec})] \;=\; \delta\,(\mathcal{E}_n-\mathcal{E}) \tag{2.14}$$

where

$$\mathcal{E} \;=\; \omega - T_N - E_S - E_{\rm rec} \tag{2.15}$$

is the so–called "missing energy", fixed by the external kinematics, and $T_N$ the kinetic energy of the outgoing nucleon.

To proceed further we introduce now into the formalism three quantities, related to different one–body operators, which allow to express the exclusive cross–section in a compact form.

The first of these is the general (off–diagonal) *spectral function* defined as follows

$$\begin{aligned} < \vec{p}, s|S(\mathcal{E})|\vec{p}\,', s' > &= \sum_n < A|\hat{a}^\dagger(\vec{p}\,', s')|n > \delta[\mathcal{E} - (E^n_{A-1} - E^0_{A-1})] < n|\hat{a}(\vec{p}, s)|A > \\ &= < A|\hat{a}^\dagger(\vec{p}\,', s)\delta(\widehat{\mathcal{H}} - \mathcal{E})\hat{a}(\vec{p}, s)|A > . \end{aligned} \tag{2.16}$$

In the above closure has been applied and $\widehat{\mathcal{H}}$ is the Hamiltonian whose eigenvalues are the excitation energies of the residual nucleus.

Upon integration over the excitation energy, the spectral function, diagonal in the spin indices for parity conserving interactions, yields

$$\int d\mathcal{E} \; < \vec{p}, s|S(\mathcal{E})|\vec{p}\,', s > = < A|\,\hat{a}^\dagger(\vec{p}\,', s)\hat{a}(\vec{p}, s)|A > \tag{2.17}$$

which, for $\vec{p} = \vec{p}\,'$, is just the momentum distribution of the nucleons inside the nucleus. Thus the latter becomes experimentally accessible providing the data span a range of missing energy large enough.

Obviously for the diagonal part of the spectral function the well–known sum rule

$$\int \frac{d\vec{p}}{(2\pi)^3} \int d\mathcal{E} < \vec{p}, s|S\,(\mathcal{E})\,|\vec{p}, s > = A, \tag{2.18}$$

holds, $A$ being the number of nucleons in the nucleus and a spin–isospin summation being implicitly assumed.

Next the matrix elements in momentum space

$$< \vec{p}\,'|\hat{\rho}_N|\vec{p} > = < \vec{p}\,'|\widetilde{\vec{p}}_N > < \widetilde{\vec{p}}_N|\vec{p} > \tag{2.19}$$

of the *one–body projection operator* $\hat{\rho}_N$ should be brought into the formalism. The operator $\hat{\rho}_N$ embodies the distortion of the outgoing nucleon's wave and it is clearly of central relevance for the present treatment. In PWIA the outgoing nucleon state is a plane wave and thus (2.19) becomes:

$$< \vec{p}\,'|\hat{\rho}_N|\vec{p} > = (2\pi)^6 \delta\,(\vec{p}\,' - \vec{p}_N)\,\delta\,(\vec{p}_N - \vec{p}) \; . \tag{2.20}$$

In the next three sections we shall study in detail both $S$ and $\hat{\rho}_N$ in the semi–classical framework.

Finally the one-body operator associated with the electromagnetic vertices, namely the *single nucleon tensor*

$$< \vec{p}\,', s'|W^{\mu\nu}_{sn}\;(\vec{q})|\vec{p}, s > = j^{\mu *}_{s_N\, s'}\;(\vec{p}\,' + \vec{q}, \vec{p}\,')\; j^{\nu}_{s_N\, s}\;(\vec{p} + \vec{q}, \vec{p})\,, \tag{2.21}$$

is the third element which enters into the physics of the electromagnetic exclusive processes. Concerning the latter, we shall assume it to be on the mass–shell. Furthermore we shall discuss in the following Section the condition under which the off–diagonal matrix elements of $W_{sn}^{\mu\nu}$ can be disregarded. The diagonal $W_{sn}^{\mu\nu}$ directly relates to the physical "on shell" electron–nucleon cross–section according to

$$\left(\frac{d\sigma}{d\Omega_e}\right)_{sn} = 2\alpha^2 \frac{1}{Q_\mu^4} \frac{k'}{k} \eta_{\mu\nu} < \vec{p}, s|W_{sn}^{\mu\nu}(\vec{q})|\vec{p}, s > . \tag{2.22}$$

# 3 The semi–classical approach

For the details of the method we refer the reader to ref. [4]. Here it suffices to remind the definition of the Wigner transform (WT) of a one–body operator O, namely

$$O_W(\vec{R}, \vec{p}) = \int \frac{d\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{R}} \left\langle \vec{p} + \frac{\vec{k}}{2} \middle| \hat{O} \middle| \vec{p} - \frac{\vec{k}}{2} \right\rangle \tag{3.1}$$

and

$$< \vec{p}\,|\,\hat{O}\,|\,\vec{p}\,' >= \int d\vec{R}\; e^{i(\vec{p}-\vec{p}\,')\cdot\vec{R}}\; O_W\left(\vec{R}, \frac{\vec{p}+\vec{p}\,'}{2}\right) . \tag{3.2}$$

The core of the semi–classical approach lies in the systematic expansion in $\hbar$ (or in the gradient with respect to $\vec{R}$ or $\vec{p}$) of the Wigner transform of operators; in our case only the leading order of the expansion will be kept: accordingly the Wigner transform of the product of operators reduces to the product of the Wigner transforms of each factor. We shall repeatedly exploit this rule in the following.

Let us now start from the exclusive cross–section (2.10) which we rewrite as follows

$$\begin{aligned} \frac{d^4\sigma}{d\Omega_e d\epsilon' d\vec{p}_N} &= \frac{2\alpha^2}{(2\pi)^3} \frac{1}{Q_\mu^4} \frac{k'}{k} \eta_{\mu\nu} \int \frac{d\vec{p}}{(2\pi)^3} \int \frac{d\vec{p}\,'}{(2\pi)^3} \sum_{s_N, s', s} < \vec{p}\,' + \vec{q}|\hat{\rho}_N|\vec{p} + \vec{q} > \\ &\quad \times < \vec{p}, s|S(\mathcal{E})|\vec{p}\,', s' >< \vec{p}\,', s'|W_{sn}^{\mu\nu}(\vec{q})|\vec{p}, s > . \end{aligned} \tag{3.3}$$

Performing the change of variables

$$\vec{p} = \vec{K} + \frac{\vec{u}}{2} \quad \text{and} \quad \vec{p}\,' = \vec{K} - \frac{\vec{u}}{2} \tag{3.4}$$

and introducing the Wigner transforms, we can recast the above expression into the form

$$\frac{d^4\sigma}{d\Omega_e d\epsilon' d\vec{p}_N} = \frac{2\alpha^2}{(2\pi)^3} \frac{1}{Q_\mu^4} \frac{k'}{k} \eta_{\mu\nu} \int \frac{d\vec{K}}{(2\pi)^3} \int \frac{d\vec{u}}{(2\pi)^3} \sum_{s_N s' s}$$

$$\times \int d\vec{R}\, e^{-i\vec{u}\cdot\vec{R}} \; [\hat{\rho}_{N}]_W \left(\vec{R}, \vec{K}+\vec{q}\right) \; \int d\vec{S}\, e^{i\vec{u}\cdot\vec{S}} \; [S_{ss'}(\mathcal{E})]_W \left(\vec{S}, \vec{K}\right)$$

$$\times \left\langle \vec{K} - \frac{\vec{u}}{2}, s' \;\middle|\; W^{\mu\nu}_{sn}(\vec{q}) \;\middle|\; \vec{K} + \frac{\vec{u}}{2}, s \right\rangle \tag{3.5}$$

$$= \frac{2\alpha^2}{(2\pi)^3} \frac{1}{Q^4_\mu} \frac{k'}{k} \eta_{\mu\nu} \sum_{s's} \int \frac{d\vec{K}}{(2\pi)^3} \int d\vec{R}\, d\vec{S} \; [S_{s's}(\mathcal{E})]_W (\vec{S}, \vec{K})$$

$$\times [\hat{\rho}_{N}]_W (\vec{R}, \vec{K}+\vec{q}) \; \left[W^{\mu\nu}_{s',s}(\vec{q})\right]_W (\vec{S}-\vec{R}, \vec{K})$$

where

$$\left[W^{\mu\nu}_{s',s}(\vec{q})\right]_W (\vec{S}-\vec{R}, \vec{K}) = \int \frac{d\vec{t}}{(2\pi)^3} \, e^{i\vec{t}\cdot(\vec{S}-\vec{R})}$$

$$\times \sum_{s_N} \left[ j^\mu_{s_N s'} \left( \vec{K} + \frac{\vec{t}}{2} + \vec{q} \,, \vec{K} + \frac{\vec{t}}{2} \right) \right]^* j^\nu_{s_N s} \left( \vec{K} - \frac{\vec{t}}{2} + \vec{q} \,, \vec{K} - \frac{\vec{t}}{2} \right)$$

$$\simeq \delta\left(\vec{R}-\vec{S}\right) < \vec{K}, s \mid W^{\mu\nu}_{sn}(\vec{q}) \mid \vec{K}, s > . \tag{3.6}$$

To leading order in $\hbar$ the $t$ dependence in the electromagnetic current can be disregarded. Hence the Wigner transform in eq.(3.6) depends only on the diagonal nucleonic matrix elements. This approximation leads to a local expression for the exclusive cross-section.

By inserting (3.6) into (3.5) one finally obtains for the exclusive cross–section the semi–classical expression

$$\frac{d^4\sigma}{d\Omega_e d\epsilon' d\vec{p}_N} = \frac{2\alpha^2}{(2\pi)^3} \frac{1}{Q^4_\mu} \frac{k'}{k} \eta_{\mu\nu} \int \frac{d\vec{K}}{(2\pi)^3} < \vec{K}, s| \; W^{\mu\nu}_{sn}(\vec{q}) |\vec{K}, s >$$

$$\times \int d\vec{R} \, [S_{ss}(\mathcal{E})]_W (\vec{R}, \vec{K}) \, [\hat{\rho}_{N}]_W (\vec{R}, \vec{K}+\vec{q}) \tag{3.7}$$

$$= \frac{1}{(2\pi)^3} \int \frac{d\vec{K}}{(2\pi)^3} \left(\frac{d\sigma}{d\Omega_e}\right)_{sn} (\vec{K}, \vec{q}) \int d\vec{R} \; [S_{ss}(\mathcal{E})]_W (\vec{R}, \vec{K}) \; [\hat{\rho}_{N}]_W (\vec{R}, \vec{K}+\vec{q})$$

(repeated indices are meant to be summed) where

$$\left[\frac{d\sigma}{d\Omega_e}\right]_{sn} (\vec{p}, \vec{q}) = \sigma_{Mott} \frac{M_N}{E(p)} \frac{M_N}{E(|\vec{p}+\vec{q}\,|)} v_L R_L \tag{3.8}$$

$\sigma_{Mott}$ being the Mott cross–section, $E(p) = \sqrt{p^2 + M^2_N}$,

$$v_L = \left(\frac{Q^2_\mu}{q^2}\right)^2 \equiv \left(\frac{\tau}{\kappa^2}\right)^2 , \tag{3.9}$$

$$R_L = \frac{\kappa^2}{\tau} \left\{ G^2_E(\tau) + W_2(\tau)\chi^2 \right\} \tag{3.10}$$

and

$$W_2(\tau) = \frac{1}{1+\tau}\left[G_E^2(\tau) + \tau G_M^2(\tau)\right] . \tag{3.11}$$

Furthermore the dimensionless variables $\chi = p\sin\theta/M_N$ ($\theta$ being the angle between $\vec{p}$ and $\vec{q}$), $\kappa = q/2M_N$, $\lambda = \omega/2M_N$, $\tau = \kappa^2 - \lambda^2$ are utilized, and $G_E$ and $G_M$ are the Sach's electric and magnetic nucleon's form factors (only the longitudinal part of the cross–section is considered here, for the sake of illustration).

Formula (3.7) clearly shows how the PWIA scheme has been improved. Indeed, in the framework of the PWIA, only one nucleon inside the nucleus, with a given momentum $\vec{K}$, takes part in the exclusive process leading to a final nucleon with momentum $\vec{p}_N = \vec{K} + \vec{q}$, whereas now, because of the FSI embodied in $\hat{\rho}_N$, the momentum of the nucleon actually involved into the process might take any value: hence the integration over the variable $\vec{K}$.

The above formulae also transparently show how the Wigner transform operates: it replaces the off–diagonal momentum matrix elements of the one–body operators entering into the expression for the exclusive cross–section with diagonal "space-dependent" matrix elements. Moreover the WT of the matrix elements of the one–body operators associated with the distortion of the outgoing nucleon wave, the spectral function and the single nucleon cross–section are each evaluated at "different places" into the nucleus and finally their product is integrated over the whole nuclear volume.

This statement is rigorously true when one does not employ the diagonal approximation (3.6). When the latter is adopted, however, then the various ingredients of the exclusive cross–section are actually evaluated at the same place. It thus appears that the exclusive process, in the semi–classical framework, is viewed as being built up "non locally" (but "locally" in the present, leading order approximation) from elementary contributions arising from different regions of the nucleus. The interference among these is neglected in first order, but it can be accounted for through higher order terms in the $\hbar$ expansion.

Before ending this section we write the expression for the semi–classical PWIA cross–section; introducing into (3.3) the explicit form (2.20) for the matrix element of the one–body projection operator and applying again the WT to the spectral function and the single nucleon tensor, one easily ends up with:

$$\left(\frac{d^4\sigma}{d\Omega_e d\epsilon' d\vec{p}_N}\right)_{PWIA} = \frac{1}{(2\pi)^3}\left(\frac{d\sigma}{d\Omega_e}\right)_{sn}(\vec{p}_m, \vec{q}) \int d\vec{R}\, \left[S_{ss}(\mathcal{E})\right]_W (\vec{R}, \vec{p}_m) \tag{3.12}$$

where the so–called missing momentum $\vec{p}_m = \vec{p}_N - \vec{q}$ has been introduced. The above formula clearly shows that in PWIA the cross–section is directly proportional to the spectral function.

In the next two sections we shall calculate the WT of the spectral function and of the distortion operator in leading order.

# 4 The diagonal semi–classical spectral function

Let us consider a closed shell nucleus having $A$ nucleons sitting in the lowest orbits of a potential well V(R) (in principle the Hartree–Fock (HF) mean field) and an $(A-1)$ daughter nucleus obtained by creating a hole in a generic occupied level of the former. The $(A-1)$ nucleus thus obtained will generally be in an excited state with an energy given by (neglecting the small recoil energy)

$$\begin{aligned} E_{A-1} &\equiv E_{A-1,h} \qquad (4.1)\\ &= \sum_{\beta<\epsilon_F} t_\beta + \frac{1}{2}\sum_{\beta,\beta'} < \beta\beta'|v|\beta\beta' >_a + AM_N - t_h - \sum_{\beta} < h\beta|v|h\beta >_a - M_N \end{aligned}$$

in the non–relativistic HF approximation. In (4.1) the first three terms on the RHS correspond to the energy of the nucleus with mass number $A$, $v$ is a suitable two–body interaction and the subtracted terms represent the energy of a particle in the $h$ orbit. Accordingly we can recast (4.1) as follows

$$E_{A-1,h} = M_A - M_N - \epsilon_h\,, \qquad (4.2)$$

which defines the hole energy $\epsilon_h$. Of course in the HF scheme the ground state energy of the $A-1$ daughter nucleus is obtained by removing a particle at the Fermi energy, namely

$$E^0_{A-1} = M_A - M_N - \epsilon_F \equiv M_{A-1}\,. \qquad (4.3)$$

from where the relation $\epsilon_F = -E_S$ follows. We then get for the (positive) excitation energy

$$\mathcal{E} = E_{A-1,h} - M_{A-1} = \epsilon_F - \epsilon_h\,. \qquad (4.4)$$

In the semi–classical approximation the "negative" Fermi energy is

$$\epsilon_F = \frac{k_F^2(R)}{2M_N^*} + V(R) \qquad (4.5)$$

$R$ being the radial variable; a nucleon effective mass $M_N^*$ has been introduced to account, together with the potential well V(R), for the Hartree–Fock mean field. Equation (4.5) locally defines a Fermi momentum $k_F(R)$ for $R \le R_c$, $R_c$ being the classical turning point fixed by the equation

$$\epsilon_F = V(R_c)\,. \qquad (4.6)$$

We now express the spectral function in the basis of the eigenfunctions of the single particle hamiltonian

$$h = \frac{p^2}{2M_N^*} + V(R)\,, \qquad (4.7)$$

rather than in the basis of the momentum eigenfunctions. The former of course obey the equation

$$h|\alpha>=\epsilon_\alpha|\alpha> \tag{4.8}$$

$\epsilon_\alpha$ being the associated eigenvalues.

We thus get

$$\begin{aligned}<\vec{p}\mid S(\mathcal{E})\mid \vec{p}\,'>&=\sum_{\alpha\alpha'}<\vec{p}\,'|\alpha'><A\mid \hat{a}^\dagger_{\alpha'}\,\delta\left(\widehat{\mathcal{H}}-\mathcal{E}\right)\,\hat{a}_\alpha\,|A><\alpha\mid\vec{p}\;>\\ &=\sum_{\alpha\alpha'}<\vec{p}\,'|\;\alpha'><\alpha'\mid\delta\left[\mathcal{E}-(\epsilon_F-\epsilon_\alpha)\right]\theta(\epsilon_F-\epsilon_\alpha)|\;\alpha><\alpha\mid\vec{p}\;>\\ &=<\vec{p}\,'|\;\theta(\epsilon_F-h)\;\delta\left[\mathcal{E}-(\epsilon_F-h)\right]\;|\;\vec{p}\;>\;.\end{aligned} \tag{4.9}$$

By applying then the definition (3.1) of the Wigner transform it is an easy matter to verify that, in the leading order of the $\hbar$ expansion, one has

$$\left[\theta(\epsilon_F-h)\right]_W(\vec{R},\vec{p})=\theta\left[\epsilon_F-\left(\frac{p^2}{2M^*_N}+V(R)\right)\right]+O(\hbar^2) \tag{4.10}$$

and

$$\left\{\delta\left[\mathcal{E}-(\epsilon_F-h)\right]\right\}_W(\vec{R},\vec{p})=\delta\left[\mathcal{E}-\left(\epsilon_F-\frac{p^2}{2M^*_N}-V(R)\right)\right]+O(\hbar^2)\,. \tag{4.11}$$

Hence it follows that, in leading order, the WT of the spectral function is just the product of the WT (4.10) and (4.11), namely

$$\left[S(\mathcal{E})\right]_W(\vec{R},\vec{p})=\theta\left(\epsilon_F-\frac{p^2}{2M^*_N}-V(R)\right)\;\delta\;\left[\mathcal{E}-\left(\epsilon_F-\frac{p^2}{2M^*_N}-V(R)\right)\right] \tag{4.12}$$

or, recalling (4.5),

$$\left[S(\mathcal{E})\right]_W(\vec{R},\vec{p})=\theta\left(\frac{k_F^2(R)}{2M^*_N}-\frac{p^2}{2M^*_N}\right)\;\delta\;\left[\mathcal{E}-\left(\frac{k_F^2(R)}{2M^*_N}-\frac{p^2}{2M^*_N}\right)\right]\;. \tag{4.13}$$

An integration over the whole nucleus yields then for the diagonal spectral function in the semi–classical approximation the expression

$$S(\vec{p},\mathcal{E})=\int d\vec{R}\,\theta\left(\frac{k_F^2(R)}{2M^*_N}-\frac{p^2}{2M^*_N}\right)\;\delta\;\left[\mathcal{E}-\left(\frac{k_F^2(R)}{2M^*_N}-\frac{p^2}{2M^*_N}\right)\right] \tag{4.14}$$

which displays a striking similarity with the one of a Fermi gas. Actually, in leading order, the HF semi–classical approximation of the spectral function for a finite nucleus might be viewed as arising from a superposition of a large set of FG each one characterized by a different $k_F$. The latter is locally defined according to the prescription

$$k_F(R) = \sqrt{2M_N^* \left(\epsilon_F - V(R)\right)}\,. \tag{4.15}$$

Note that, as previously mentioned, there is no interference among the contributions of the different volume elements to the spectral function in leading order.

To illustrate the method we report in the Appendix the calculation of the diagonal semi–classical spectral function for a few specific single particle potentials.

# 5 The distortion operator

The distortion operator

$$\widehat{\rho}_N = \mid \tilde{\vec{p}}_N >< \tilde{\vec{p}}_N \mid \tag{5.1}$$

obeys, in the HF scheme, the equation

$$h\ \widehat{\rho}_N = \frac{p_N^2}{2M_N}\ \widehat{\rho}_N \tag{5.2}$$

where, on the RHS, the free nucleon mass appears.

Taking the WT of the above one gets, in first order,

$$(h\widehat{\rho}_N)_W\ \cong\ (h)_W\ (\widehat{\rho}_N)_W\ =\ \frac{p_N^2}{2M_N}\ (\widehat{\rho}_N)_W \tag{5.3}$$

or, focussing on the dependence upon the variables $\vec{R}$ and $\vec{p}$,

$$\left[(h)_W\ (\vec{R},\vec{p}) - \frac{p_N^2}{2M_N}\right]\ (\widehat{\rho}_N)_W\ (\vec{R},\vec{p}) = 0\,. \tag{5.4}$$

Now since

$$(h)_W\ (\vec{R},\vec{p})\ =\ \frac{p^2}{2M_N^*}\ +\ V(R), \tag{5.5}$$

V(R) being the chosen potential well, one can set

$$(\widehat{\rho}_N)_W\ (\vec{R},\vec{p})\ =\ Z\ (\vec{R},\vec{p};\vec{p}_N)\ \delta\ \left(\frac{p_N^2}{2M_N} - V(R)\ -\ \frac{p^2}{2M_N^*}\right) \tag{5.6}$$

where the factor $Z(\vec{R},\vec{p};\vec{p}_N)$ can be partially fixed by closure.

The latter indeed requires

$$\int \frac{d\vec{p}_N}{(2\pi)^3} \mid \tilde{\vec{p}}_N >< \tilde{\vec{p}}_N \mid\ + \sum_\alpha\ \mid \alpha >< \alpha \mid \theta(-h) = 1 \tag{5.7}$$

the sum being extended over the bound HF orbits. In Wigner transform (5.7) becomes

$$\int \frac{d\vec{p}_N}{(2\pi)^3} \, (\widehat{\rho}_N)_W(\vec{R},\vec{p}) = \int \frac{d\vec{p}_N}{(2\pi)^3} Z(\vec{R},\vec{p};\vec{p}_N) \, \delta \, \left( \frac{p_N^2}{2M_N} - V(R) - \frac{p^2}{2M_N^*} \right)$$
$$= \theta \left( \frac{p^2}{2M_N^*} + V(R) \right) \qquad (5.8)$$

or, after performing the integration over the modulus of $\vec{p}_N$,

$$M_N p_N(R,p) \int \frac{d\widehat{p}_N}{(2\pi)^3} \, Z\Big(\vec{R},\vec{p};\vec{p}_N(R,p)\Big) = \theta \left( \frac{p^2}{2M_N^*} + V(R) \right) \qquad (5.9)$$

the integral being over the direction of $\vec{p}_N(R,p) \equiv p_N(R,p)\widehat{p}_N$, with

$$p_N(R,p) = \sqrt{\frac{M_N}{M_N^*} \, p^2 + 2M_N V(R)} \, . \qquad (5.10)$$

In a quantum framework one would obtain the states $|\widetilde{\vec{p}_N} >$ from the positive energy solutions of an appropriate optical potential: the distortion operator, as well as its Wigner transform (5.6), would then be fixed. Here, in the spirit of the semi–classical approach, we heuristically set

$$Z\left(\vec{R},\vec{p};\vec{p}_N(R,p)\right) = \frac{(2\pi)^3}{M_N p_N(R,p)} \, F(\widehat{p}_N,\widehat{p}) \, \theta \left( \frac{p^2}{2M_N^*} + V(R) \right) , \qquad (5.11)$$

with the additional condition

$$\int d\widehat{p}_N \, F(\widehat{p}_N,\widehat{p}) = 1 \, . \qquad (5.12)$$

As an illustration we shall consider in the following two extreme assumptions for $F$, namely:

- $$F(\widehat{p}_N,\widehat{p}) = \delta\left(\widehat{p}_N - \widehat{p}\right) \qquad (5.13)$$

  which corresponds to the *eikonal approximation* and therefore is expected to be valid only for large enough energies of the outgoing nucleon. In this case in fact the final nucleon should be little deflected from the direction of the initial one, which has absorbed the photon inside the nucleus;

- $$F(\widehat{p}_N,\widehat{p}) = \frac{1}{4\pi} \qquad (5.14)$$

  which obviously corresponds to a final nucleon escaping the nucleus with the same probability in all directions (*uniform approximation*).

The true physics should of course lie in between the predictions of (5.13) and (5.14), respectively.

## 6 The semi–classical exclusive cross–section

In this Section we derive explicit expressions for the exclusive cross–section in the semi–classical approximation, treating the distortion of the outgoing nucleon wave in the HF approximation as discussed in Section 5. We use the harmonic oscillator (cut at the classical turning point) and the Woods–Saxon wells.

To settle the basis for this scope, we insert into the expression of the exclusive cross–section (3.7) the distortion operator as given by (5.6) and (5.11). Then the $\delta$–function appearing into the latter allows us to perform the integration over the modulus of the momentum variable. We thus obtain:

$$\begin{aligned}\frac{d^4\sigma}{d\Omega_e d\epsilon' d\vec{p}_N} &= \frac{1}{(2\pi)^3}\left(\frac{M_N^*}{M_N}\right)\int d\hat{p}\int d\vec{R}\left[\frac{d\sigma}{d\Omega_e}\right]_{sn}(\mathcal{P}(R)\hat{p}-\vec{q},\vec{q}) \\ &\times [S_{ss}(\mathcal{E})]_W\,(\vec{R},\mathcal{P}(R)\hat{p}-\vec{q})\frac{\mathcal{P}(R)}{p_N}F(\hat{p}_N,\hat{p}) \qquad (6.1)\end{aligned}$$

with

$$\mathcal{P}(R) = \sqrt{\frac{M_N^*}{M_N}}\sqrt{p_N^2 - 2M_N V(R)}\,. \qquad (6.2)$$

To proceed further we exploit the general expression (4.12) for the semi–classical spectral function in Wigner transform. Then (6.1) can be recast as follows

$$\begin{aligned}\frac{d^4\sigma}{d\Omega_e d\epsilon' d\vec{p}_N} &= \frac{1}{2\pi^2}\left(\frac{M_N^*}{M_N}\right)\int R^2 dR\,\frac{\mathcal{P}(R)}{|\vec{q}-\vec{p}_m|}\int d\hat{p}\,F\left(\widehat{\vec{q}-\vec{p}_m},\hat{p}\right)\left[\frac{d\sigma}{d\Omega_e}\right]_{sn}(\mathcal{P}(R)\hat{p}-\vec{q},\vec{q}) \\ &\times\delta\left[\mathcal{E}-\left(\epsilon_F - \frac{(\vec{q}-\vec{p}_m)^2}{2M_N} - \frac{q^2}{2M_N^*} + \frac{\mathcal{P}(R)}{M_N^*}\hat{p}\cdot\vec{q}\right)\right]\theta(\mathcal{E}) \qquad (6.3)\end{aligned}$$

where the trivial integration over the angles of the vector $\vec{R}$ has been performed and the "missing momentum" variable $\vec{p}_m = \vec{q} - \vec{p}_N$ has been used.

We now separately investigate the exclusive cross section in the two limiting approximations for the "distortion function" $F$ discussed at the end of Section 5 (for simplicity here and in the following we shall ignore the effective mass, setting $M_N^* = M_N$).

To start with we consider the *eikonal approximation* [formula (5.13)]. In this case it is straightforward to obtain the following one–dimensional integral expression for the exclusive cross–section

$$\begin{aligned}\frac{d^4\sigma}{d\Omega_e d\epsilon' d\vec{p}_N} &= \frac{1}{2\pi^2}\int_0^{R_c} R^2 dR\frac{\mathcal{P}(R)}{|\vec{q}-\vec{p}_m|}\left[\frac{d\sigma}{d\Omega_e}\right]_{sn}\left(\frac{\mathcal{P}(R)(\vec{q}-\vec{p}_m)}{|\vec{q}-\vec{p}_m|}-\vec{q},\vec{q}\right) \\ &\times\delta\left\{\mathcal{E}-\left[\epsilon_F - \frac{p_m^2}{2M_N} + \left(\frac{q^2-\vec{q}\cdot\vec{p}_m}{M_N}\right)\left(\frac{\mathcal{P}(R)}{|\vec{q}-\vec{p}_m|}-1\right)\right]\right\}\theta(\mathcal{E})\,. \qquad (6.4)\end{aligned}$$

Note that the "exclusive variables" $\mathcal{E}$ and $p_m$ appear explicitly in (6.4); among the "inclusive" ones only $q$ does, whereas the transferred energy $\omega$ is hidden in the scalar product $\vec{q}\cdot\vec{p}_m$. Moreover the upper limit of the $R$–integration in (6.4) is set by the equation (4.6), which clearly entails $V(R_c) < 0$ and $\sqrt{1 - 2M_{\scriptscriptstyle N}V(R)/(\vec{q}-\vec{p}_m)^2} > 1$. Hence, since the hole energy

$$\epsilon_h = \frac{p_m^2}{2M_{\scriptscriptstyle N}} - \frac{q^2 - \vec{q}\cdot\vec{p}_m}{M_{\scriptscriptstyle N}}\left(\frac{\mathcal{P}(R)}{|\vec{q}-\vec{p}_m|} - 1\right) \tag{6.5}$$

is obviously negative, it follows that in the semi–classical eikonal approximation the projection of the missing momentum $\vec{p}_m$ on the momentum transfer $\vec{q}$ has to be less than $q$.

It remains now to exploit the energy conserving $\delta$–function in (6.4) to perform the $R$–integration. This is easily achieved by taking advantage of the identity

$$\delta\left[\mathcal{E} - (\epsilon_F - \epsilon_h)\right] = \frac{(\vec{q}-\vec{p}_m)^2}{(q^2 - \vec{q}\cdot\vec{p}_m)}\,\frac{\delta(R-\tilde{R})}{|dV/dR|}\,\frac{\mathcal{P}(R)}{|\vec{q}-\vec{p}_m|}\,, \tag{6.6}$$

$\tilde{R}$ being the root of the equation

$$\mathcal{A}(\tilde{R}) = \mathcal{A} \equiv \frac{\mathcal{E} - \epsilon_F + p_m^2/2M_{\scriptscriptstyle N}}{(q^2 - \vec{q}\cdot\vec{p}_m)/M_{\scriptscriptstyle N}} + 1 \tag{6.7}$$

where

$$\mathcal{A}(R) \equiv \frac{\mathcal{P}(R)}{|\vec{q}-\vec{p}_m|} = \sqrt{1 - \frac{V(R)}{(\vec{q}-\vec{p}_m)^2/2M_{\scriptscriptstyle N}}}\,. \tag{6.8}$$

One then obtains the following "analytic" expression for the semi–classical exclusive cross–section in the eikonal approximation for the mean field distortion operator:

$$\frac{d^4\sigma}{d\Omega_e d\epsilon' d\vec{p}_{\scriptscriptstyle N}} = \frac{1}{2\pi^2}\tilde{R}^2\mathcal{A}^2(\tilde{R})\frac{1}{|dV/dR|_{R=\tilde{R}}}\,\frac{(\vec{q}-\vec{p}_m)^2}{(q^2 - \vec{q}\cdot\vec{p}_m)}\left[\frac{d\sigma}{d\Omega_e}\right]_{sn}\left[\mathcal{A}(\vec{q}-\vec{p}_m) - \vec{q},\vec{q}\right]. \tag{6.9}$$

For a full exploitation of formula (6.9) an expression for the cosine of the angle between $\vec{p}_m$ and $\vec{q}$ is still needed: it is most easily obtained by applying the energy conservation to the right sector of the diagram displayed in Fig. 2. One gets:

$$\cos\theta_{p_m q} = \frac{M_{\scriptscriptstyle N}}{qp_m}\left(\mathcal{E} + E_S - \omega\right) + \frac{1}{2}\left[\frac{p_m}{q}\left(1 + \frac{M_{\scriptscriptstyle N}}{M_{A-1}}\right) + \frac{q}{p_m}\right]. \tag{6.10}$$

Choosing now for $V(R)$ the harmonic oscillator potential, as given by formula (A.7) with the bare nucleon's mass, we easily get the following cross–section

$$\frac{d^4\sigma}{d\Omega_e d\epsilon' d\vec{p}_{\scriptscriptstyle N}} = \frac{1}{\sqrt{2}\pi^2}\frac{1}{M_{\scriptscriptstyle N}^{3/2}\omega_o^3}\left\{\frac{(\vec{q}-\vec{p}_m)^2}{2M_{\scriptscriptstyle N}}\left(1 - \mathcal{A}^2\right) + V_o\right\}^{1/2}$$
$$\times\mathcal{A}^2\frac{(\vec{q}-\vec{p}_m)^2}{(q^2 - \vec{q}\cdot\vec{p}_m)}\left[\frac{d\sigma}{d\Omega_e}\right]_{sn}\left[\mathcal{A}(\vec{q}-\vec{p}_m) - \vec{q},\vec{q}\right] \tag{6.11}$$

where the solution of (6.7) is

$$\widetilde{R} = \frac{1}{\omega_o}\sqrt{\frac{2}{M_{\scriptscriptstyle N}}\left[\frac{(\vec{q}-\vec{p}_m)^2}{2M_{\scriptscriptstyle N}}\left(1-\mathcal{A}^2\right)+V_o\right]}\,. \tag{6.12}$$

For a Woods–Saxon well

$$V(R) = -\frac{V_1}{1+e^{(R-R_o)/a}}\,, \tag{6.13}$$

formula (6.9) holds with

$$\widetilde{R} = R_o + a\ln\left\{\frac{2M_{\scriptscriptstyle N}V_1}{(\vec{q}-\vec{p}_m)^2\left(\mathcal{A}^2-1\right)}-1\right\}\,. \tag{6.14}$$

Next we turn to the *uniform approximation* for the distortion function $F$ [formula (5.14)]. In this case a fully analytic expression for the exclusive cross–section cannot be achieved. Indeed by exploiting the $\delta$–function of the distortion operator and the azimuthal angle independence of the elementary single nucleon cross–section, one gets

$$\begin{aligned}\frac{d^4\sigma}{d\Omega_e d\epsilon' d\vec{p}_{\scriptscriptstyle N}} &= \frac{1}{(2\pi)^2}\int_0^{R_c} R^2 dR\,\frac{\mathcal{P}(R)}{|\vec{q}-\vec{p}_m|}\int d\cos\theta\left[\frac{d\sigma}{d\Omega_e}\right]_{sn}\left(\mathcal{P}(R)\hat{p}-\vec{q},\vec{q}\right)\\ &\times\delta\left\{\mathcal{E}-\left[\epsilon_F-\frac{(\vec{q}-\vec{p}_m)^2}{2M_{\scriptscriptstyle N}}-\frac{q^2}{2M_{\scriptscriptstyle N}}+\frac{\mathcal{P}(R)}{M_{\scriptscriptstyle N}}\,q\cos\theta\right]\right\}\end{aligned} \tag{6.15}$$

where $\mathcal{P}(R)$ and $R_c$ are again fixed by eq.(6.2) and (4.6), and $\theta$ is the angle between $\vec{q}$ and $\vec{p}$. The integration over the latter variable is trivial and one finally obtains for the semi–classical exclusive cross–section, in the uniform approximation for the distortion operator, the following one–dimensional integral expression

$$\begin{aligned}\frac{d^4\sigma}{d\Omega_e d\epsilon' d\vec{p}_{\scriptscriptstyle N}} &= \frac{1}{(2\pi)^2}\frac{M_{\scriptscriptstyle N}}{q}\frac{1}{|\vec{q}-\vec{p}_m|}\int_0^{R_c} R^2 dR\\ &\times\left[\frac{d\sigma}{d\Omega_e}\right]_{sn}\left(\mathcal{P}(R)\hat{p}-\vec{q},\vec{q}\right)\Big|_{\cos\theta=y_0(R)}\theta\left(1-|y_0(R)|\right)\end{aligned} \tag{6.16}$$

where

$$y_0(R) = \frac{M_{\scriptscriptstyle N}}{\mathcal{P}(R)q}\left[\mathcal{E}-\left(\epsilon_F-\frac{(\vec{q}-\vec{p}_m)^2}{2M_{\scriptscriptstyle N}}-\frac{q^2}{2M_{\scriptscriptstyle N}}\right)\right]\,. \tag{6.17}$$

Notably the one–body potential confining the nucleons into the nucleus does not explicitly appear in (6.16): it is however hidden in the equations fixing $R_c$, $\epsilon_F$ and $\mathcal{P}(R)$.

## 7 The semi–classical $t$–inclusive cross–section

To get the inclusive cross–section in the $t$–channel one should integrate the exclusive one, obtained in the previous Section, over the outgoing nucleon's momentum. Indeed in the $t$–inclusive scattering only the final electron is detected: accordingly the momentum $\vec{q}$ transferred to the nucleus is kept fixed in the process. By contrast in the $u$–channel, where the outgoing nucleon only is detected, the vector $\vec{\xi} = \vec{p}_{_N} - \vec{k}$ is kept fixed, whereas $\vec{q}$ varies.

We shall apply the semi–classical formalism to the $u$–inclusive scattering in a forthcoming paper: here we focuss instead on the $t$–inclusive channel, where the vast majority of the electron scattering experiments have been performed.

To start with we first show that by integrating the exclusive cross–section over the momentum of the emitted nucleon we recover the correct inclusive cross–section if and only if the distortion of the outgoing particle is properly accounted for. For this purpose the integral of (3.7) over $\vec{p}_{_N}$,

$$\begin{aligned}\frac{d^2\sigma}{d\Omega_e d\epsilon'} &= \int \frac{d\vec{p}_{_N}}{(2\pi)^3} \int \frac{d\vec{K}}{(2\pi)^3} \left(\frac{d\sigma}{d\Omega_e}\right)_{sn} \left(\vec{K}, \vec{q}\right) \\ &\times \int d\vec{R} \left[S_{ss}(\mathcal{E})\right]_W \left(\vec{R}, \vec{K}\right) (\widehat{\rho}_{_N})_W \left(\vec{R}, \vec{K} + \vec{q}\right), \end{aligned} \tag{7.1}$$

is carried out by exploiting the semi–classical expressions (4.13) for the spectral function and (4.5) for the Fermi energy. We get

$$\begin{aligned}\frac{d^2\sigma}{d\Omega_e d\epsilon'} &= \int \frac{d\vec{p}_{_N}}{(2\pi)^3} \int \frac{d\vec{K}}{(2\pi)^3} \left(\frac{d\sigma}{d\Omega_e}\right)_{sn} \left(\vec{K}, \vec{q}\right) \int d\vec{R} (\rho_{_N})_W \left(\vec{R}, \vec{K} + \vec{q}\right) \\ &\times \theta \left[\epsilon_F - \frac{K^2}{2M_N^*} - V(R)\right] \delta \left\{\mathcal{E} - \left[\epsilon_F - \frac{K^2}{2M_N^*} - V(R)\right]\right\} \end{aligned} \tag{7.2}$$

Now, by taking advantage of the $\delta$ explicitly embodied in the distortion operator [see (5.6)] and since (2.15) implies (neglecting the recoil energy)

$$\mathcal{E} = \omega + \epsilon_F - V(R) - \frac{(\vec{q} + \vec{K})^2}{2M_N^*}\,, \tag{7.3}$$

we obtain

$$\begin{aligned}\frac{d^2\sigma}{d\Omega_e d\epsilon'} = &\int \frac{d\vec{p}_{_N}}{(2\pi)^3} \int \frac{d\vec{K}}{(2\pi)^3} \left(\frac{d\sigma}{d\Omega_e}\right)_{sn} \left(\vec{K}, \vec{q}\right) \int d\vec{R}\, (\widehat{\rho}_{_N})_W \left(\vec{R}, \vec{K} + \vec{q}\right) \\ &\times \theta \left[\epsilon_F - \frac{K^2}{2M_N^*} - V(R)\right] \delta \left\{\omega - \left(\frac{q^2}{2M_N^*} + \frac{\vec{q}\cdot\vec{K}}{M_N^*}\right)\right\}. \end{aligned} \tag{7.4}$$

Then the integration over $\vec{p}_N$ is immediately done with the help of (5.8) and we end up with

$$\frac{d^2\sigma}{d\Omega_e d\epsilon'} = \int \frac{d\vec{K}}{(2\pi)^3} \int d\vec{R} \left(\frac{d\sigma}{d\Omega_e}\right)_{sn} \left(\vec{K}, \vec{q}\right) \theta \left[\frac{|\vec{K}+\vec{q}|^2}{2M_N^*} + V(R)\right]$$
$$\times \theta \left[\epsilon_F - \frac{K^2}{2M_N^*} - V(R)\right] \delta \left\{\omega - \left(\frac{q^2}{2M_N^*} + \frac{\vec{q}\cdot\vec{K}}{M_N^*}\right)\right\}, \tag{7.5}$$

which is the well–known semi–classical expression for the inclusive cross–section, limited however to the continuum spectrum for the emitted nucleon. The internal consistency of the semi–classical approach is thus proved. In connection with this result we note that it extends the PWIA of ref.[2] where it is shown that for the fully quantum mechanical relativistic Fermi gas the integral of the exclusive cross–section leads to the inclusive one only in the non–Pauli blocked domain.[1] The expression (7.5) instead fully respects the Pauli principle: of course it does not account for the contribution to the inclusive cross section arising from the unoccupied bound states lying in between the Fermi energy and the continuum.

For the actual evaluation of (7.5), however, we choose here to follow the approach of integrating over the "exclusive" variables $\mathcal{E}$ and $p_m$. For this purpose we observe that, owing to the independence of the exclusive cross–section upon the azimuthal angle of $\vec{p}_N$, the integration over the latter can be converted into an integration over an appropriate domain of the missing energy – missing momentum plane. Indeed the following relationship holds:

$$\frac{d^2\sigma}{d\Omega_e d\epsilon'} = 2\pi \frac{M_N}{q} \int_\Gamma p_m dp_m \, d\mathcal{E} \frac{d^4\sigma}{d\Omega_e d\epsilon' d\vec{p}_N}, \tag{7.6}$$

the boundaries of the integration domain $\Gamma$ being given (in the positive quadrant of the $(\mathcal{E}, p_m)$ plane) by the curves

$$\mathcal{E}^- = \omega - E_S - \frac{(q-p_m)^2}{2M_N} - \frac{p_m^2}{2M_{A-1}} \tag{7.7}$$

$$\mathcal{E}^+ = \omega - E_S - \frac{(q+p_m)^2}{2M_N} - \frac{p_m^2}{2M_{A-1}}, \tag{7.8}$$

[1] The same result is obtained in the present framework, when the PWIA for $\hat{\rho}_N$, expression (2.20), is employed; from the latter, indeed, one gets:

$$\frac{d^2\sigma}{d\Omega_e d\epsilon'} = \int \frac{d\vec{K}}{(2\pi)^3} \int d\vec{R} \left(\frac{d\sigma}{d\Omega_e}\right)_{sn} \left(\vec{K}, \vec{q}\right)$$
$$\times \theta \left[\epsilon_F - \frac{K^2}{2M_N^*} - V(R)\right] \delta \left\{\omega - \left(\frac{q^2}{2M_N^*} + \frac{\vec{q}\cdot\vec{K}}{M_N^*}\right)\right\}.$$

which are obtained by setting $\cos\theta_{p_m q} = -1$ and $\cos\theta_{p_m q} = +1$, respectively, in equation (6.10). They represent the larger ($\mathcal{E}^-$) and the lower ($\mathcal{E}^+$) excitation energy of the residual nucleus compatible with the kinematical constraints in an exclusive process.

We remind that, while $\mathcal{E}^-$ always extends to the first quadrant of the $(\mathcal{E}, p_m)$ plane, this might not be the case for $\mathcal{E}^+$. Indeed, for $\omega \leq q^2/2M_N + E_S \equiv \widetilde{\omega}$, $\mathcal{E}^+$ lives entirely outside the first quadrant: in this case the lower limit of integration over the variable $\mathcal{E}$ should simply be set equal to zero. On the other hand, for $\omega \geq \widetilde{\omega}$, $\mathcal{E}^+$ extends to the first quadrant as well.

Let's first consider the eikonal approximation for the distortion function, with the harmonic oscillator as a binding potential. In this instance we obtain for the semi–classical inclusive cross–section:

$$\begin{aligned}\frac{d^2\sigma}{d\Omega_e d\epsilon'} &= \sqrt{\frac{2}{M_N}}\frac{1}{\pi q \omega_o^3}\left\{\theta\left(\omega-\widetilde{\omega}\right)\left[\int_0^{p^+_{m,max}} p_m dp_m \int_{\mathcal{E}^+}^{\mathcal{E}^-} d\mathcal{E} + \int_{p^+_{m,max}}^{p^-_{m,max}} p_m dp_m \int_0^{\mathcal{E}^-} d\mathcal{E}\right]\right. \\ &\quad \left. +\theta\left(\widetilde{\omega}-\omega\right)\int_{p^-_{m,min}}^{p^-_{m,max}} p_m dp_m \int_0^{\mathcal{E}^-} d\mathcal{E}\right\} \\ &\quad \times\left\{\frac{(\vec{q}-\vec{p}_m)^2}{2M_N}\left[1-\mathcal{A}^2(\mathcal{E},p_m)\right]+V_o\right\}^{1/2}\mathcal{A}^2(\mathcal{E},p_m) \\ &\quad \times\frac{(\vec{q}-\vec{p}_m)^2}{(q^2-\vec{q}\cdot\vec{p}_m)}\left[\frac{d\sigma}{d\Omega_e}\right]_{sn}[\mathcal{A}(\mathcal{E},p_m)(\vec{q}-\vec{p}_m)-\vec{q},\vec{q}]\end{aligned} \tag{7.9}$$

where one should recall that $\mathcal{A}$, defined in eq. (6.7), explicitly depends upon $\mathcal{E}$ and $p_m$. A similar expression holds for the Woods–Saxon potential well, the integration of the corresponding exclusive cross–section extending over the same $(\mathcal{E}, p_m)$ domain.

The limits on the missing momentum variable appearing in the above integrals are most easily deduced by setting $\mathcal{E}^- = 0$, which yields

$$p^-_{m,min} = \frac{1}{1+M_N/M_{A-1}}\left(q-\sqrt{q^2-\left(1+\frac{M_N}{M_{A-1}}\right)[q^2-2M_N(\omega-E_S)]}\right) \tag{7.10}$$

and

$$p^-_{m,max} = \frac{1}{1+M_N/M_{A-1}}\left(q+\sqrt{q^2-\left(1+\frac{M_N}{M_{A-1}}\right)[q^2-2M_N(\omega-E_S)]}\right) \tag{7.11}$$

and $\mathcal{E}^+ = 0$, which yields:

$$p^+_{m,min} = -p^-_{m,max} \qquad \text{(always negative)} \tag{7.12}$$
$$p^+_{m,max} = -p^-_{m,min}\,. \tag{7.13}$$

In connection with these formulas we remind the reader that the scaling variable $y$ is customarily defined as the opposite of the lower limit of the integration over the missing momentum variable. One thus sees that $y$ vanishes at $\omega = \widetilde{\omega}$, being positive (negative) for frequencies smaller (larger) than $\widetilde{\omega}$.

We turn now to evaluate the $t$ inclusive cross–section in the uniform approximation for the distortion function $F$ [see eqs. (5.14) and (6.17)]. In this case we get

$$\frac{d^2\sigma}{d\Omega_e d\epsilon'} = \frac{1}{2\pi}\frac{M_N}{q}\left\{\theta\left(\omega-\widetilde{\omega}\right)\left[\int_0^{p^+_{m,max}} p_m dp_m \int_{\mathcal{E}^+}^{\mathcal{E}^-} d\mathcal{E} + \int_{p^+_{m,max}}^{p^-_{m,max}} p_m dp_m \int_0^{\mathcal{E}^-} d\mathcal{E}\right] + \right.$$
$$\left. +\theta\left(\widetilde{\omega}-\omega\right)\int_{p^-_{m,min}}^{p^-_{m,max}} p_m dp_m \int_0^{\mathcal{E}^-} d\mathcal{E}\right\} \qquad (7.14)$$
$$\times\frac{1}{|\vec{q}-\vec{p}_m|}\int_0^{R_c} R^2 dR \left[\frac{d\sigma}{d\Omega_e}\right]_{sn} \left(\mathcal{P}(R)\hat{p}-\vec{q},\vec{q}\right)\Big|_{\cos\theta=y_0}\theta\left(1-|y_0|\right)$$

where $\mathcal{P}(R)$ is again given by (6.2) and $y_0$ by (6.17), while the upper limit of the integral over $R$ is found by solving the equation

$$V(R_c) = \frac{(\vec{q}-\vec{p}_m)^2}{2M_N} \qquad (7.15)$$

which, for the harmonic oscillator potential, yields

$$R_c = \frac{1}{\omega_o}\sqrt{\frac{1}{M_N}}\left\{2\left[V_o + (\omega-\mathcal{E}-E_S)\right] - \frac{p_m^2}{M_{A-1}}\right\} . \qquad (7.16)$$

Analogous expressions hold for the Woods–Saxon well.

# 8 Results

In this section the predictions of our theory are numerically appraised. We first consider the exclusive cross–sections. They are displayed in Fig. 3a–d as a function of the missing momentum for various missing energies at $q = 300$ MeV/c (Figs. 3a and 3b) and at $q = 500$ MeV/c (Figs. 3c and d); results both in the eikonal (Figs. 3a and c) and in the uniform (Figs. 3b and d) approximation for the distortion of the outgoing nucleon are shown. The energy transfer $\omega$ has been choosen here to be close to the quasi–elastic peak ($\omega \simeq q^2/2M_N + E_S$). All the figures displayed in this first set refer to the harmonic oscillator well, with $V_o = 55$ MeV and $\hbar\omega_0 = 41/A^{1/3} \simeq 12$ MeV for the $^{40}$Ca nucleus.

The corresponding results for the Woods–Saxon well (in the same kinematical conditions and in the two approximations employed for the distortion operator) are shown in Figs. 4a–d; we use $V_1 = 49.8$ MeV, $R_o = 1.2A^{1/3}$ fm and $a = 0.65$ fm.

A few features are worth commenting: one is related to the striking differences between the exclusive cross–sections evaluated in the uniform and in the eikonal approximations. In the first case the cross–sections are seen to be fairly constant over the whole range of the kinematically allowed values of the missing momentum $p_m$, *independently* from the potential well binding the nucleons into the nucleus. In the second case the cross–sections show a tendency to peak at low missing momenta and to fall off quite rapidly; moreover they turn out to be restricted to a rather limited range of $p_m$, much smaller than the one allowed by the pure kinematics. These outcomes clearly reflect the drastically different distortion functions $F$ of the two cases.

A second feature refers to the exclusive cross–section in the eikonal approximation, which is quite sensitive to the shape of the potential well: indeed in this approximation the outgoing particle keeps memory of the initial momentum and hence of the spectral function inside the nucleus. The eikonal cross–sections turn out in fact to reflect quite closely the structure of $S(p, \mathcal{E})$, which is illustrated in the Appendix A.

The effects of the distortion operator can be even better appreciated by comparing the above exclusive cross–sections with the corresponding ones, evaluated in PWIA [see equation (3.12)]: this is done in Figs. 5a and 5b, for the harmonic oscillator and the Woods–Saxon well, respectively. In both cases $q = 300$ MeV/c, $\omega$ is close to the quasi–elastic peak and $\mathcal{E} = 10$ MeV/c. The PWIA turns out to be closer, but still appreciably different from the distorted cross–section evaluated in eikonal approximation, spanning a range of missing momenta ($p_m$) larger than in the eikonal approximation itself, but much smaller than in the uniform approximation. The sensitivity to the shape of the mean field is strong also in PWIA.

Concerning the inclusive cross–sections, they are displayed in Figs. 6a and 6b at $q = 200$ MeV/c and $q = 500$ MeV/c, respectively, for both the harmonic oscillator and the Woods–Saxon well; they are obtained via the integration in the $(\mathcal{E}, p_m)$ plane of the exclusive cross–section, using for the single nucleon cross–section $\sigma_{Mott}$ only. The calculation has been performed both in the uniform and in the eikonal approximations, which however cannot be distinguished in the figures, owing to their identity. The expected numerical coincidence of the inclusive cross–sections obtained by integrating the two, markedly different, exclusive cross–sections is thus seen to be realized, providing that higher order relativistic effects in the single nucleon cross–section are ignored, which in turn amounts to keep only the Mott cross–section. When however the fully relativistic expression for the single nucleon cross–section, eq.(3.8), is employed, then the above mentioned coincidence between inclusive cross–sections no longer holds, especially near the maximum, but the discrepancy remains mild up to $q = 500$ MeV/c, as illustrated in Fig. 7a and 7b, which exhibit the same cases as in Fig. 6a and 6b.

Also displayed in Figs. 6a and 6b are the inclusive cross–sections obtained through the polarization propagator method. These differ in the low energy side from the cross sections produced by the integration of the exclusive processes in the missing energy–missing momentum plane: at low $q$ the difference is sizeable, but it becomes negligible as $q$ increases. In general the inclusive cross sections obtained through the polarization propagator turn out to be larger, as discussed in the introduction (we remind that the difference arises from the bound unoccupied orbits). The effect disappears at large $q$, as it should.

As a consequence of these findings, we confirm that little can be learned from the inclusive electron scattering, treated within the mean field approximation, about the mechanism responsible for the distortion of the outgoing nucleon wave [18]. Furthermore only a very weak dependence of the inclusive cross–sections on the shape of the potential well is observed.

Accordingly one is lead to conclude that while the nucleon–nucleon correlations ( of short and long range) in the initial state affect both the inclusive and exclusive inelastic electron scattering, other details of the dynamics of the emitted nucleon, mainly reflecting the shape of the mean field and how it affects the way of the nucleon out of the nucleus, are more conveniently studied with exclusive processes.

# 9 Conclusions

In this paper we have applied the semiclassical approach to the exclusive electron scattering, going beyond the PWIA. Indeed the FSI has been accounted for in the mean field approximation including antisymmetrization.

A few outcomes of the present study are worth to be recalled: the first one is the strong sensitivity of the exclusive process to the distortion of the outgoing nucleon wave: the stronger is the latter, the weakest are the remnants of the nuclear mean field in the cross–section, as it happens for the rather extreme situation described by the uniform approximation. A "weak" distortion, like the one entailed by the eikonal approximation, yields cross–sections more strictly related to the spectral function and hence closer to the PWIA, as it should: in this framework we have shown that the exclusive cross–sections are quite different in shape, whether we adopt the harmonic oscillator or the Woods–Saxon well for the nuclear mean field.

A second interesting result relates to the strong correlation between the distortion mechanism and the domain, in the missin energy – missing momentum plane, where the exclusive cross–section exists. We have found that in the eikonal approximation this domain is quite restricted, whereas in the uniform one it essentially covers the whole kinematically allowed region.

Concerning the inclusive cross–section, obtained by integrating the exclusive ones in the appropriate domain of the $(\mathcal{E}, p_m)$ plane, we have shown that they are quite insensitive both to the distortion of the outgoing nucleon, as we prove also with an analytic calculation, and to the specific shape of the mean binding field, in agreement with previous direct evaluations and measurements of the quasi–elastic inclusive process. Yet we should mention a small difference between the inclusive cross–section obtained by integrating, as discussed above, the exclusive ones, and the ones obtained by a direct calculation based on the so–called polarization propagator. Indeed, in this last instance, bound excited states are embodied, which are not allowed to the outgoing nucleon in exclusive processes. Finally we have observed, in the inclusive cross–section, a mild dependence upon the distortion mechanism when higher order relativistic effects are taken into account in the expression of the single nucleon cross–section: this discrepancy, which is quite small up to the largest momenta ($500 \div 600$ MeV/c) considered here, signals the need of a fully consistent relativistic approach, both in the currents and in the mean field utilized to describe the nucleon dynamics inside the nucleus.

We have found that the semiclassical method yields sensible results, but of course the validity of the scheme can only be ultimately assessed by testing it against both the experiment and fully quantum mechanical calculations.

Yet we consider the semiclassical method attractive by itself on three counts, namely:

1. for its simplicity,
2. for allowing to grasp the role of the FSI in a remarkably transparent way (this follows either by comparing the eikonal with the uniform approximations for the distortion or by a comparison with the PWIA),
3. for retaining in the exclusive process the simplicity of the Fermi gas model but taking into account the local features of the nuclear mean field through the folding of Fermi gases at different densities. Quantum mechanical interferences between different points in the nucleus, not considered here, occur in the second and higher orders of the $\hbar$ expansion.

With reference to point 2 we note that in our formalism the mechanism of the distortion is embedded into the function $F(\hat{p}_N, \hat{p})$ of Section 5, at variance with traditional (quantum mechanical) approaches, where it is included in the distorted wave function of the outgoing nucleon.

Clearly, before attempting to test the semiclassical approach against the experimental data, a realistic expression for the function $F$ should be developed: indeed the ones discussed in this paper were just meant as an illustration.

Concerning point 3 it is worth pointing out the correspondence with the quantum mechanical approach of ref.[2], where a different Fermi gas is associated with each individual orbit of the nucleons inside the nucleus rather than with elementary volumes into which the nucleus is ideally split.

In the present work we have left out important dynamical effects arising from

1. the nucleon–nucleon correlations
2. the meson exchange currents
3. the two–step processes, which occur when the nucleon which has absorbed the photon scatters, in its way out, with another one in the nucleus or when a second nucleon is emitted from the excited daughter nucleus and comes out almost simultaneously to the one directly hit by the impinging photon. These mechanisms can of course contribute to both the $(e, e'p)$ and to the $(e, e'NN)$ exclusive cross–sections.

Undoubtedly these processes can be, and in fact have partly been, treated in a fully quantum mechanical framework[19, 20]. Yet, as in the case of the FSI in the mean field approximation, we believe that the semi–classical method can be advantageously employed in dealing also with this far from simple physics to gain at least an orientation on its role in shaping the remarkably complex structure of the exclusive response in the missing energy–missing momentum plane.

# A Appendix

We calculate here explicitly the diagonal semi–classical spectral function for a few typical potential wells.

1. **Square potential well**
   Let's call $V_o(> 0)$ the depth of the well and $\bar{R}$ its range:

$$V(R) = \begin{cases} -V_o & \text{for} \quad 0 \leq R \leq \bar{R} \\ 0 & \text{for} \quad R > \bar{R} \end{cases} \tag{A.1}$$

   Then one gets for the Fermi momentum, only defined for $R < \bar{R}$,

$$k_F = \sqrt{2M_N^*(\epsilon_F + V_o)}. \tag{A.2}$$

   An easy calculation yields then for the spectral function the following expression

$$S_{sw}(p, \mathcal{E}) = \frac{4}{3}\pi \bar{R}^3 \theta \left( \sqrt{2M_N^*(\epsilon_F + V_o)} - p \right) \delta \left[ \mathcal{E} - \left( \epsilon_F + V_o - \frac{p^2}{2M_N^*} \right) \right] \tag{A.3}$$

   which looks indeed like the FG one.

   The support of (A.3) in the $(\mathcal{E}, p)$ plane is a line. Note that

$$4 \int_0^\infty d\mathcal{E}\, S_{sw}(p, \mathcal{E}) = 4 \frac{4}{3}\pi \bar{R}^3 \theta \left( \sqrt{2M_N^*(\epsilon_F + V_o)} - p \right) \tag{A.4}$$

provides the momentum distribution of the nucleons inside the nucleus, while

$$4\int \frac{d\vec{p}}{(2\pi)^3} \int d\mathcal{E}\ S_{sw}(p,\mathcal{E}) = \frac{4}{3}\pi\ \bar{R}^3\ \frac{2k_F^3}{3\pi^2} = A \tag{A.5}$$

[with $k_F$ given by (A.2)] is the number of nucleons, the factor of 4 accounting for the spin–isospin degeneracy. Of course the normalisation (A.5) holds valid if the Fermi energy fulfills the condition

$$\epsilon_F = \frac{1}{8}\ \frac{(9\pi A)^{2/3}}{M_N^* \bar{R}^2} - V_o. \tag{A.6}$$

2. **Harmonic oscillator potential well**

Setting

$$V(R) = \frac{1}{2} M_N^* \omega_o^2 R^2 - V_o\,, \tag{A.7}$$

$V_o$ being a positive constant of, say, 55 MeV and $\omega_o$ the harmonic oscillator frequency, one easily derives the spectral function

$$S_{ho}(p,\mathcal{E}) = \sqrt{2}\ \frac{4\pi}{(M_N^* \omega_o^2)^{3/2}}\ \theta(\mathcal{E})\sqrt{\epsilon_F + V_o - p^2/2M_N^* - \mathcal{E}} \tag{A.8}$$

whose support is no longer a line in the ($\mathcal{E}$, $p$) plane, but rather a two–dimensional domain bound by the curve

$$\mathcal{E} = \epsilon_F + V_o - p^2/2M_N^*\,, \tag{A.9}$$

the range of the allowed momenta being

$$0 \le p \le p_{ho}^{max} \tag{A.10}$$

with

$$p_{ho}^{max} \ = \ \sqrt{2M_N^*(\epsilon_F + V_o)}\,. \tag{A.11}$$

Although (A.9) is reminiscent of the Fermi gas, the actual support of $S_{ho}$ is more closely related to the one of a quantum mechanical harmonic oscillator. Indeed in the latter case the spectral function is substantially different from zero only in the region where $S_{ho}(p,\mathcal{E})$ does not vanish, although it lives only on a set of parallel lines corresponding to the discrete harmonic oscillator eigenvalues.

In Fig. 8 the semi–classical harmonic oscillator spectral function is displayed for the case of $^{40}$Ca ($\omega_o \simeq 12$ MeV). At variance with the quantum mechanical situation where $\mathcal{E}$ is quantized and $p$ is a continuous variable, now both $\mathcal{E}$ and

$p$ are continuous. Furthermore the semi–classical $S_{ho}\,(p, \mathcal{E})$ vanishes abruptly along the line (A.9) whereas in the quantum case there is a small leakage across this line. Finally the wild oscillations characterizing the quantum momentum distribution of the nucleons in a given shell are now replaced by a smooth behaviour with $p$. By comparing this figure with Fig.8 of ref.[2], which is also obtained with an harmonic oscillator potential, one can better appreciate the quantum mechanical versus semi–classical features of the spectral function. Yet the semi–classical approach succeeds in filling up the region of the $(\mathcal{E}, p)$ plane where the strength of the mean field spectral function is expected to occur, keeping, to a large extent, the simplicity of the Fermi gas treatment, but without leading to the naive and extreme momentum distribution of the latter.

Indeed this quantity reads now

$$
\begin{aligned}
& 4\int_0^\infty d\mathcal{E}\ S_{ho}(p, \mathcal{E}) = \qquad\qquad (A.12)\\
& = 4\sqrt{2}\ \frac{4\pi}{(M_N^*\omega_0^2)^{3/2}}\ \int_0^{\mathcal{E}_F + V_o - p^2/2M_N^*} d\mathcal{E}\sqrt{\epsilon_F + V_o - p^2/_{2M_N^*} - \mathcal{E}}\\
& = \frac{16\pi}{3}\ \frac{1}{M_N^*\omega_0^3}\ \left(2M_N^*(\epsilon_F + V_o) - p^2\right)^{3/2} = \ \frac{16\pi}{3}\ \frac{1}{M_N^{*3}\omega_0^3}\ \left[(p_{ho}^{max})^2 - p^2\right]^{3/2}
\end{aligned}
$$

with the normalization

$$
\begin{aligned}
4\int \frac{d\vec{p}}{(2\pi)^3}\ d\mathcal{E}\ S_{ho}(p, \mathcal{E}) \ &= \ \frac{16\pi}{3}\ \frac{1}{2\pi^2}\ \frac{p_{h.o}^{max}}{(M_N^*\omega_0)^3}\ \int_0^1\ dx\ x^2(1-x^2)^{3/2}\\
&= \frac{4}{3\pi}\ \left(\frac{(p_{ho}^{max})^2}{M_N^*\omega_0}\right)^3\ B\left(\frac{3}{2}\ ,\ \frac{5}{2}\right) = \frac{2}{3}\ \left(\frac{\epsilon_F + V_o}{\omega_0}\right)^3 \qquad (A.13)
\end{aligned}
$$

where $B$ is the Euler function of second kind.

Since, according to the nuclear shell model,

$$
\omega_0 = \frac{41}{A^{1/3}}\ \text{MeV}\,, \qquad (A.14)
$$

it follows that the momentum distribution is normalized to the number of nucleons A providing the Fermi energy is fixed by

$$
\epsilon_F = (32)^{1/3}\, 41 - V_o \simeq 47 - V_o. \qquad (A.15)
$$

For the sake of comparison, we illustrate in Fig. 9 the semi–classical spectral function associated with the Woods–Saxon well, which reads:

$$
S_{WS}(p, \mathcal{E}) = 4\pi R_\delta^2 \theta\left[\varepsilon_F - \frac{p^2}{2M_N} - V(R_\delta)\right]\frac{1}{|dV/dR|_{R=R_\delta}} \qquad (A.16)
$$

where

$$R_\delta = R_o + a \ln \left( \frac{V_1}{\mathcal{E} + p^2/2M_N - \varepsilon_F} - 1 \right) \tag{A.17}$$

and $dV/dR$ is the derivative of the Woods–Saxon well (6.13). The spectral function (A.16) has a different distribution of strength with respect to the one obtained with the harmonic oscillator: in particular its strength is quite sizeable close to the boundary of the region, in the $(\mathcal{E}, p)$ plane, where it is confined; on the other hand the latter is not much different from the one of the harmonic oscillator well.

3. **Power–law potential well**

For the potential

$$V(R) = -V_o + (\epsilon_F + V_o) \left( \frac{R}{R_c} \right)^n, \tag{A.18}$$

$V_o$ and $R_c$ being positive constants and $n = 1, 2, 3...$, one finds

$$\begin{aligned} S_{plw}(p, \mathcal{E}) \quad &= \frac{4}{3} \pi R_c^3 \, \frac{3}{n(\epsilon_F + V_o)} \left( 1 - \frac{\mathcal{E} + p^2/2M_N^*}{\epsilon_F + V_o} \right)^{3/n-1} \\ &\times \theta \left[ \epsilon_F + V_o - \left( \mathcal{E} + \frac{p^2}{2M_N^*} \right) \right] \end{aligned} \tag{A.19}$$

the Fermi energy being linked to the potential through the relation (4.6).

As in the previous instances $S_{plw}(p, \mathcal{E})$ lives in a domain of the plane $(\mathcal{E}, p)$ bound by the curve

$$\mathcal{E} = \epsilon_F + V_o - \frac{p^2}{2M_N^*} \tag{A.20}$$

on which it vanishes, and within the range of momenta

$$0 \leq p \leq p_{plw}^{max} \tag{A.21}$$

with, as before,

$$p_{plw}^{max} = \sqrt{2M_N^* \, (\epsilon_F + V_o)}\,. \tag{A.22}$$

Note that for $n = 3$ the spectral function is constant. Furthermore it becomes closer and closer to the one of the square well (and, hence, of the FG) as $n$ becomes large.

The momentum distribution is given by

$$\int_0^\infty d\mathcal{E} \; S_{plw}(p, \mathcal{E}) = \frac{16 \pi R_c^3}{3} \left( \frac{\epsilon_F + V_o - p^2/2M_N^*}{\epsilon_F + V_o} \right)^{3/n} \tag{A.23}$$

with normalization

$$\int \frac{d\vec{p}}{(2\pi)^3} \int_0^\infty d\mathcal{E}\, S_{plw}(p,\mathcal{E}) = \frac{8}{3\pi} \left(R_c p_{plw}^{max}\right)^3 \int_0^1 dt\; t^2(1-t^2)^{3/n}$$

$$= \frac{4}{3}\,(p_{plw}^{max} R_c)^3\; B\left(\frac{3}{2}, \frac{3}{n}+1\right)\,, \qquad \text{(A.24)}$$

and yields the nucleons' number $A$ when the Fermi energy is fixed according to

$$\epsilon_F = \frac{1}{2M_N^* R_c^2} \left(\frac{3\pi A}{4B(3/2, 3/n+1)}\right)^{2/3} - V_o\,. \qquad \text{(A.25)}$$

Finally worth pointing out is that (A.18) interpolates between various forms of the mean field.

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Figure 1: The Feynman diagram representing the inclusive $(e, e')$ process.

Figure 2: Diagrammatic representation of an exclusive $(e, e'N)$ process.

Figure 3: a–d Exclusive cross sections as a function of the missing momentum $p_m$ (in MeV/c) at $q = 300$ MeV/c (a and b) and $q = 500$ MeV/c (c and d); $\omega = q^2/2M_N + E_S$. The eikonal (on the left) and the uniform (on the right) approximations for the distortion operator are used; the harmonic oscillator potential is employed. In all figures curves corresponding to three different values for the missing energy are displayed: $\mathcal{E} = 10$ MeV (continuous line), $\mathcal{E} = 20$ MeV (dashed line) and $\mathcal{E} = 30$ MeV (dot–dashed line).

Figure 4: a–d The same as in Fig. 3a–d. but employing the Woods–Saxon potential well.

Figure 5: a,b Exclusive cross–sections as a function of the missing momentum $p_m$ (in MeV/c), for $q = 300$ MeV/c, $\omega = q^2/2M_N + E_S$, $\mathcal{E} = 10$ MeV, in the eikonal (continuous line) and uniform (dashed line) DWIA, as well as in PWIA (dot–dashed line). In (a) the harmonic oscillator potential is employed, in (b) the Woods–Saxon well.

Figure 6: a,b Inclusive cross–sections at $q = 200$ MeV/c (a) and $q = 500$ MeV/c (b), as a function of the energy transfer $\omega$ (in MeV): the single nucleon cross section coincides with $\sigma_{Mott}$. The continuous and dashed (coincident) lines refer to the integral of the exclusive cross–section in eikonal and uniform approximation, respectively, using the harmonic oscillator potential. The dot–dashed (eikonal) and dotted (uniform) lines are obtained with the Woods–Saxon well. The long–dashed line is the direct evaluation (with the harmonic oscillator potential) of the inclusive cross–section, through the polarization propagator.

Figure 7: a,b The same as in Fig. 6a,b but using the fully relativistic single nucleon cross section (and omitting the direct calculation).

Figure 8: Semi–classical spectral function obtained with the harmonic oscillator potential, as a function of missing energy and missing momentum.

Figure 9: Semi–classical spectral function obtained with the Woods–Saxon potential, as a function of missing energy and missing momentum.

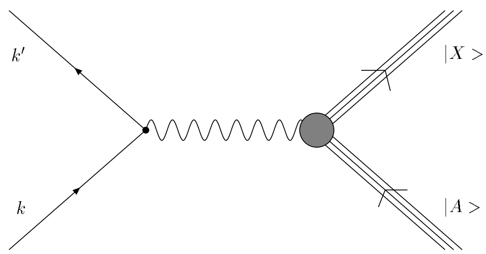



Fig. 1

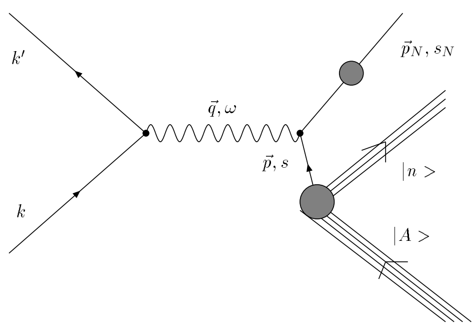
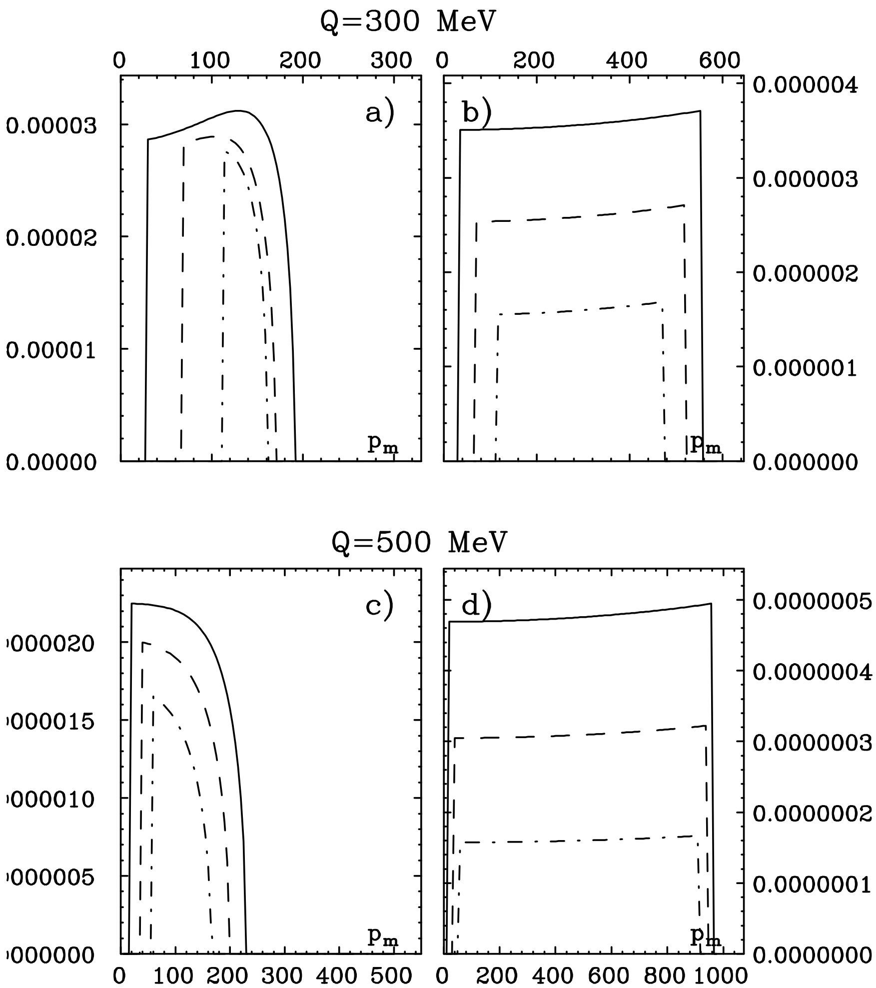



Fig. 2

1



Q=300 MeV
0
100
200
300
0
200
400
600
0.00003
0.00002
0.00001
0.00000
0.000004
0.000003
0.000002
0.000001
0.000000
a)
b)
$p_m$
$p_m$
Q=500 MeV
c)
d)
0.0000005
0.0000004
0.0000003
0.0000002
0.0000001
0.0000000
0
100
200
300
400
500
0
200
400
600
800
1000


Figure 3

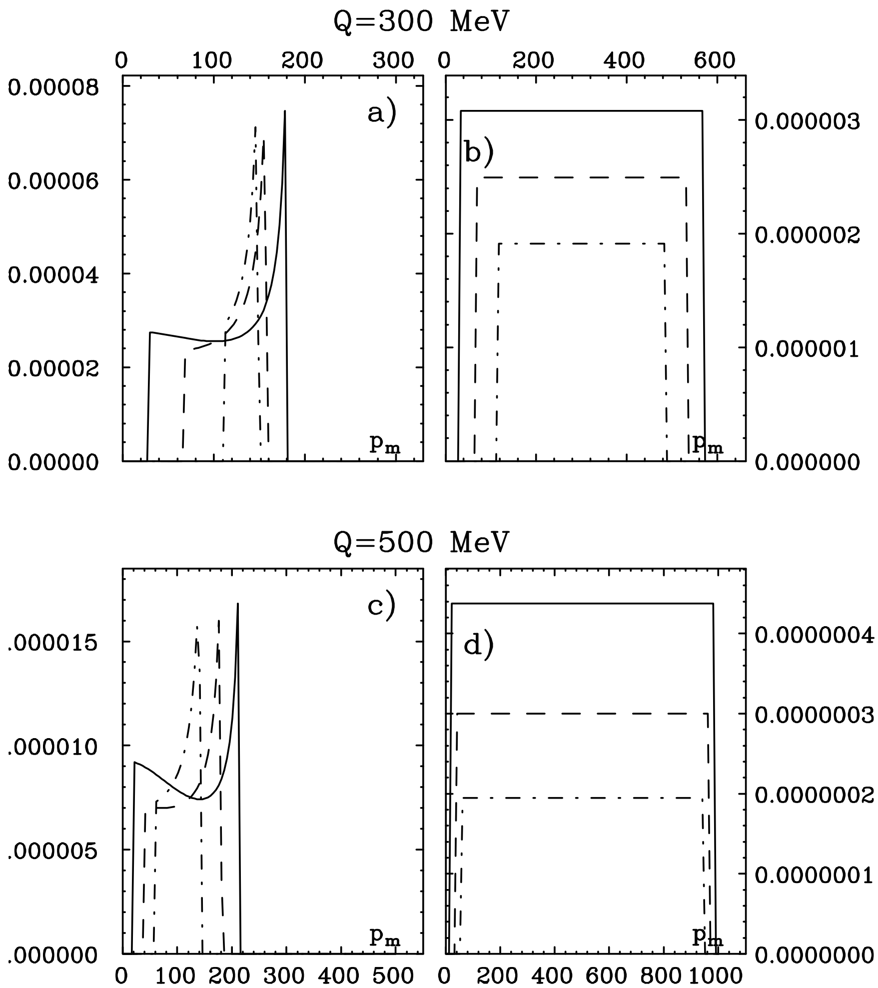
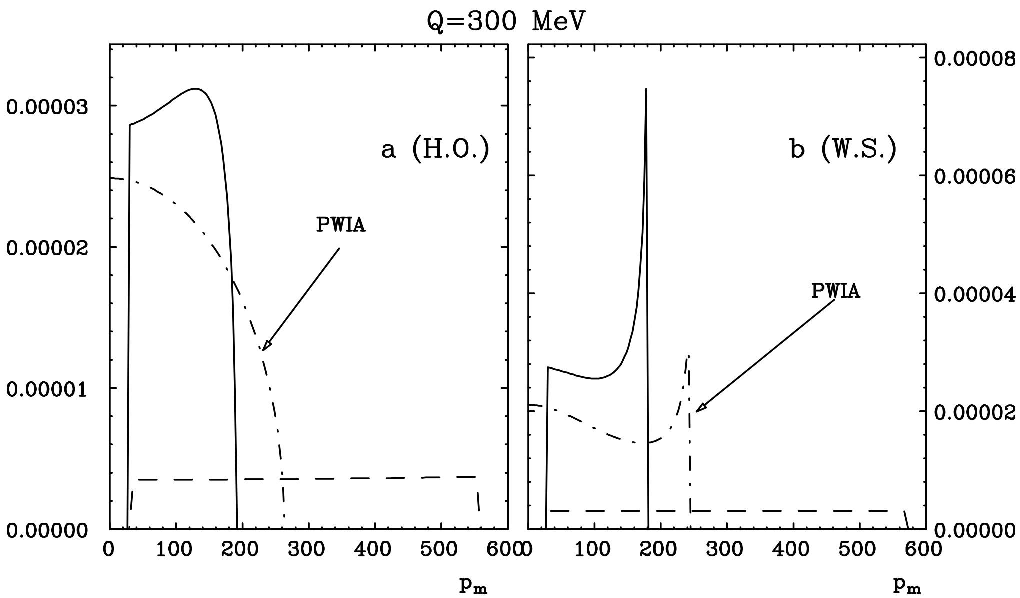



Figure 4



Q=300 MeV
a (H.O.)
b (W.S.)
PWIA
PWIA
0.00003
0.00002
0.00001
0.00000
0.00008
0.00006
0.00004
0.00002
0.00000
0 100 200 300 400 500 600
0 100 200 300 400 500 600
$p_m$
$p_m$


Figure 5

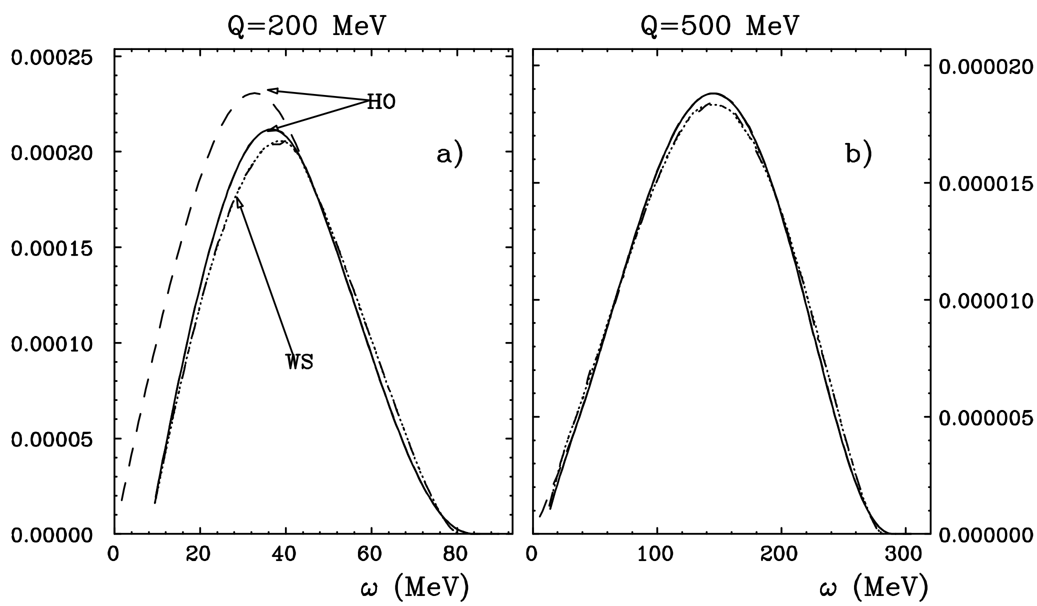

Q=200 MeV
Q=500 MeV
0.00025
0.00020
0.00015
0.00010
0.00005
0.00000
0
20
40
60
80
ω (MeV)
HO
WS
a)
b)
0.000020
0.000015
0.000010
0.000005
0.000000
0
100
200
300
ω (MeV)

Figure 6

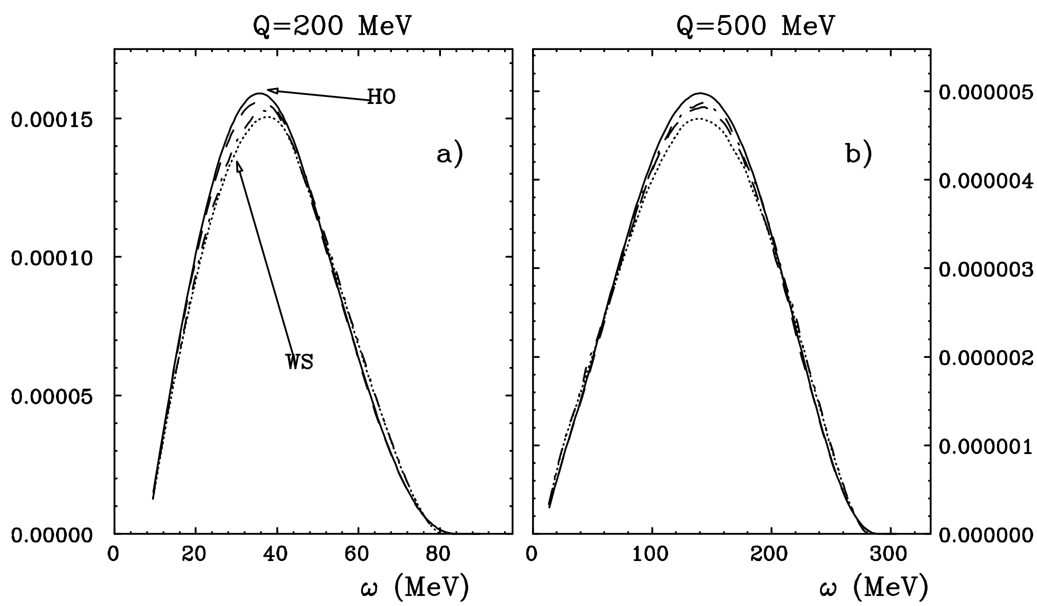

Q=200 MeV
Q=500 MeV
HO
WS
a)
b)
0.00015
0.00010
0.00005
0.00000
0.000005
0.000004
0.000003
0.000002
0.000001
0.000000
0
20
40
60
80
100
200
300
ω (MeV)
ω (MeV)

Figure 7

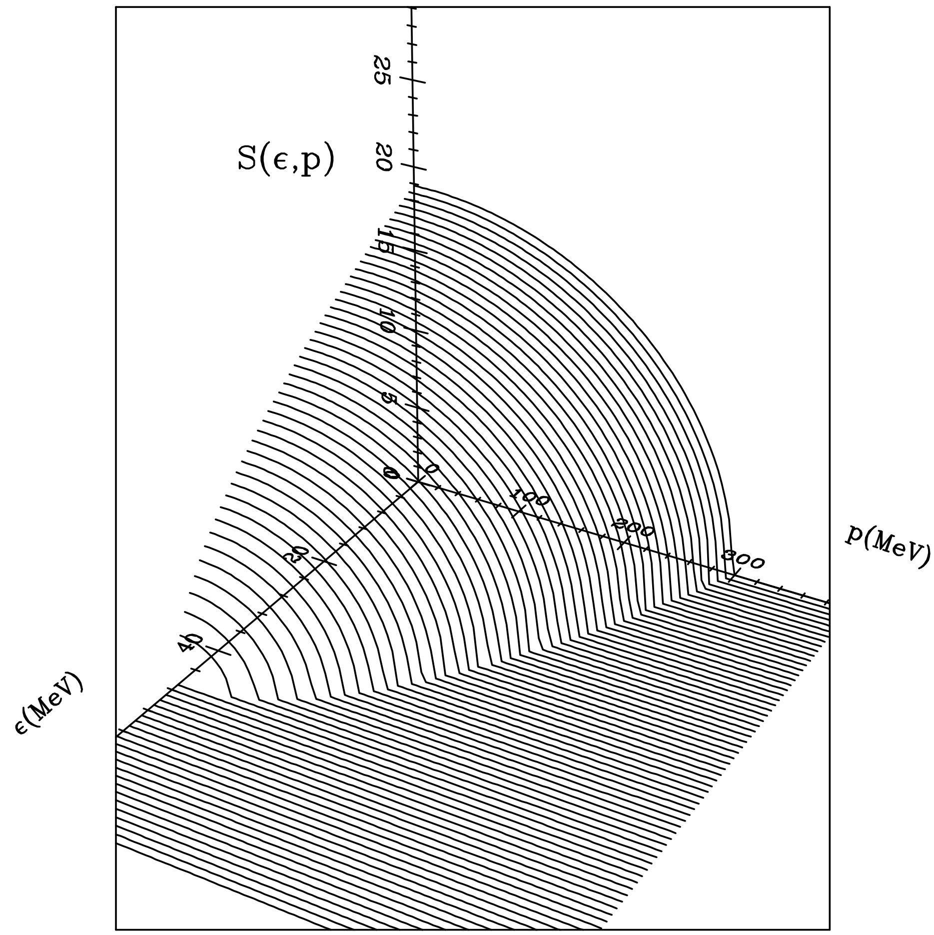



Figure 8

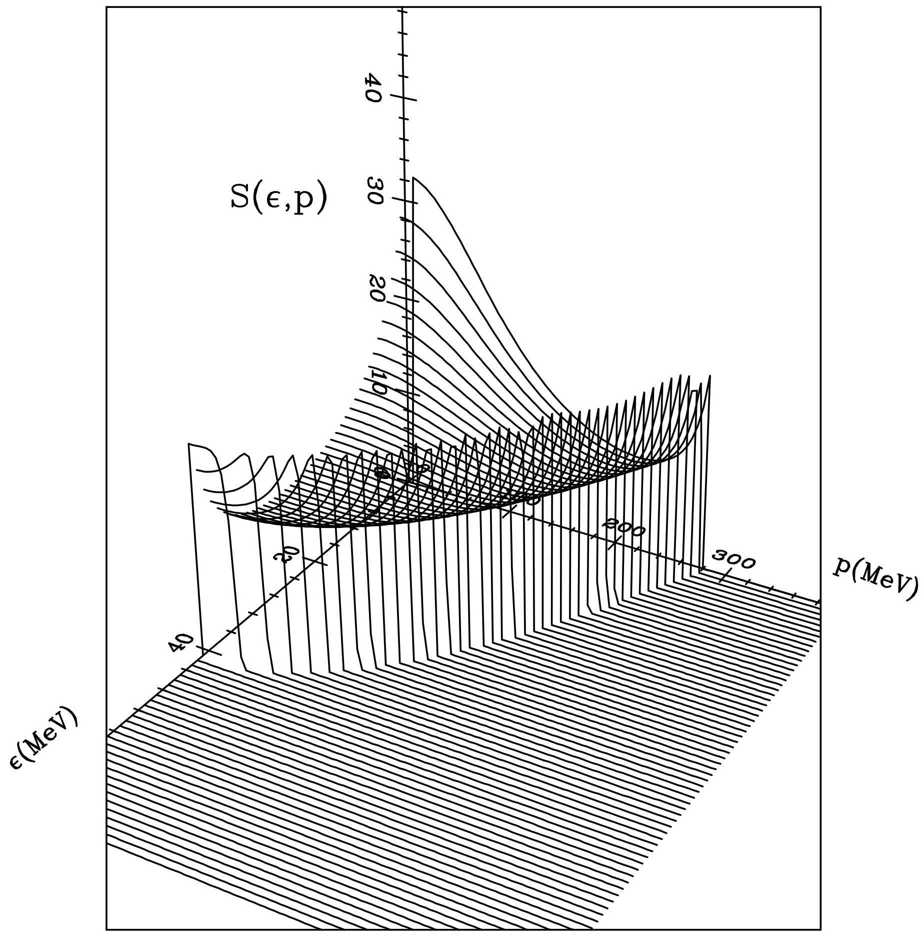



Figure 9