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hep-th/9708062 **R^2 CORRECTIONS AND NON-PERTURBATIVE DUALITIES
OF $N = 4$ STRING GROUND STATES**A. GREGORI¹, E. KIRITSIS¹, C. KOUNNAS^{1,*}, N.A. OBERS¹,
P.M. PETROPOULOS^{1,2,◇} and B. PIOLINE^{1,◇}¹ *Theory Division, CERN †*
1211 Geneva 23, Switzerland

and

² *Institut de Physique Théorique, Université de Neuchâtel*
*2000 Neuchâtel, Switzerland***Abstract**

We compute and analyse a variety of four-derivative gravitational terms in the effective action of six- and four-dimensional type II string ground states with $N = 4$ supersymmetry. In six dimensions, we compute the relevant perturbative corrections for the type II string compactified on $K3$. In four dimensions we do analogous computations for several models with $(4, 0)$ and $(2, 2)$ supersymmetry. Such ground states are related by heterotic–type II duality or type II–type II U -duality. Perturbative computations in one member of a dual pair give a non-perturbative result in the other member. In particular, the exact CP-even R^2 coupling on the $(2, 2)$ side reproduces the tree-level term plus NS 5-brane instanton contributions on the $(4, 0)$ side. On the other hand, the exact CP-odd coupling yields the one-loop axionic interaction $aR \wedge R$ together with a similar instanton sum. In a subset of models, the expected breaking of the $SL(2, Z)_S$ S -duality symmetry to a $\Gamma(2)_S$ subgroup is observed on the non-perturbative thresholds. Moreover, we present a duality chain that provides evidence for the existence of heterotic $N = 4$ models in which $N = 8$ supersymmetry appears at strong coupling.

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* On leave from *Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, CNRS, 24 rue Lhomond, 75231 Paris Cedex 05, France.*

◇ On leave from *Centre de Physique Théorique, Ecole Polytechnique, CNRS, 91128 Palaiseau Cedex, France.*

† e-mail addresses: agregori, kiritsis, kounnas, obers, petro, bpioline@mail.cern.ch.

1 Introduction

There has been intriguing evidence that different perturbative string theories might be non-perturbatively equivalent [1, 2, 3, 4]. In six dimensions, there is a conjectured duality between the heterotic string compactified on T^4 and the type IIA string compactified on $K3$ [1, 2]. Both theories have $N = 2$ supersymmetry and 20 massless vector multiplets in six dimensions. Several arguments support this duality:

i) The tree-level two-derivative actions of the two theories (in the Einstein frame) are related by a duality transformation. In particular, the field strength of the heterotic anti-symmetric tensor (with the gauge Chern–Simons form included) is dual to that of the type II string.

ii) The relation described above implies that the heterotic string is a magnetic or solitonic string of the type II theory and vice versa. This is also supported by the following facts [5]. There is a singular string solution of the heterotic theory, electrically charged under the antisymmetric tensor, which can be identified with the perturbative heterotic string [6]. There is also a magnetically charged solitonic (regular at the core) string solution, which has the correct zero-mode structure to be identified with the type II string. Upon the duality map, their role is interchanged in the type II theory [5].

iii) Anomaly cancellation of the heterotic string implies that there should be a one-loop R^2 term in the type II theory. Such a term was found by direct calculation in [7]. Its one-loop threshold correction upon compactification to four dimensions [8] implies instanton corrections on the heterotic side due to 5-branes wrapped on the six-torus.

iv) Upon toroidal compactification to four dimensions, heterotic–type II duality translates into $S \leftrightarrow T$ interchange [1]. As a consequence, perturbative T -duality of the type II string implies S -duality [9] of the heterotic string (and vice versa). Electrically charged states are interchanged with magnetically charged states.

v) The six-dimensional heterotic–type II duality implies by the adiabatic argument [10] non-perturbative dualities in lower-dimensional models obtained as freely-acting orbifolds of the original pair.

vi) More general (non-free) symmetric orbifolds still give rise to $N = 2$ heterotic–type II dual pairs in four dimensions [11, 12, 13]. On the heterotic side they can be viewed as compactifications on $K3$, while on the type IIA side they correspond to compactifications on $K3$ -fibred Calabi–Yau manifolds. On the heterotic side the dilaton is in a vector multiplet and the vector moduli space receives both perturbative and non-perturbative corrections. On the other hand, the hypermultiplet moduli space does not receive perturbative corrections, or non-perturbative ones, if $N = 2$ is assumed to be unbroken. On the type II side the dilaton is in a hypermultiplet and the prepotential for the vector multiplets comes only from the tree level. This fact provides a quantitative test of duality; this was shown in Refs. [11, 13], where the tree-level type II prepotential was computed and shown to give the correct one-loop heterotic result and to predict the non-perturbative corrections on the heterotic side. This quantitative test is not applicable to $N = 4$ string duality.

There is another class of non-perturbative duality symmetries known as U -duality [2],

which relates type II vacua with maximal supersymmetry. They are obtained from the convolution of the $SL(2, Z)$ symmetry of type IIB in ten dimensions and $O(d, d, Z)$ duality upon compactification. Using freely-acting orbifolds, the supersymmetry can be reduced but there should still be a U -duality symmetry [10]. In Ref. [14] a class of models with $N = 4$ and $N = 2$ supersymmetry was discussed; these are related by U -duality. Again, there are several arguments in favour of U -duality, but no quantitative test to our knowledge.

In this paper we shall focus on the implications of heterotic–type IIA and U -dualities for higher-derivative gravitational terms in the effective action, namely R^2 couplings and variations thereof. These terms have the property that in vacua with 16 supercharges ($N = 4$ in four dimensions) they only receive contribution from short representations of the supersymmetry algebra, the so-called BPS multiplets. This property becomes obvious once these terms are written in terms of helicity supertraces [15], which are known to count only BPS states. Therefore, R^2 terms in $N = 4$ vacua are very similar to the terms in the two-derivative action for vacua with 8 supercharges ($N = 2$ in four dimensions). In fact, the two-derivative action can be shown to be uncorrected both perturbatively and non-perturbatively in $N = 4$ vacua, so that these couplings are the first terms where quantum corrections manifest themselves, in a still controllable way, though. The F^4 [15] and R^4 terms in vacua with $(4, 0)$ supersymmetry also belong to this class of BPS-saturated couplings, together with higher-derivative terms constructed out of the Riemann tensor and the graviphoton field strengths [16].

Contributions to R^2 couplings depend on the type of $N = 4$ vacua we are considering: $(2, 2)$ vacua, where two supersymmetries come from the left-movers and two from the right-movers, or $(4, 0)$ vacua, where all four supersymmetries come from the left-movers only. All heterotic ground states with $N = 4$ supersymmetry are of the $(4, 0)$ type, but $(4, 0)$ type II vacua can also be constructed [2, 17]. In that case, the axion–dilaton corresponds to the complex scalar in the gravitational multiplet in four dimensions and, as such, takes values in an $SU(1, 1)/U(1)$ coset space, while the other scalars form an $SO(6, N_V)/(SO(6) \times SO(N_V))$ manifold, where N_V is the number of vector multiplets in four dimensions. On the other hand $(2, 2)$ models only exist in type II and have a different structure: the dilaton is now part of the $SO(6, N_V)/(SO(6) \times SO(N_V))$ manifold, while the $SU(1, 1)/U(1)$ coset is spanned by a perturbative modulus. Duality always maps a $(2, 2)$ ground state to a $(4, 0)$ ground state [2]. We shall argue that R^2 couplings are exactly given by their one-loop result in $(2, 2)$ vacua. Translated into the dual $(4, 0)$ theory, the exact R^2 coupling now appears to arise from non-perturbative effects, identified with NS 5-brane instantons in Ref. [8]. Here we shall carry the work of [8] further, and extend it to more general gravitational couplings and to other $N = 4$ models, obtained as freely-acting orbifolds of the usual type IIA on $K3 \times T^2$ and heterotic on T^6 theories. These exotic ground states possess a number of vector multiplets smaller than that of their parents ($N_V = 22$), and we shall generically refer to them as *reduced-rank $N = 4$ models*. They reduce to standard $N = 4$ or $N = 8$ models in proper decompactification limits, and are invariant under reduced groups of T - or S -dualities.

To be specific, we will consider the following $N = 4$ models:

- a) Type II theory compactified on $K3 \times T^2$ with $(2, 2)$ supersymmetry and 22 vector

multiplets. We will denote this ground state by $\text{II}_{22}^{(2,2)}$. It is conjectured to be dual to the heterotic string compactified on T^6 (denoted henceforth by HET_{22}) via $S \leftrightarrow T$ interchange [2]. The R^2 coupling has already been considered in the case [8]. We will reconsider it here in order to compute also the thresholds of other four-derivative terms, as well as to compare it with the six-dimensional thresholds once we decompactify the T^2 .

b) Type II theory compactified on a six-dimensional manifold with $SU(2)$ holonomy, which is locally but *not globally* $K3 \times T^2$. The supersymmetry is still $(2, 2)$. We present examples with $N_V = 6, 10, 14$. The class of models with N_V reduced was initially constructed in Ref. [17], using a fermionic construction [18, 19]. Here we shall construct them by starting with the $K3 \times T^2$ model, going to a subspace of $K3$ with a Z_2 (non-freely-acting) symmetry and orbifolding with this symmetry, accompanied by a lattice shift w on the two-torus (w is a four-dimensional vector of $\text{mod}(2)$ integers with $w^2 = 0$). In practice, we consider the T^4/Z_2 orbifold limit of $K3$. The Z_2 symmetry we use is a subgroup of the $(D_4)^4$ symmetry of T^4/Z_2 . Appropriately choosing this Z_2 subgroup allows the construction of $(2, 2)$ ground states with $N_V = 6, 10, 14$ vector multiplets. Such a Z_2 symmetry has the property that if we orbifold by it, without a T^2 shift, it reproduces the $K3 \times T^2$ models at a different point in the $K3$ moduli space. Moreover, because of the shift on the T^2 , the $SL(2, Z)_T$ duality symmetry is broken to a $\Gamma(2)_T$ subgroup. We will denote these ground states by $\text{II}_{N_V}^{(2,2)}(w)$.

Since the orbifold acts freely, by the adiabatic argument, the new model should be dual to a corresponding orbifold of the heterotic string on T^6 with reduced rank, which we will denote by $\text{HET}_{N_V}(w)$. Duality will then imply that such $N = 4$ ground states have a reduced S -duality group, $\Gamma(2)_S \subset SL(2, Z)_S$. This property is reflected in the non-invariance of the R^2 threshold under the full $SL(2, Z)_S$ group. When the shift vector w involves projections on momenta only on the type II side, then its action is perturbatively visible on the heterotic side, since it acts again on momenta. If, on the other hand, it contains projections on the winding numbers on the type II side, then in heterotic language the projection is on non-perturbative states carrying magnetic charges.

c) A $(2, 2)$ type II model obtained by orbifolding the type II string compactified on T^6 (maximal $N = 8$ supersymmetry). We split $T^6 = T^2 \times T^4$ and the Z_2 orbifold action is an inversion on T^4 and a shift w on T^2 . This is a ground state, where $N = 8$ supersymmetry is *spontaneously* broken to $N = 4$. It has $N_V = 6$ vector multiplets and will be denoted by $\overline{\text{II}}_6^{(2,2)}(w)$.

d) A $(4, 0)$ type II model, constructed by freely orbifolding by $(-1)^{F_L}$ times a Z_2 lattice shift w on T^6 ($(-1)^{F_L}$ is the left-moving fermion number). Such a ground state has $N_V = 6$ and we will denote it by $\text{II}_6^{(4,0)}(w)$. Here again $N = 8$ supersymmetry is spontaneously broken to $N = 4$. It was argued in [14] to be U -dual to the $\overline{\text{II}}_6^{(2,2)}(w)$ ground state of the previous paragraph via $S \leftrightarrow T$ interchange. There is a map of the two-torus electric and magnetic charges similar to the case of string-string duality.

String-string duality and U -duality imply that the aforementioned models are related through

$$\text{HET}_{N_V}(w)(S, T) = \text{II}_{N_V}^{(2,2)}(w')(T, S), \quad (1.1a)$$

$$\Pi_6^{(4,0)}(w)(S, T) = \bar{\Pi}_6^{(2,2)}(w')(T, S), \quad (1.1b)$$

where the lattice shift w' is obtained from w through the duality map. For the particular case of a shift vector w^* acting on the momenta only, $w'^* = w^*$.

Moreover, we shall prove that, at least in the weak-coupling regime $S_2 \rightarrow \infty$, the two models $\bar{\Pi}_6^{(2,2)}(w)$ and $\Pi_6^{(2,2)}(w)$ are actually identical, up to relabelling of perturbative moduli. In particular, for $w^* = w_I \equiv (0, 0, 1, 0)$:

$$\bar{\Pi}_6^{(2,2)}(w_I)(S, T, U) = \Pi_6^{(2,2)}(w_I)(S, -2/T, -1/2U). \quad (1.2)$$

We also have the following decompactification limits, at least in the perturbative regime:

$$\Pi_6^{(2,2)}(w_I)(T_2 \rightarrow \infty) = \bar{\Pi}_6^{(2,2)}(w_I)(T_2 \rightarrow 0) = \text{type IIA on } K3, \quad (1.3)$$

$$\Pi_6^{(2,2)}(w_I)(T_2 \rightarrow 0) = \bar{\Pi}_6^{(2,2)}(w_I)(T_2 \rightarrow \infty) = \text{type IIA on } T^4. \quad (1.4)$$

Now making use of the string-string duality (1.1a) we obtain that, at least in the *large-radius* limit,

$$\text{HET}_6(w^*)(S_2 \rightarrow \infty) = \text{heterotic on } T^4, \quad (1.5)$$

$$\text{HET}_6(w^*)(S_2 \rightarrow 0) = \text{type IIA on } T^4. \quad (1.6)$$

Thus we find that at weak coupling the $\text{HET}_6(w^*)$ has $N = 4$ supersymmetry while at strong coupling $N = 8$ supersymmetry is restored. It is known that the heterotic string can be viewed as a (non-freely-acting) Z_2 orbifold of M-theory [20]. Here, however, we find a ground state of the heterotic string in which $N = 8$ supersymmetry is *spontaneously* broken to $N = 4$. The extra gravitinos are magnetic solitons, with masses scaling as the inverse of the heterotic coupling constant. Therefore, $N = 8$ supersymmetry is restored in the strong-coupling limit where these gravitinos become massless. This is similar to the situation described in [21, 22, 23].

The structure of this paper is as follows: in Section 2 we describe the potential perturbative and non-perturbative contributions to the R^2 couplings in the various string models we analyse further. In Section 3 we consider the type IIA, B string compactified to six dimensions on $K3$ and compute the perturbative corrections to the four-derivative couplings involving the metric, the NS-NS antisymmetric tensor and the dilaton. In Section 4 we describe the calculation of the one-loop R^2 threshold in generic type II orbifold models with $N = 4$ supersymmetry. In Section 5 we reconsider the R^2 thresholds of type II string on $K3 \times T^2$ and its dual heterotic theory. In Section 6 we analyse the BPS spectrum and R^2 thresholds of the various models with $N = 4$ supersymmetry and reduced rank. Section 7 contains our conclusions. In Appendix A we describe the kinematics of on-shell string vertices relevant for our threshold calculations, and in Appendix B the calculation of helicity supertraces that count BPS states in string ground states with $N = 4$ supersymmetry and some associated ϑ -function identities. In Appendix C we calculate the relevant fundamental-domain integrals appearing in the one-loop calculation of the thresholds. Details of one-loop string calculations are left to Appendix D.

2 Perturbative and instanton corrections to R^2 couplings

In this paper we shall be interested in four-derivative gravitational couplings in the low-energy effective action of superstring vacua with 16 supercharges ($N = 4$ in four dimensions, or $N = 2$ in six dimensions). The prototype of these terms is $R^2 \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, but we shall also consider couplings involving the NS antisymmetric tensor $B_{\mu\nu}$ and the dilaton Φ .

At tree level, such terms can be obtained directly from the relevant ten-dimensional calculations (see [24]) upon compactification on the appropriate manifold, $K3$, $K3 \times T^2$ or T^6 . They turn out to be non-zero in $(4, 0)$ ground states (heterotic or type II) and zero for $(2, 2)$ ground states. They may a priori also receive higher-loop perturbative corrections, but $(4, 0)$ ground states appear to have no perturbative corrections at all, while the perturbative corrections in $(2, 2)$ vacua are expected to come only from one loop owing to the presence of extended supersymmetry.

These terms are related by supersymmetry to eight-fermion couplings. As such they may receive non-perturbative corrections from instantons having not more than 8 fermionic zero-modes. This rules out generic instanton configurations, which break all of the 16 supersymmetric charges and therefore possess at least 16 zero-modes. However, there exist particular configurations that preserve one half of the supersymmetries (this is the only possibility in six dimensions where the supersymmetry is $N = 2$), thereby possessing 8 fermionic zero-modes¹. These configurations correspond to the various p -brane configurations of the original ten-dimensional theory: a Euclidean p -brane can generate an instanton when its $(p + 1)$ -dimensional world-volume wraps around some appropriate submanifold of the compactification manifold ($K3$). All superstrings in ten dimensions have in common the NS 5-brane that couples to the dual of the NS–NS antisymmetric tensor and breaks half of the ten-dimensional supersymmetry. Type II superstrings also have D p -branes that are charged under the various R–R forms and their duals: $p = 0, 2, 4, 6, 8$ for type IIA theory, $p = -1, 1, 3, 5, 7$ for type IIB. Obviously, D-branes are absent from heterotic ground states, but also from the $(4, 0)$ type II model we shall consider, since the latter has no massless R–R fields. We conjecture that this is in fact true for any $(4, 0)$ vacuum. The only instanton configuration for such vacua is therefore the NS 5-brane, which only starts to contribute for dimensions less than or equal to four.

In $(2, 2)$ models the situation is a bit more involved. Let us consider first the type IIA, B string compactified on $K3$ to six dimensions. Since $K3$ is four-dimensional, only branes with $p+1 \leq 4$ need be considered as instantons. Wrapped in a generic fashion around submanifolds of $K3$ they break all supersymmetries and thus do not contribute, in our calculation. There are, however, supersymmetric 0, 2 and 4 cycles in $K3$. The relevant instantons will then have $p+1 = 0, 2, 4$, found only in type IIB. Thus in type IIA theory we do not expect any instanton corrections. In type IIB theory, all scalar fields span an $SO(5, 21)/\left(SO(5) \times SO(21)\right)$ coset space. The perturbative T -duality symmetry $O(4, 20, Z)$ combines with the $SL(2, Z)$ symmetry in ten dimensions into an $O(5, 21, Z)$ U -duality symmetry group. The exact non-perturbative threshold should therefore be an $O(5, 21, Z)$ -invariant function of the moduli

¹Instantons with less than 8 zero-modes do not exist, in agreement with the absence of corrections to the two-derivative or four-fermion action.

and, as argued in [25], it can be written as linear combinations of the Eisenstein–Poincaré series. However, all such series have distinct and non-zero perturbative terms when expanded in terms of any modulus, in disagreement with the fact that all perturbative corrections should vanish. We thus conclude that the R^2 threshold is non-perturbatively zero also in type IIB on $K3$.

There is an independent argument pointing to the same result. Consider compactifying type IIA, B on $K3 \times S^1$. Then IIA and IIB are related by inverting the circle radius. From the type IIA point of view there are now potential instanton corrections from the $p = 0, 2, 4$ -branes wrapping around a $0, 2, 4$ $K3$ cycle times S^1 . However, on the heterotic side we are still in a dimension larger than four so we still have no perturbative or non-perturbative corrections. This implies that the contribution of the IIA instantons still vanishes, as it does for the IIB instantons, which are just the same as the six-dimensional ones. The instanton contributions in six dimensions thus also have to vanish.

Compactifying further to four dimensions on an extra circle, the scalar manifold becomes $SU(1, 1)/U(1) \times SO(6, 22)/(SO(6) \times SO(22))$ and the duality group $SL(2, Z) \times O(6, 22, Z)$. The instanton contributions can come from 5-branes wrapped around $K3 \times T^2$ as well as, in type IIB, from D 3-branes wrapped around T^2 times a $K3$ 2-cycle, and (p, q) D 1-branes wrapped around T^2 . The 1-brane contribution is zero since it is related via $SL(2, Z)$ duality to that of the fundamental string world-sheet instantons, which vanish from the one-loop result². All other instanton corrections depend non-trivially on the $O(6, 22)$ moduli. Again, it is expected that an $O(6, 22)$ -invariant result would imply perturbative corrections depending on the $O(6, 22)$ moduli, which are absent, as we will show. Therefore, we again obtain that the non-perturbative corrections vanish in IIB, and also in IIA. This can also be argued via type II–heterotic–type I triality. On the heterotic and type I side these corrections come from the 5-brane wrapped on T^6 . The world-volume action of the D 5-brane in type II theory is known and wrapping it around T^6 and translating to heterotic variables produces a result depending only on the S field. Thus on the heterotic side we do not expect $O(6, 22)$ -dependent corrections, and therefore no instanton contributions in type II.

The upshot of the above discussion is that, in $(2, 2)$ models, various dualities imply that on the type II side instanton corrections to R^2 terms are absent in six, five and four dimensions. Similar arguments apply to the other reduced-rank $(2, 2)$ models considered in this paper, since their instanton corrections are related to the ones above by applying some selection rules on the possible brane-wrappings. We can therefore restrict ourselves to a one-loop computation on both type IIA and type IIB $(2, 2)$ models.

3 One-loop corrections in six-dimensional type IIA and IIB theories

In this section, we compute the one-loop four-derivative terms in the effective action for type IIA and IIB theory compactified to six dimensions on the $K3$ manifold. We will work in the Z_2 orbifold limit of $K3$ in order to be explicit but, as we will show, the result will be valid

²This is equivalent to the statement that in IIB the one-loop threshold only depends on the complex structure U of the torus. This no longer holds for other thresholds such as $\nabla H \nabla H$ and, accordingly, we shall find that those are non-perturbatively corrected even in type II.

for all values of the $K3$ moduli.

3.1 Type II superstring on $K3$: a reminder

The one-loop partition function of type IIA, B theory compactified on the T^4/Z_2 orbifold is

$$\begin{aligned} Z_{\text{II}}^{\text{six dim}} &= \frac{1}{\tau_2^2 |\eta|^{24}} \frac{1}{2} \sum_{a,b=0}^1 (-1)^{a+b+ab} \vartheta^2 \begin{bmatrix} a \\ b \end{bmatrix} \frac{1}{2} \sum_{\bar{a},\bar{b}=0}^1 (-1)^{\bar{a}+\bar{b}+\mu\bar{a}\bar{b}} \bar{\vartheta}^2 \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} \\ &\times \frac{1}{2} \sum_{h,g=0}^1 \vartheta \begin{bmatrix} a+h \\ b+g \end{bmatrix} \vartheta \begin{bmatrix} a-h \\ b-g \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a}+h \\ \bar{b}+g \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a}-h \\ \bar{b}-g \end{bmatrix} \Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix}, \end{aligned} \quad (3.1)$$

where the T^4 orbifold blocks $\Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix}$ are given by

$$\Gamma_{4,4} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \Gamma_{4,4}, \quad \Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix} = 16 \frac{|\eta|^{12}}{|\vartheta \begin{bmatrix} 1+h \\ 1+g \end{bmatrix}|^4}, \quad (h, g) \neq (0, 0), \quad (3.2)$$

where $\Gamma_{4,4}$ is the $(4, 4)$ lattice sum. The parameter μ takes the value 0 or 1 for type IIA or IIB superstrings, respectively, as it determines the sign of the antiholomorphic Ramond sector and hence the space-time chirality of the fermions. We shall also use the notations $\varepsilon = (-1)^\mu$ and denote by left and right the holomorphic and antiholomorphic side, respectively. On each side the sum over spin structures splits into three even structures $(a, b) \in \{(0, 0), (0, 1), (1, 0)\}$ and one odd $(a, b) = (1, 1)$ according to the number of fermionic zero-modes on the world-sheet.

To compute the massless spectrum we need the following geometric data of $K3$: the Einstein metric on $K3$ is parametrized by 58 scalars, and the non-zero Betti numbers are $b_0 = b_4 = 1$ and $b_2 = 22$. Out of the 22 two-forms, 3 are self-dual, while the remaining 19 are anti-self-dual. At the T^4/Z_2 orbifold point of $K3$, those correspond to the $3 + 3$ Z_2 -even two-forms $dx^i \wedge dx^j$ and to 16 anti-self-dual two-forms supported by the two-sphere that blows up each of 16 fixed points. With this in mind, it is easy to derive the massless spectrum:

Type IIA. The ten-dimensional bosonic massless spectrum consists of the NS–NS fields G_{MN} , B_{MN} , Φ and of the R–R three-form and one-form potentials A_{MNR} and A_M . Compactification on $K3$ then gives in the NS–NS sector $G_{\mu\nu}$ and 58 scalars, $B_{\mu\nu}$ and 22 scalars, and the dilaton Φ ; in the R–R sector we have $A_{\mu\nu\rho}$ and 22 vectors in addition to A_μ . In six dimensions, $A_{\mu\nu\rho}$ can be dualized into a vector, so all in all the bosonic fields comprise a graviton, 1 antisymmetric two-form tensor, 24 $U(1)$ vectors and 81 scalars. Hence, we end up with the following supermultiplets of six-dimensional $(1, 1)$ (non-chiral) supersymmetry:

$$1 \text{ supergravity multiplet, } 20 \text{ vector multiplets,} \quad (3.3)$$

where we recall that:

– the $(1, 1)$ supergravity multiplet comprises a graviton, 2 Weyl gravitinos of opposite chirality, 4 vectors, 4 Weyl spinors of opposite chirality, 1 antisymmetric tensor, 1 real scalar;

– a vector multiplet comprises 1 vector, 2 Weyl spinors of opposite chirality, 4 scalars. The scalars parametrize $R \times SO(4, 20) / (SO(4) \times SO(20))$, where the first factor corresponds to the dilaton up to a global $O(4, 20, Z)$ T -duality identification.

Type IIB. The ten-dimensional massless bosonic spectrum consists of the NS–NS fields G_{MN} , B_{MN} , Φ , and the self-dual four-form A_{MNRs}^+ , the two-form A_{MN} and the zero-form A from the R–R sector. Compactification on $K3$ then gives in the NS–NS sector the same as for type IIA. In the R–R sector, we obtain respectively $A_{\mu\nu\rho\sigma}^+$ (which is not physical), 22 $B_{\mu\nu}^{R-R}$ (of which 19 anti-self-dual and 3 self-dual) and 1 scalar, $A_{\mu\nu}$ and 22 scalars, and the scalar A itself. If we decompose both $B_{\mu\nu}$ and $A_{\mu\nu}$ into a self-dual and an anti-self-dual part, the bosonic content comprises a graviton, 5 self-dual and 21 anti-self-dual antisymmetric tensors and 105 scalars. Hence, we end up with the following six-dimensional $(2, 0)$ (chiral) supermultiplets:

$$1 \text{ supergravity multiplet, } 21 \text{ tensor multiplets,} \quad (3.4)$$

where we recall that:

- the $(2, 0)$ supergravity multiplet comprises a graviton, 5 self-dual antisymmetric tensors, 2 left Weyl gravitinos, 2 Weyl fermions;
- a $(2, 0)$ tensor multiplet comprises 1 anti-self-dual antisymmetric tensor, 5 scalars, 2 Weyl fermions of chirality opposite to that of the gravitinos.

The scalars including the dilaton parametrize the coset space $SO(5, 21) / (SO(5) \times SO(21))$, and the low-energy supergravity has a global $O(5, 21, R)$ symmetry [26].

3.2 One-loop three-graviton scattering amplitude in six dimensions

We must consider the piece quartic in momenta of the one-loop three-point function:

$$\mathcal{I} = \epsilon_{1\mu\nu}\epsilon_{2\kappa\lambda}\epsilon_{3\rho\sigma} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \int \prod_{i=1}^3 \frac{d^2z_i}{\pi} \langle V^{\mu\nu}(p_1, \bar{z}_1, z_1) V^{\kappa\lambda}(p_2, \bar{z}_2, z_2) V^{\rho\sigma}(p_3, \bar{z}_3, z_3) \rangle. \quad (3.5)$$

Here the space-time indices run over $\mu = 0, \dots, 5$ (see Appendix A for conventions), and the vertex operators in the 0-picture are

$$V^{\mu\nu}(p, \bar{z}, z) = \left(\bar{\partial} X^\mu(\bar{z}, z) + ip \cdot \bar{\psi}(\bar{z}) \bar{\psi}^\mu(\bar{z}) \right) \left(\partial X^\nu(\bar{z}, z) + ip \cdot \psi(z) \psi^\nu(z) \right) e^{ip \cdot X(\bar{z}, z)}, \quad (3.6)$$

where the polarization tensor $\epsilon_{\mu\nu}$ is symmetric traceless for a graviton ($\rho \equiv 1$) and antisymmetric for an antisymmetric two-form gauge field ($\rho \equiv -1$).

Altogether the physical conditions are

$$\epsilon_{\mu\nu} = \rho \epsilon_{\nu\mu}, \quad p^\mu \epsilon_{\mu\nu} = 0, \quad p^\mu p_\mu = 0, \quad p_1 + p_2 + p_3 = 0. \quad (3.7)$$

Note that they imply $p_i \cdot p_j = 0$ for all i, j . Were the p_i 's real and the metric Minkowskian, this would indicate that the momenta are in fact collinear, and all three-point amplitudes would vanish due to kinematics. This can be evaded by going to complex momenta in Euclidean space.

The expression (3.6) gives the form for all the vertex operators when we take the even spin structure both on the left and the right. When one spin structure (say left) is odd, though,

the presence of a conformal Killing spinor together with a world-sheet gravitino zero-mode requires one of the vertex operators (say the last one) be converted to the -1 -picture on the left

$$V^{\mu\nu}(p, \bar{z}, z) = \left(\bar{\partial} X^\mu(\bar{z}, z) + ip \cdot \bar{\psi}(\bar{z}) \bar{\psi}^\mu(\bar{z}) \right) \psi^\nu(z) e^{ip \cdot X(\bar{z}, z)}, \quad (3.8)$$

and a left-moving supercurrent

$$G_F = \partial X^\gamma \psi_\gamma + G_F^{\text{int}} \quad (3.9)$$

be inserted at an arbitrary point on the world-sheet [27].

There are four possible spin-structure combinations to consider, which can be grouped in two pairs according to whether they describe CP-even or CP-odd couplings,

$$\text{CP-even: } \begin{cases} \bar{e}-e \\ \bar{o}-o \end{cases} \quad \text{CP-odd: } \begin{cases} \bar{e}-o \\ \bar{o}-e, \end{cases} \quad (3.10)$$

where we denote e (o) the even (odd) spin structure on the left and the barred analogues for those on the right.

Because of the physical conditions (3.7), the only kinematic structures that can appear at four-derivative order are, in an index-free notation (see Appendix A):

$$(p_1 \epsilon_2 p_3)(p_2 \epsilon_1 \epsilon_3 p_2) \quad \text{and} \quad p_1 \wedge p_2 \wedge p_1 \epsilon_2 \wedge p_2 \epsilon_1 \wedge \epsilon_3 \quad (3.11)$$

up to permutations of $(1, 2, 3)$.

The low-energy action can then be determined by finding Lorentz-invariant terms that yield the same vertices on shell. Depending on the polarization of the incoming particles, the string amplitude can be reproduced by the following terms in the effective action (see Appendix A for more details):

$$\begin{aligned} R^2 &\equiv R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = (p_1 h(p_2) p_3) (p_2 h(p_1) h(p_3) p_2) \\ \nabla H \nabla H &\equiv \nabla_\mu H_{\nu\rho\sigma} \nabla^\mu H^{\nu\rho\sigma} = 6 (p_1 h(p_2) p_3) (p_2 b(p_1) b(p_3) p_2) \\ B \wedge R \wedge R &\equiv \epsilon^{\mu\nu\kappa\lambda\rho\sigma} B_{\mu\nu} R_{\kappa\lambda}^{\alpha\beta} R_{\rho\sigma\alpha\beta} = -2 p_1 \wedge p_2 \wedge p_1 h(p_2) \wedge p_2 h(p_1) \wedge b(p_3) \\ B \wedge \nabla H \wedge \nabla H &\equiv \epsilon^{\mu\nu\kappa\lambda\rho\sigma} B_{\mu\nu} \nabla_\kappa H_\lambda^{\alpha\beta} \nabla_\rho H_{\sigma\alpha\beta} = -2 p_1 \wedge p_2 \wedge p_1 b(p_2) \wedge p_2 b(p_1) \wedge b(p_3) \\ H \wedge H \wedge R &\equiv \epsilon^{\mu\nu\kappa\lambda\rho\sigma} H_{\mu\nu\kappa} H_\lambda^{\alpha\beta} R_{\rho\sigma\alpha\beta} = 6 p_2 \wedge p_3 \wedge p_2 h(p_3) \wedge p_3 b(p_2) \wedge b(p_1). \end{aligned} \quad (3.12)$$

In these expressions, $h(p_i)$ and $b(p_i)$ denote the Fourier components of the graviton and antisymmetric tensor, which we identify with the polarizations ϵ_i in the string calculation, $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}$ is the field strength of the two-form potential, and the left-hand side defines a short-hand notation for the corresponding term (in agreement with standard notation up to factors of $\sqrt{-g}$).

The precise meaning to be attributed to Eq. (3.12) is, for instance:

$$\int d^6 x \sqrt{-g} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \underset{\text{on shell}}{=} \int \frac{d^6 p_1 d^6 p_2 d^6 p_3}{(2\pi)^{12}} \delta^{(6)}(p_1 + p_2 + p_3) (p_1 h(p_2) p_3) (p_2 h(p_1) h(p_3) p_2). \quad (3.13)$$

Note that other four-derivative terms such as squared Ricci tensor or squared scalar curvature do not contribute at three-graviton scattering in traceless gauge, so that their coefficient cannot be fixed at this order. That this remains true at four-graviton scattering was proved in [28]; it can be seen as a consequence of the field redefinition freedom $g_{\mu\nu} \rightarrow g_{\mu\nu} + aR_{\mu\nu} + bRg_{\mu\nu}$, which generates R^2 and $R_{\mu\nu}R^{\mu\nu}$ couplings from the variation of the Einstein term. Similarly, the coupling of two antisymmetric tensors and one graviton could as well be reproduced by a variety of RHH terms, equivalent under field redefinitions.

We now defer the interested reader to Appendix D for the actual detailed evaluation of the string amplitude, and merely state the salient results:

- The $\bar{e}-e$ sector manifestly receives $O(p^4)$ contributions from contractions of four fermions on both sides, and the resulting terms in the effective action are

$$\mathcal{I}_{\text{eff}}^{\bar{e}-e} = 32\pi^3 \int d^6x \sqrt{-g} \left(R^2 + \frac{1}{6} \nabla H \nabla H \right). \quad (3.14)$$

- In the $\bar{o}-o$ sector we find the same result, but with an overall minus sign depending on whether we consider type IIA or IIB:

$$\mathcal{I}_{\text{eff}}^{\bar{o}-o} = 32\pi^3 \varepsilon \int d^6x \sqrt{-g} \left(R^2 + \frac{1}{6} \nabla H \nabla H \right). \quad (3.15)$$

Therefore, one-loop string corrections generate R^2 and $\nabla H \nabla H$ terms in the effective action of type IIA superstring on $K3$, while no such terms appear in the type IIB superstring.

- The CP-odd sectors $\bar{e}-o$ and $\bar{o}-e$ again lead to the same vertices up to a sign depending on type IIA, B but also on the nature of the particles involved. This leaves

$$\mathcal{I}_{\text{eff, IIA}}^{\text{CP-odd}} = 32\pi^3 \int d^6x \sqrt{-g} \frac{1}{2} (B \wedge R \wedge R + B \wedge \nabla H \wedge \nabla H), \quad (3.16a)$$

$$\mathcal{I}_{\text{eff, IIB}}^{\text{CP-odd}} = -32\pi^3 \int d^6x \sqrt{-g} \frac{1}{6} H \wedge H \wedge R. \quad (3.16b)$$

Summarizing, we can put the results (3.14), (3.15) for the CP-even terms and (3.16) for the CP-odd terms together, and we record the one-loop four-derivative terms in the six-dimensional effective action for type IIA and IIB:

$$\mathcal{I}_{\text{eff, IIA}} = 32\pi^3 \mathcal{N}_6 \int d^6x \sqrt{-g} \left(2R^2 + \frac{1}{3} \nabla H \nabla H + \frac{1}{2} B \wedge (R \wedge R + \nabla H \wedge \nabla H) \right), \quad (3.17a)$$

$$\mathcal{I}_{\text{eff, IIB}} = -32\pi^3 \mathcal{N}_6 \int d^6x \sqrt{-g} \frac{1}{6} H \wedge H \wedge R, \quad (3.17b)$$

where we introduced an overall normalization constant \mathcal{N}_6 .

As a check note that the type IIA theory should be invariant under a combined space-time (P) and world-sheet parity (Ω). Since the Levi-Civita ϵ tensor changes sign under P while the B field changes sign under Ω , we verify the correct invariance under $P\Omega$. On the other hand, the type IIB theory is correctly invariant under the world-sheet parity Ω , since the interactions contain an even number of antisymmetric tensor fields.

We should stress here that these thresholds, although they were computed at the T^4/Z_2 orbifold point of $K3$ are valid for any value of the $K3$ moduli. The reason is that the threshold is proportional to the elliptic genus of $K3$ (which in this case is equal to the $K3$ Euler number) and thus is moduli-independent. It can also be seen directly in the T^4/Z_2 calculation as follows. The result is obviously independent of the $(4, 4)$ orbifold moduli. All the other moduli have vertex operators that are proportional to the twist fields of the orbifold. The correlator of three gravitons or antisymmetric tensors and one of the extra moduli is identically zero, since the symmetry changes the sign of twist fields. Thus, the derivatives of the threshold with respect to the extra moduli are zero.

4 One-loop gravitational corrections in four-dimensional type II models

Further compactification of six-dimensional $N = 2$ type IIA, B string theory on a two-torus yields $N = 4$ string theories in four dimensions. Six-dimensional duality between heterotic string on T^4 and type IIA string on $K3$ is expected to descend to a duality between the corresponding four-dimensional $N = 4$ compactified theories. Moreover, the two compact flat dimensions make it possible to construct more exotic compactifications, preserving $N = 4$ in four dimensions [17, 22], via the fermionic construction or constructions based on freely-acting asymmetric orbifolds. As we will see in Section 6, the models obtained in this way may have heterotic S -duals or type II U -duals. In the following we shall be interested in computing the four-dimensional counterparts of the six-dimensional four-derivative gravitational terms for generic $N = 4$ ground states. Before that, however, we shall briefly recall some features of $N = 4$ supersymmetry.

4.1 Four-dimensional $N = 4$ supersymmetry and its BPS states

Massless multiplets of $N = 4$ four-dimensional supersymmetry, with helicity less than or equal to 2, are the gravity multiplet (1 graviton, 4 gravitinos, 6 graviphotons, 4 fermions, 1 complex scalar) and the vector multiplet (1 photon, 4 fermions and 6 real scalars). In particular, the six-dimensional $N = (1, 1)$ gravity multiplet decomposes under reduction into the four-dimensional $N = 4$ gravity multiplet plus two $N = 4$ vector multiplets, while the six-dimensional chiral $N = (2, 0)$ gravity multiplet yields one four-dimensional $N = 4$ gravity multiplet plus one $N = 4$ vector multiplet (upon dualization of four-dimensional two-form potentials into scalars). On the other hand, both the six-dimensional $N = (1, 1)$ vector and $N = (2, 0)$ tensor multiplets reduce to one $N = 4$ vector multiplet each.

The generic massive L^j representation of $N = 4$ supersymmetry contains 128 bosonic plus 128 fermionic states generated by the action of eight fermionic raising operators on a spin $j \in Z/2$ vacuum (j denotes the representation of the $SO(3)$ little group of massive

representations). However, when the central charge matrix degenerates, only 6 or 4 of the raising operators survive, respectively yielding intermediate BPS representations I^j of dimension 64 or short BPS representations³ S^j of dimension 16. Such BPS states can be traced by using helicity supertraces, which behave as “indices” counting unpaired BPS multiplets [15]. More details about the actual computation of helicity supertraces can be found in Appendix B.

4.2 Gravitational thresholds in four dimensions

The two-derivative low-energy effective action for $N = 4$ theories is believed to be exact at tree level, but higher-derivative terms can receive perturbative and non-perturbative one-loop corrections. We will be interested in computing the moduli dependence of the four-derivative terms involving the graviton, antisymmetric tensor and dilaton, more generally called gravitational thresholds. The terms of interest are therefore:

$$\begin{aligned} \mathcal{I}_{\text{eff}} = \int d^4x \sqrt{-g} \quad & \left(\Delta_{\text{gr}}(T, U) R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \Theta_{\text{gr}}(T, U) \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} \right. \\ & + \Delta_{\text{as}}(T, U) \nabla_\mu H_{\nu\rho\sigma} \nabla^\mu H^{\nu\rho\sigma} + \Theta_{\text{as}}(T, U) \epsilon^{\mu\nu\rho\sigma} \nabla_\mu H_{\nu\alpha\beta} \nabla_\rho H_\sigma{}^{\alpha\beta} \\ & + \Delta_{\text{dil}}(T, U) \nabla_\mu \nabla_\nu \Phi \nabla^\mu \nabla^\nu \Phi + \Theta_{\text{dil-as}}(T, U) \epsilon^{\mu\nu\rho\sigma} \nabla_\mu \nabla_\alpha \Phi \nabla^\alpha H^{\nu\rho\sigma} \\ & \left. + \Theta_{\text{gr-as}}(T, U) \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\alpha\beta} \nabla_\rho H_\sigma{}^{\alpha\beta} \right). \end{aligned} \quad (4.1)$$

Again, we shall use a short-hand notation for each term appearing in the above expression: R^2 , $R \wedge R$, $\nabla H \nabla H$, $\nabla H \wedge \nabla H$, $\nabla \nabla \Phi \nabla \nabla \Phi$, $\nabla \nabla \Phi \wedge \nabla H$, $R \wedge \nabla H$. Note that there is no non-vanishing on-shell RH -coupling between one graviton and one two-form, nor any $\nabla \nabla \Phi \wedge \nabla \nabla \Phi$ or $\nabla \nabla \Phi \wedge R$ couplings. The various terms in Eq. (4.1) will turn out to be expressible in terms of helicity supertraces and, as such, will receive contributions from BPS states only. They therefore offer a reliable window into the strong-coupling regime.

We will now concentrate on the derivation of general formulas for gravitational thresholds in four-dimensional type II models descending from type II six-dimensional vacua compactified on $K3$. At first, one might think that such thresholds could be evaluated by computing two-graviton scattering. Such an amplitude, however, vanishes on shell, and is potentially infrared-divergent. A rigorous and unambiguous way to deal with this problem was described in [29, 30] and further analysed in [31]; this amounts to regularizing the infrared by turning on background fields that provide the theory with a mass gap. This method preserves some of the original supersymmetries of the theory: up to $N = 2$ for heterotic ground states, and up to $(p, q) = (2, 2)$ for type II ground states, where we denote by p and q the number of supersymmetries coming from the left and the right. However, this procedure does not allow us to discriminate the various interaction terms appearing in (4.1), by lack of a sufficient number of marginal operators that could be turned on as background fields.

Here, however, we shall only be interested in the (T, U) moduli dependence of the four-derivative gravitational couplings in the effective action; it will therefore be sufficient to compute the scattering amplitude between two gravitons (or two two-forms or two dilatons) and moduli fields. This will give access to $\partial_\phi \Delta$ and $\partial_\phi \Theta$, which are infrared-finite.

³The massless representations are always short representations.

The same comments as in the six-dimensional case apply to the choice of vertices in Eq. (4.1) for describing the string amplitude. In particular, one may add to this expression terms such as $R_{\mu\nu}R^{\mu\nu}$ or R^2 without changing the S-matrix, and for instance choose instead of $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ the Gauss–Bonnet combination ($R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$), which has the advantage of being a total derivative at second order in h and therefore does not correct the graviton propagator⁴. This would be useful if one were to look at four-particle scattering, where field theory subtraction enters into play [24]. Also, it will turn out that the naive $R\wedge R$, $\nabla H\wedge\nabla H$ and $\nabla\nabla\Phi\wedge\nabla H$ terms, chosen to represent the CP-odd interaction of gravitons with moduli, are inadequate and have to be supplemented by Chern–Simons couplings. With these provisos, the kinematical structures contributing to gravitational thresholds read:

$$\begin{aligned}
R^2 &= (p_1 h(p_2) p_1) (p_2 h(p_1) p_2) - 2(p_1 p_2) (p_2 h(p_1) h(p_2) p_1) + (p_1 p_2)^2 (h(p_1) h(p_2)) \\
\nabla H \nabla H &= 6(p_1 p_2) (p_2 b(p_1) b(p_2) p_1) - 3(p_1 p_2)^2 (b(p_1) b(p_2)) \\
R \wedge R &= -2h(p_2) p_1 \wedge h(p_1) p_2 \wedge p_1 \wedge p_2 - 2(p_1 p_2) h(p_1) h(p_2) \wedge p_1 \wedge p_2 \\
\nabla H \wedge \nabla H &= -2b(p_2) p_1 \wedge b(p_1) p_2 \wedge p_1 \wedge p_2 + 2(p_1 p_2) b(p_1) b(p_2) \wedge p_1 \wedge p_2 \\
R \wedge \nabla H &= -2b(p_2) p_1 \wedge h(p_1) p_2 \wedge p_1 \wedge p_2 + 2(p_1 p_2) h(p_1) b(p_2) \wedge p_1 \wedge p_2 \\
\nabla\nabla\Phi\nabla\nabla\Phi &= (p_1 p_2)^2 \\
\nabla\nabla\Phi \wedge \nabla H &= 3(p_1 p_2) p_1 \wedge p_2 \wedge b(p_2)
\end{aligned} \tag{4.2}$$

and it is readily checked that these expressions are consistent with gauge invariance $\epsilon \rightarrow \epsilon + p \otimes k + \rho k \otimes p$ with $k \cdot p = 0$.

The second equation in (4.2) shows that the $\nabla H \nabla H$ coupling cannot be revealed by a three-particle amplitude. This forces us to look at scattering amplitudes involving at least two gravitons (or two two-forms or two dilatons) and two moduli. In fact, the insertion of any number of moduli remains tractable as long as two complex-conjugated moduli are not simultaneously present, and we shall therefore keep with the general case of N moduli.

4.3 Two-graviton– N -moduli scattering amplitude

The class of $(2, 2)$ supersymmetric models descending from six-dimensional type II string on $K3$ can be generically described at the Z_2 orbifold point of $K3$ by the following partition function:

$$\begin{aligned}
Z_{\text{II}}^{\text{four dim}} &= \frac{1}{\tau_2 |\eta|^{12}} \frac{1}{2} \sum_{a,b=0}^1 (-1)^{a+b+ab} \vartheta^2 \begin{bmatrix} a \\ b \end{bmatrix} \frac{1}{2} \sum_{\bar{a},\bar{b}=0}^1 (-1)^{\bar{a}+\bar{b}+\mu\bar{a}\bar{b}} \bar{\vartheta}^2 \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} \\
&\quad \times \frac{1}{2} \sum_{h,g=0}^1 \vartheta \begin{bmatrix} a+h \\ b+g \end{bmatrix} \vartheta \begin{bmatrix} a-h \\ b-g \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a}+h \\ \bar{b}+g \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a}-h \\ \bar{b}-g \end{bmatrix} Z_{6,6} \begin{bmatrix} h \\ g \end{bmatrix}
\end{aligned}$$

⁴The Gauss–Bonnet combination in four dimensions is a total derivative to any order in h , so one might wonder how it could describe vertices at all. The answer is that the vertices derived from it only vanish when taking into account all kinematical restrictions on momenta and polarizations special to four dimensions. Those no longer exist when going to Euclidean complex momenta. We thank R. Woodard for discussions on this point.

$$\equiv \sum_{a,b=0,1} \sum_{\bar{a},\bar{b}=0,1} Z \begin{bmatrix} \bar{a} & a \\ \bar{b} & b \end{bmatrix}, \quad (4.3)$$

where $Z_{6,6} \begin{bmatrix} h \\ g \end{bmatrix}$ are generic⁵ orbifold blocks whose structure depends on the specific way the Z_2 group acts on the various states of the spectrum.

For such a vacuum, we shall need to extract the four-momenta part from the following amplitude:

$$\mathcal{I}_\phi = \epsilon_{1\mu\nu} \epsilon_{2\kappa\lambda} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \int \prod_{i=1}^{N+2} \frac{d^2z_i}{\pi} \left\langle V^{\mu\nu}(p_1, \bar{z}_1, z_1) V^{\kappa\lambda}(p_2, \bar{z}_2, z_2) \prod_{j=3}^{N+2} V_{\phi_j}(p_j, \bar{z}_j, z_j) \right\rangle, \quad (4.4)$$

containing two gravitons or antisymmetric tensor fields (depending on the polarization tensors $\epsilon_{i\mu\nu}$) and N two-torus moduli fields. In contrast to Section 3, the space-time indices now run over $\mu = 0, \dots, 3$, but the vertex operators of the space-time fields are identical to those given in Eq. (3.6) for the 0-picture, Eq. (3.8) for the -1 -picture on the left, etc. In close analogy, in the 0-picture the vertex operators of the moduli fields are given by

$$V_\phi(p, \bar{z}, z) = v_{IJ}(\phi) \left(\bar{\partial} X^I(\bar{z}, z) + ip \cdot \bar{\psi}(\bar{z}) \bar{\psi}^I(\bar{z}) \right) \left(\partial X^J(\bar{z}, z) + ip \cdot \psi(z) \psi^J(z) \right) e^{ip \cdot X(\bar{z}, z)}, \quad (4.5)$$

where

$$v_{IJ}(\phi) = \partial_\phi (G_{IJ} + B_{IJ}), \quad I, J = 1, 2. \quad (4.6)$$

In particular, in the standard (T, U) parametrization recalled in Appendix C, we have

$$v_{IJ}(T) \bar{\partial} X^I \partial X^J = \frac{1}{2iU_2} \bar{\partial} X \partial \tilde{X}, \quad v_{IJ}(U) \bar{\partial} X^I \partial X^J = \frac{iT_2}{2U_2^2} \bar{\partial} \tilde{X} \partial \tilde{X} \quad (4.7)$$

with $X = X^4 + UX^5$, $\tilde{X} = X^4 + \bar{U}X^5$ (and similarly $\Psi = \psi^4 + U\psi^5$), while the vertices for \bar{T}, \bar{U} are obtained by complex conjugation. Note that chiral moduli (T, U) have $\partial \tilde{X}$ as left-moving part, while the antichiral ones (\bar{T}, \bar{U}) have ∂X instead.

The modifications for -1 -picture on the right and/or left are as described in Section 3, so that for example for the -1 -picture on the left we have

$$V_T(p, \bar{z}, z) = \frac{1}{2iU_2} \left(\bar{\partial} X + ip \cdot \bar{\psi} \bar{\Psi} \right) \tilde{\Psi} e^{ip \cdot X(\bar{z}, z)} \quad (4.8)$$

together with an insertion of the left-moving supercurrent

$$G_F = \partial X^\mu \psi_\mu + G_{KL} \partial X^K \psi^L = \partial X^\mu \psi_\mu + \partial X \tilde{\Psi} + \partial \tilde{X} \Psi, \quad (4.9)$$

where we omitted the $K3$ internal part of G_F .

We will again defer the details of the computation to Appendix D, and simply outline the calculation here. A drastic simplification occurs thanks to a selection rule that forbids contractions not conserving the $U(1)$ charge of the T^2 superconformal theory:

$$\langle XX \rangle = \langle \tilde{X} \tilde{X} \rangle = \langle \Psi \Psi \rangle = \langle \tilde{\Psi} \tilde{\Psi} \rangle = 0. \quad (4.10)$$

⁵In particular, they are not necessarily lattice partition functions, and may carry dependence on several untwisted or twisted moduli.

Except when a pair of complex-conjugated moduli occurs, only the zero-mode of the bosonic part of the moduli vertices contributes and generates for each insertion a derivative with respect to the corresponding modulus (together, in the odd structure, with a sign depending on the nature of the last modulus). Supersymmetry then demands that the fermionic part of the two gravitons be contracted together, yielding the four powers of momenta as desired. The $\bar{e}-e$ and $\bar{o}-o$ kinematics turn out to be equal in the two-graviton case and opposite in the two-antisymmetric-tensor case (zero in the graviton–two-form case).

Our final result for the one-loop moduli dependence of the four-derivative gravitational couplings in Eq. (4.1) is summarized by

$$\partial_\phi \Delta_{\text{gr}}(T, U) = \frac{\mathcal{N}_4}{\pi^4} \int_{\mathcal{F}} d^2\tau \frac{1}{2} \partial_\phi (\kappa^{\bar{e}e} Z^{\bar{e}e} - \kappa^{\bar{o}o} Z^{\bar{o}o}) \quad (4.11a)$$

$$\partial_\phi \Delta_{\text{as}}(T, U) = \frac{\mathcal{N}_4}{\pi^4} \int_{\mathcal{F}} d^2\tau \frac{1}{12} \partial_\phi (\kappa^{\bar{e}e} Z^{\bar{e}e} + \kappa^{\bar{o}o} Z^{\bar{o}o}) \quad (4.11b)$$

$$\partial_\phi \Delta_{\text{dil}}(T, U) = \frac{\mathcal{N}_4}{\pi^4} \int_{\mathcal{F}} d^2\tau \frac{1}{2} \partial_\phi (\kappa^{\bar{e}e} Z^{\bar{e}e} + \kappa^{\bar{o}o} Z^{\bar{o}o}) \quad (4.11c)$$

$$\partial_\phi \Theta_{\text{gr}}(T, U) = -\frac{\mathcal{N}_4}{\pi^4} \int_{\mathcal{F}} d^2\tau \frac{1}{4} \partial_\phi (\kappa^{\bar{e}o} Z^{\bar{e}o} + \kappa^{\bar{o}e} Z^{\bar{o}e}) \quad (4.11d)$$

$$\partial_\phi \Theta_{\text{as}}(T, U) = -\frac{\mathcal{N}_4}{\pi^4} \int_{\mathcal{F}} d^2\tau \frac{1}{4} \partial_\phi (\kappa^{\bar{e}o} Z^{\bar{e}o} + \kappa^{\bar{o}e} Z^{\bar{o}e}) \quad (4.11e)$$

$$\partial_\phi \Theta_{\text{gr-as}}(T, U) = -\frac{\mathcal{N}_4}{\pi^4} \int_{\mathcal{F}} d^2\tau \frac{1}{2} \partial_\phi (\kappa^{\bar{e}o} Z^{\bar{e}o} - \kappa^{\bar{o}e} Z^{\bar{o}e}) \quad (4.11f)$$

$$\partial_\phi \Theta_{\text{dil-as}}(T, U) = -\frac{\mathcal{N}_4}{\pi^4} \int_{\mathcal{F}} d^2\tau \frac{1}{3} \partial_\phi (\kappa^{\bar{e}o} Z^{\bar{e}o} - \kappa^{\bar{o}e} Z^{\bar{o}e}), \quad (4.11g)$$

where \mathcal{N}_4 is a normalization constant that we will fix later. The derivative ∂_ϕ stands for the product $\prod_{j=3}^N \partial_{\phi_j}$. The $\kappa^{\bar{i}j}$ are numerical coefficients that depend on the choice of type II string as well as on the choice of moduli:

$$\kappa^{\bar{i}j} = \begin{cases} 1, & \bar{i}, j = \bar{e}, e \\ -\sigma_\phi \varepsilon, & \bar{i}, j = \bar{o}, o \\ i\chi_\phi, & \bar{i}, j = \bar{e}, o \\ i\sigma_\phi \chi_\phi \varepsilon, & \bar{i}, j = \bar{o}, e, \end{cases} \quad (4.12)$$

where (χ_ϕ, σ_ϕ) specifies the nature of the last modulus (see Eqs. (D.32a,b)) and the conformal blocks $Z^{\bar{i}j}$ are expressed in terms of the blocks $Z\left[\begin{smallmatrix} \bar{a} & a \\ \bar{b} & b \end{smallmatrix}\right]$ appearing in the four-dimensional partition function (4.3):

$$Z^{\bar{i}j} = \begin{cases} 16\pi^2 \sum_{(a,b), (\bar{a},\bar{b}) \text{ even}} Z\left[\begin{smallmatrix} \bar{a} & a \\ \bar{b} & b \end{smallmatrix}\right] \partial_{\bar{\tau}} \log\left(\frac{\vartheta\left[\begin{smallmatrix} \bar{a} \\ \bar{b} \end{smallmatrix}\right]}{\eta}\right) \partial_{\tau} \log\left(\frac{\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}{\eta}\right), & \bar{i}, j = \bar{e}, e \\ 'Z'\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right], & \bar{i}, j = \bar{o}, o \\ -4\pi i \sum_{(\bar{a},\bar{b}) \text{ even}} \partial_{\bar{\tau}} \log\left(\frac{\vartheta\left[\begin{smallmatrix} \bar{a} \\ \bar{b} \end{smallmatrix}\right]}{\eta}\right) Z'\left[\begin{smallmatrix} \bar{a} & 1 \\ \bar{b} & 1 \end{smallmatrix}\right], & \bar{i}, j = \bar{e}, o \\ 4\pi i \sum_{(a,b) \text{ even}} 'Z'\left[\begin{smallmatrix} 1 & a \\ 1 & b \end{smallmatrix}\right] \partial_{\tau} \log\left(\frac{\vartheta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}{\eta}\right), & \bar{i}, j = \bar{o}, e. \end{cases} \quad (4.13)$$

In the previous expression, a prime on the left and/or the right stands for the operation in Eq. (D.3).

4.4 Gravitational thresholds and helicity supertraces

Using Riemann identity and (2, 2) supersymmetry, it is readily seen that the four blocks $Z^{\bar{i}j}$ are equal to $Z^{\bar{e}e}$. Moreover, identity (B.16) allows us to convert the ∂_τ derivative in $Z^{\bar{e}e}$ into a second-order derivative with respect to the variable v conjugate to the left helicity λ_L , as described in Appendix B. A similar statement applies to the right side, yielding:

$$Z^{\bar{i}j} = \frac{16\pi^4}{\tau_2} \langle \lambda_L^2 \lambda_R^2 \rangle = \frac{8\pi^4}{3\tau_2} \langle (\lambda_L + \lambda_R)^4 \rangle = \frac{8\pi^4}{3\tau_2} B_4. \quad (4.14)$$

Substituting in Eq. (4.11), we obtain for instance

$$\partial_\phi \Delta_{\text{gr}}(T, U) = \frac{8}{3} \mathcal{N}_4 \frac{1 + \varepsilon \sigma_\phi}{2} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \partial_\phi B_4, \quad (4.15)$$

and similar relations for the other thresholds. This makes it obvious that only short BPS states contribute to the one-loop four-derivative gravitational corrections. From now on, it will be convenient to fix the normalization constant to

$$\mathcal{N}_4 = \frac{3}{8}. \quad (4.16)$$

We note that the four different spin structures contribute in the same way to $\partial_\phi \Delta_{\text{gr}}$, but for signs depending on the type A or B of superstring and the modulus ϕ we are considering. As a result of this interference:

$$\text{type IIA: } \begin{cases} \partial_T \Delta_{\text{gr}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \partial_T B_4 \\ \partial_U \Delta_{\text{gr}} = 0 \end{cases} \quad (4.17a)$$

$$\text{type IIB: } \begin{cases} \partial_T \Delta_{\text{gr}} = 0 \\ \partial_U \Delta_{\text{gr}} = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \partial_U B_4. \end{cases} \quad (4.17b)$$

We recover in this way the well-known result that Δ_{gr} only depends on the Kähler moduli T and not on the complex-structure moduli U in type IIA, while the reverse is true in type IIB [32]. Similar interferences occur for all thresholds and yield the following moduli dependences:

$$\text{IIA: } \Delta_{\text{gr}}(T), \Delta_{\text{as}}(U), \Delta_{\text{dil}}(U), \Theta_{\text{gr}}(T), \Theta_{\text{as}}(T), \Theta_{\text{gr-as}}(U), \Theta_{\text{dil-as}}(U), \quad (4.18a)$$

$$\text{IIB: } \Delta_{\text{gr}}(U), \Delta_{\text{as}}(T), \Delta_{\text{dil}}(T), \Theta_{\text{gr}}(U), \Theta_{\text{as}}(U), \Theta_{\text{gr-as}}(T), \Theta_{\text{dil-as}}(T). \quad (4.18b)$$

The dependence of $\Delta_{\text{gr}}(T)$ is consistent with our argument that the R^2 term does not get corrections beyond one loop. However, there exists a subgroup of $SO(6, N_V, Z)$ that exchanges the (type IIA) U -modulus with the dilaton S -modulus, so that $SO(6, N_V, Z)$ duality implies that $\Delta_{\text{as}}, \Delta_{\text{dil}}, \Theta_{\text{gr-as}}, \Theta_{\text{dil-as}}$ are also S -dependent, i.e. are perturbatively and non-perturbatively corrected. The loophole in the argument of Section 2 is that, for these couplings, the world-sheet instantons of the type IIB string are non-zero (since they depend on the type IIB T -modulus), and therefore the (p, q) D 1-branes do contribute to instanton corrections. From now on we shall restrict ourselves to R^2 thresholds, for which the type II one-loop result is exact [25, 33, 34].

5 Gravitational thresholds in ordinary type II on $K3 \times T^2$

We now apply the previous formalism to the trivial reduction to four dimensions of type II string theory on $K3$. Using the considerations in Subsection 4.1 and the six-dimensional spectrum, it follows that both type IIA and type IIB on $K3 \times T^2$ have

$$1 \text{ supergravity multiplet, } 22 \text{ vector multiplets.} \quad (5.1)$$

The two theories are indeed exchanged by T -duality on one circle of T^2 , which corresponds to the exchange of T and U moduli. The scalars therefore span $SU(1,1)/U(1) \times SO(6,22)/(SO(6) \times SO(22))$, where the $SU(1,1)/U(1)$ factor corresponds to the complex scalar in the gravitational multiplet (T for IIA, U for IIB) [17].

Type II string theory on $K3 \times T^2$ at the T^4/Z_2 orbifold point is described by the following partition function:

$$\begin{aligned} \Pi_{22}^{(2,2)} : Z = & \frac{1}{\tau_2 |\eta|^{24}} \frac{1}{2} \sum_{a,b=0}^1 (-1)^{a+b+ab} \vartheta^2 \begin{bmatrix} a \\ b \end{bmatrix} \frac{1}{2} \sum_{\bar{a},\bar{b}=0}^1 (-1)^{\bar{a}+\bar{b}+\mu\bar{a}\bar{b}} \bar{\vartheta}^2 \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} \\ & \times \frac{1}{2} \sum_{h,g=0}^1 \vartheta \begin{bmatrix} a+h \\ b+g \end{bmatrix} \vartheta \begin{bmatrix} a-h \\ b-g \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a}+h \\ \bar{b}+g \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a}-h \\ \bar{b}-g \end{bmatrix} \Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix} \Gamma_{2,2}(T, U), \end{aligned} \quad (5.2)$$

where we use the same $\Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix}$ blocks as in (3.2).

5.1 Helicity supertraces and R^2 corrections

The helicity supertrace B_4 entering in the threshold (4.15) can be readily computed from (5.2) using the methods of Appendix B, with the result:

$$B_4 = 36 \Gamma_{2,2}. \quad (5.3)$$

It is easy to check the $\tau_2 \rightarrow \infty$ limit, where only short BPS massless states contribute, with the result:

$$B_4|_{\text{massless}} = 1 \times 3 + 22 \times \frac{3}{2} = 36, \quad (5.4)$$

where we used the contributions in Eq. (B.2) for the supergravity and vector multiplets. The expression in Eq. (5.3) further shows that the rest of the contributions to B_4 come from the tower of massive short BPS multiplets whose vertex operators are those of the massless states plus momenta and windings of the two-torus. The matching condition implies that we should have $\vec{m}\vec{n} = 0$ for these states and they are in $N = 4$ supermultiplets similar to the massless ones. This result is expected, since we know that a left-moving state breaks half of the two left-moving supersymmetries. Thus states that are ground states both on the left and right (plus momentum of the two-torus) are expected to break half out of the total of four supersymmetries in agreement with the helicity supertrace.

Using Eq. (B.20), not much more work is required to extract the B_6 supertrace

$$B_6 = 90 \Gamma_{2,2}, \quad (5.5)$$

whose $\tau_2 \rightarrow \infty$ limit again agrees with the massless (short BPS) spectrum since $1 \times \frac{195}{4} + 22 \times \frac{15}{8} = 90$. However, although we know that intermediate multiplets, corresponding to states that are ground states on the left only, but with arbitrary oscillator excitations on the right (or reversed) could contribute to B_6 , they turn out to cancel as a consequence of identity (B.22). We therefore conclude that intermediate BPS multiplets come in combinations that can always be paired into long massive multiplets and thus do not contribute to B_6 . Their multiplicities and mass formulae are therefore not protected from quantum corrections. This example indicates that one has to be careful when invoking non-renormalization theorems for BPS states. Only BPS states having non-zero “index” are protected from quantum corrections.

We now insert B_4 into Eq. (4.15) and use the fundamental-domain integral (C.8) to obtain the R^2 thresholds:

$$\text{type IIA: } \Delta_{\text{gr}}(T) = -36 \log \left(T_2 |\eta(T)|^4 \right) + \text{const.}, \quad (5.6a)$$

$$\text{type IIB: } \Delta_{\text{gr}}(U) = -36 \log \left(U_2 |\eta(U)|^4 \right) + \text{const.}, \quad (5.6b)$$

where the constant is undetermined in our scheme. The above result is in agreement with [8]. Note that the one-loop thresholds are respectively invariant under $SL(2, Z)_T$ and $SL(2, Z)_U$, as they should. Moreover, since only the twisted sectors $(h, g) \neq (0, 0)$ of T^4/Z_2 contribute to B_4 , Δ_{gr} as well as the other thresholds are independent of the untwisted moduli of $K3$, and therefore of all $K3$ moduli. Consequently, the result obtained at the orbifold point T^4/Z_2 is valid everywhere in the moduli space of $K3$.

5.2 Decompactification limit of CP-even couplings: a puzzle

It is important to confront this result to our six-dimensional result (3.17a), which should be retrieved in the decompactification limit of the two-torus, $T_2 = \sqrt{G} \rightarrow \infty$:

$$\text{II}_{22}^{(2,2)} : \Delta_{\text{gr}}(T) \xrightarrow{T_2 \rightarrow \infty} -36 \log T_2 + 12\pi T_2 + O\left(e^{-T_2}\right) \quad (5.7)$$

in the type IIA situation. This agrees with Eq. (3.17a) provided we set

$$\mathcal{N}_6 = \frac{4\pi^2}{3}. \quad (5.8)$$

On the other hand, taking the large-volume limit in the type IIB theory does not affect the U -dependent threshold. However, only terms of order T_2 (the volume of the torus) can be seen in the decompactification limit, so this agrees with the vanishing of R^2 coupling in six dimensions (3.17b).

We can repeat the same discussion for the four-dimensional $\nabla H \nabla H$ threshold, which has the same behaviour up to $T \leftrightarrow U$ interchange, and predict that the six-dimensional coupling $\nabla H \nabla H$ should occur only in type IIB and not in type IIA, in contrast to R^2 . This is in disagreement with our six-dimensional result, which showed that cancellation between $\bar{e}-e$ and $\bar{o}-o$ spin structures had to occur in the same way for both R^2 and $\nabla H \nabla H$. Note that

we could also have performed the three-graviton–two-form scattering calculation directly in four dimensions, finding the same result for $\bar{e}-e$ as in six dimensions, but a vanishing $\bar{o}-o$ contribution. We would have concluded that R^2 and $\nabla H \nabla H$ have to occur with the same (T, U) -dependent coupling, in both types IIA and IIB. This shows that the three-particle amplitude has to be interpreted with great care.

5.3 CP-odd couplings and holomorphic anomalies

Moving on to the CP-odd couplings and focusing on the IIA case for definiteness, Eq. (4.11d) yields

$$\partial_T \Theta_{\text{gr}} = -18i \partial_T \log \left(T_2 |\eta(T)|^4 \right), \quad \partial_{\bar{T}} \Theta_{\text{gr}} = 18i \partial_{\bar{T}} \log \left(T_2 |\eta(T)|^4 \right). \quad (5.9)$$

Would the non-harmonic T_2 term be absent, those two equations could be easily integrated and would give

$$\Theta_{\text{gr}}(T) = 18 \text{Im} \log \eta^4(T). \quad (5.10)$$

However, in the presence of the T_2 term the notation $\partial_T \Theta$ and $\partial_{\bar{T}} \Theta$ for CP-odd couplings between two gravitons and one modulus no longer makes sense. This non-integrability of CP-odd couplings has already been encountered before [35]. This problem can be evaded simply by rewriting the CP-odd coupling as

$$\mathcal{I}_{\text{gr}}^{\text{CP-odd}} = \int \Omega \wedge (Z_T dT + Z_{\bar{T}} d\bar{T}), \quad (5.11)$$

where Ω is the gravitational Chern–Simons three-form, such that $d\Omega = R \wedge R$. In the special case $Z_T = \partial_T \Theta(T, \bar{T})$, $Z_{\bar{T}} = \partial_{\bar{T}} \Theta(T, \bar{T})$, one retrieves by partial integration the usual integrable CP-odd coupling. In the case at hand,

$$Z_T = -18i \partial_T \log \left(T_2 |\eta(T)|^4 \right), \quad Z_{\bar{T}} = 18i \partial_{\bar{T}} \log \left(T_2 |\eta(T)|^4 \right). \quad (5.12)$$

We can take advantage of the special structure of Eq. (5.12) and rewrite Eq. (5.11) as

$$\mathcal{I}_{\text{as}}^{\text{CP-odd}} = 18\pi \int \left(\text{Im} \left(\log \eta^4(T) \right) R \wedge R - \frac{1}{T_2} \Omega \wedge dT_1 \right). \quad (5.13)$$

In the decompactification limit $T_2 \rightarrow \infty$, only the first term survives and we obtain

$$\mathcal{I}_{\text{gr}}^{\text{CP-odd}} = 18 \int \left(\frac{\pi}{3} T_1 R \wedge R + O(1/T_2) \right). \quad (5.14)$$

This reproduces the six-dimensional type IIA result (3.17a):

$$\begin{aligned} \mathcal{I}_{\text{IIA}}^{\text{six dim}} &= 16\pi^3 \mathcal{N}_6 \int d^6 x \sqrt{-g_6} \epsilon^{\mu\nu\kappa\lambda\rho\sigma} B_{\mu\nu} R_{\kappa\lambda\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} \\ &\xrightarrow{\text{four dim}} 6\pi \int d^4 x \sqrt{-g_4} T_2 \epsilon^{IJ} B_{IJ} \epsilon^{\kappa\lambda\rho\sigma} R_{\kappa\lambda\alpha\beta} R_{\rho\sigma}{}^{\alpha\beta} \end{aligned} \quad (5.15)$$

since $\epsilon^{IJ} B_{IJ} = 2T_1/T_2$ and $\mathcal{N}_6 = 4\pi^2/3$.

Exactly the same feature arises for the Θ_{as} , $\Theta_{\text{gr-as}}$ and $\Theta_{\text{dil-as}}$ cases, for which, in the type IIA case, the correct coupling should instead be written as

$$\mathcal{I}_{\text{as}}^{\text{CP-odd}} = 18\pi \int \left(\text{Im} \left(\log \eta^4(T) \right) \nabla H \wedge \nabla H - \frac{1}{T_2} H \wedge \nabla H \wedge dT_1 \right) \quad (5.16a)$$

$$\mathcal{I}_{\text{gr-as}}^{\text{CP-odd}} = 36\pi \int \left(\text{Im} \left(\log \eta^4(U) \right) R \wedge \nabla H - \frac{1}{T_2} R \wedge H \wedge dT_1 \right) \quad (5.16b)$$

$$\mathcal{I}_{\text{dil-as}}^{\text{CP-odd}} = 24\pi \int \left(\text{Im} \left(\log \eta^4(U) \right) \nabla \nabla \Phi \wedge \nabla H - \frac{1}{T_2} \nabla \nabla \Phi \wedge H \wedge dT_1 \right). \quad (5.16c)$$

Note also that $\nabla H \nabla H$ correctly decompactifies to the $B \wedge \nabla H \wedge \nabla H$ of six-dimensional type IIA theory in just the same way as $R \wedge R$, while in type IIB $R \wedge \nabla H$ gives the correct $H \wedge H \wedge R = -3B \wedge R \wedge \nabla H$ six-dimensional coupling. The $\nabla \nabla \Phi \wedge \nabla H$ coupling cannot be checked here since we did not consider six-dimensional dilaton scattering.

5.4 From type II to heterotic string

Coming now to duality, it is well known that heterotic on T^4 – type IIA on $K3$ duality in six dimensions implies, after compactification, the duality of the corresponding four-dimensional theories under exchange of S and T , where S is the axion–dilaton multiplet, sitting in the gravitational multiplet on the heterotic side.

For definiteness we recall the partition function of heterotic string on T^6 :

$$\text{HET}_{22} : Z = \frac{1}{\tau_2 \eta^{12} \bar{\eta}^{24}} \frac{1}{2} \sum_{a,b=0}^1 (-1)^{a+b+ab} \vartheta^4 \begin{bmatrix} a \\ b \end{bmatrix} \Gamma_{6,22}(G, B, A), \quad (5.17)$$

where $\Gamma_{6,22}(G, B, A)$ depends on the six-dimensional metric G , the antisymmetric tensor B and the Wilson lines A . At generic points of the moduli space (i.e. with gauge group broken to $U(1)$ factors), the massless bosonic spectrum is

$$1 \text{ supergravity multiplet, } 22 \text{ vector multiplets,} \quad (5.18)$$

in agreement with (5.1), as expected by duality. Contrary to the type II case, the heterotic string theory possesses a tree-level $R \wedge R$ coupling required for anomaly cancellation through the Green–Schwarz mechanism, together with an R^2 coupling required for supersymmetry. The world-sheet fermions now have 10 zero-modes, so that the one-loop three-particle amplitude vanishes (in even spin structure, one would need four fermionic contractions to have a non-vanishing result after spin-structure summation). In particular, we conclude that there is no one-loop correction to tree-level R^2 coupling.

Following [8], we can therefore translate the type IIA result (5.6a) for the heterotic string on T^6 :

$$\text{HET}_{22} : \Delta_{\text{gr}}(S) = -36 \log \left(S_2 |\eta(S)|^4 \right) \xrightarrow{S_2 \rightarrow \infty} -36 \log S_2 + 12\pi S_2 + O \left(e^{-S_2} \right). \quad (5.19)$$

The $S_2 \rightarrow \infty$ heterotic weak-coupling limit exhibits the tree-level R^2 coupling together with a non-perturbatively seen logarithmic divergence. The latter was omitted in Ref. [8], where

only the Wilsonian effective action was investigated, but is also present in other instances [25]. The full threshold is manifestly invariant under $SL(2, Z)_S$, and could in fact be inferred from $SL(2, Z)_S$ completion of the tree-level result. The exponentially suppressed terms in Eq. (5.19) were identified in [8] with the instanton contributions of the neutral heterotic NS 5-brane wrapped on T^6 , the only instanton configuration that can possibly occur in four-dimensional heterotic string.

The same mapping can be executed for the CP-odd $R \wedge R$ coupling from Eq. (5.16):

$$\mathcal{I}_{\text{gr}}^{\text{CP-odd}} = 18 \int \left(\text{Im} \left(\log \eta^4(S) \right) R \wedge R - \frac{1}{S_2} \Omega \wedge dS_1 \right). \quad (5.20)$$

There, however, in addition to the tree-level term and instead of the logarithmic divergence, we find a coupling between the axion and the gravitational Chern–Simons form. Dualizing the axion into a two-form and keeping track of the powers of the heterotic coupling S_2 , this translates into a *one-loop* coupling $H_{\mu\nu\rho} \Omega^{\mu\nu\rho}$ between one two-form and two gravitons, precluded by a one-loop heterotic calculation. Happily enough, the Chern–Simons form is co-closed, so that this coupling is a total derivative.

6 Reduced-rank $N = 4$ models and breaking of S -duality

Although the most studied $N = 4$ dual string pair is the standard heterotic on T^6 – type IIA on $K3 \times T^2$ pair with generic gauge group $U(1)^{28}$, more exotic models with a lower gauge-group rank do exist. Since all $N = 4$ matter multiplets have to transform into the adjoint representation of the gauge group, their expectation values cannot break it to a group with lower rank, and those theories therefore have to live in disconnected moduli spaces.

On the type II side, such models can be easily obtained by compactifying the six-dimensional IIA on $K3$ theory at orbifold points of $K3$ using a generalized Scherk–Schwarz mechanism [36, 37, 38] to give a (moduli-dependent) mass [21, 22, 23] to part of the vector multiplets originating from the twisted sectors of $K3$. This can be implemented by orbifolding the IIA on $K3 \times T^2$ theory by a translation on the torus accompanied by an action on the twisted sector.

On the heterotic side, such models have been constructed in Ref. [39] with fermionic characters, but it is difficult to identify them with models dual to the above type II, since that would require identifying the point in heterotic moduli space corresponding to the orbifold points of $K3$. Nevertheless, if one trusts six-dimensional heterotic–type IIA duality, such heterotic duals are guaranteed to exist.

The construction on the type II side makes it clear that T -duality is *broken* to a subgroup by the precise translation vector on the $\Gamma_{2,2}$ lattice, which translates in heterotic variables into a *breaking* of S -duality. This breaking modifies the non-perturbative instanton corrections in lower-rank heterotic or type II theories discussed below. In the following, we shall examine the four-derivative perturbative gravitational corrections in various type II models, and translate them in terms of non-perturbative effects on the heterotic side.

6.1 The $\overline{\text{II}}_6^{(2,2)} - \text{II}_6^{(4,0)}$ U -dual type II pair

Here we consider a variation of the type II over $T^4/Z_2 \times T^2$ compactification described above (see model $\text{II}_{22}^{(2,2)}$, (5.2)). The Z_2 will now act both as a twist on the T^4 and as a shift on the two-torus. This model is a spontaneously broken $N = 8 \rightarrow 4$ theory with $(2, 2)$ supersymmetry, as will become clear shortly, and will be denoted by $\overline{\text{II}}_6^{(2,2)}(w)$. The partition function reads:

$$\begin{aligned} \overline{\text{II}}_6^{(2,2)}(w) : Z = & \frac{1}{\tau_2 |\eta|^{24}} \frac{1}{2} \sum_{a,b=0}^1 (-1)^{a+b+ab} \vartheta^2 \begin{bmatrix} a \\ b \end{bmatrix} \frac{1}{2} \sum_{\bar{a},\bar{b}=0}^1 (-1)^{\bar{a}+\bar{b}+\mu\bar{a}\bar{b}} \bar{\vartheta}^2 \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} \\ & \times \frac{1}{2} \sum_{h,g=0}^1 \vartheta \begin{bmatrix} a+h \\ b+g \end{bmatrix} \vartheta \begin{bmatrix} a-h \\ b-g \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a}+h \\ \bar{b}+g \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a}-h \\ \bar{b}-g \end{bmatrix} \Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix} \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}, \end{aligned} \quad (6.1)$$

where $\Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix}$ are the twisted $(4, 4)$ lattice sums (see Eq. (3.2)) and $\Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}$ are the shifted $(2, 2)$ lattice sums given in Appendix C. Modular invariance requires the shift vector w to satisfy $w^2 = 0$. The 16 twisted vector multiplets from the $T^4/Z_2 \times T^2$ model now acquire a mass of the order of the inverse radii of T^2 , so that the massless spectrum becomes:

$$1 \text{ supergravity multiplet, } 6 \text{ vector multiplets.} \quad (6.2)$$

The scalars of the 6 vector multiplets parametrize $SO(6, 6)/(SO(6) \times SO(6))$, while the complex scalar in the gravitational multiplet corresponds to the T modulus (resp. U) in type IIA (resp. B) theories.

It is now straightforward to compute helicity supertraces directly from Eq. (6.1) and using (B.19), (B.20). They read⁶

$$B_4 = 12 \sum'_{(h,g)} \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \sim 12, \quad (6.3a)$$

$$B_6 = 15 \sum'_{(h,g)} \left(2 + \text{Re } H \begin{bmatrix} h \\ g \end{bmatrix} \right) \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \sim 60, \quad (6.3b)$$

where $H \begin{bmatrix} h \\ g \end{bmatrix}$ are given in Eq. (B.21). In Eq. (6.3), we have indicated after the \sim sign the contributions of the massless states, obtained by using the fact that only the $\Gamma_{2,2}^w \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ block contains massless states, as well as the leading behaviour $H \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 + O(q)$. As a check, we observe the correct values for the contributions of the massless states,

$$B_4|_{\text{massless}} = 1 \times 3 + 6 \times \frac{3}{2} = 12, \quad (6.4a)$$

$$B_6|_{\text{massless}} = 1 \times \frac{195}{4} + 6 \times \frac{15}{8} = 60, \quad (6.4b)$$

where we used the elementary contributions (B.2) and (B.3). Moreover, we observe that

⁶The primed summation over (h, g) stands for $(h, g) \in \{(0, 1), (1, 0), (1, 1)\}$.

in contrast to the ordinary type IIA theory on $K3 \times T^2$ (model (5.2)), the intermediate multiplets do contribute to B_6 .

Inserting the result (6.3a) in Eq. (4.15) that we recall here

$$\partial_\phi \Delta_{\text{gr}}(T, U) = \frac{1 + \varepsilon \sigma_\phi}{2} \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} \partial_\phi B_4, \quad (6.5)$$

allows us to determine the gravitational thresholds in terms of B_4 . Fundamental-domain integrals involving $\Gamma_{2,2} \begin{bmatrix} h \\ g \end{bmatrix}$ are computed in Appendix C, and yield, for the type IIA case,

$$\overline{\text{II}}_6^{(2,2)}(w) : \Delta_{\text{gr}}(T) = -12 \log \left(T_2 |\vartheta_i(T)|^4 \right) + \text{const.}, \quad (6.6)$$

where $i = 2, 3, 4$, depending on the shift vector w (see Appendix C). An important consequence is that the resulting corrections break the $SL(2, Z)_T$ duality group to a $\Gamma(2)_T$ subgroup. The precise subgroup depends on i as indicated in Appendix C.

This model was argued in [14] to be U -dual to a $(4, 0)$ supersymmetric type II model, to which we now turn. This model is obtained as a Z_2 orbifold of type II on T^6 , where the Z_2 acts as $(-1)^{F_L}$ together with a translation on T^6 . Again, this model exhibits spontaneously broken $N = 8 \rightarrow 4$ supersymmetry and we will denote it by $\text{II}_6^{(4,0)}(w)$. The resulting partition function reads:

$$\begin{aligned} \text{II}_6^{(4,0)}(w) : Z &= \frac{1}{\tau_2 |\eta|^{24}} \frac{1}{2} \sum_{a,b=0}^1 (-1)^{a+b+ab} \vartheta^4 \begin{bmatrix} a \\ b \end{bmatrix} \frac{1}{2} \sum_{\bar{a}, \bar{b}=0}^1 (-1)^{\bar{a}+\bar{b}+\mu \bar{a} \bar{b}} \bar{\vartheta}^4 \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} \\ &\times \frac{1}{2} \sum_{h,g=0}^1 (-1)^{ag+bh+gh} \Gamma_{4,4} \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}. \end{aligned} \quad (6.7)$$

To compute the massless spectrum, we first recall that for $N = 8$ type II, obtained by compactifying on T^6 , the spectrum is as follows: NS–NS gives $G_{\mu\nu}$, $B_{\mu\nu}$, Φ and 12 vectors as $(6, 6)$ and $6 \times 6 = 36$ scalars; R–R gives 16 vectors as $(0, 16)$ and 32 scalars. Because of the $(-1)^{F_L}$ orbifold, the R–R sector is projected out so we are left with the NS–NS states only, which combine into the following four-dimensional $N = 4$ multiplets:

$$1 \text{ supergravity multiplet, } 6 \text{ vector multiplets,} \quad (6.8)$$

in agreement with the massless spectrum (6.2) of the dual theory. The complex scalar in the gravitational multiplet now corresponds to the axion–dilaton field.

For completeness, the helicity supertraces for this model can be computed using (B.16) and (B.17):

$$B_4 = \frac{3}{4} \Gamma_{4,4} \sum'_{(h,g)} H_4 \begin{bmatrix} h \\ g \end{bmatrix} \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \sim 12, \quad (6.9a)$$

$$B_6 = \frac{15}{8} \Gamma_{4,4} \sum'_{(h,g)} \left(H_4 \begin{bmatrix} h \\ g \end{bmatrix} + H_6 \begin{bmatrix} h \\ g \end{bmatrix} \right) \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \sim 60, \quad (6.9b)$$

where

$$H_4 \begin{bmatrix} h \\ g \end{bmatrix} = e^{i\pi hg} \frac{\vartheta^4 \begin{bmatrix} 1-h \\ 1-g \end{bmatrix}}{\eta^{12}}, \quad H_6 \begin{bmatrix} h \\ g \end{bmatrix} = \begin{cases} \frac{1}{2} \frac{\vartheta_3^8 - \vartheta_4^8}{\eta^{12}}, & (h, g) = (0, 1) \\ \frac{1}{2} \frac{\vartheta_2^8 - \vartheta_3^8}{\eta^{12}}, & (h, g) = (1, 0) \\ \frac{1}{2} \frac{\vartheta_4^8 - \vartheta_2^8}{\eta^{12}}, & (h, g) = (1, 1), \end{cases} \quad (6.10)$$

from which we see that B_4 again receives contributions only from massless and massive short BPS multiplets, while B_6 also gets contributions from intermediate ones.

However, for $(4, 0)$ supersymmetric models, a four-graviton scattering calculation shows that the one-loop corrections to R^2 terms do not involve helicity supertraces. Instead, the one-loop corrections simply vanish, and the only contributions to R^2 couplings, as argued in the introduction, are non-perturbative. Now, U -duality can be invoked to obtain the non-perturbative $(4, 0)$ result from the one-loop result (6.6) of the $(2, 2)$ dual, by identifying the T -modulus of the $(2, 2)$ theory with the S -modulus of the $(4, 0)$ theory. There is however an important subtlety involved in identifying the lattice shifts on both sides. We recall that in the full non-perturbative spectrum, states have not only electrical charges m_i, n^i under the Kaluza–Klein gauge fields of T^2 , but also have magnetic charges \tilde{m}_i, \tilde{n}^i . Under $S \leftrightarrow T$ interchange, electric and magnetic charges are mapped to each other according to [22]

$$(m_i, n^i, \tilde{m}_i, \tilde{n}^i) \rightarrow (m_i, \epsilon^{ij} \tilde{m}_j, -\epsilon_{ij} \tilde{n}^j, \tilde{n}^i). \quad (6.11)$$

In particular, a $(-1)^{n^1}$ projection on states with even electric winding n^1 on the $(2, 2)$ side translates into a $(-1)^{\tilde{m}_2}$ projection on the $(4, 0)$ side, of no effect in perturbation theory. A $(-1)^{m_1}$ projection on the other hand in the $(2, 2)$ theory translates into a perturbative $(-1)^{m_1}$ in the dual $(4, 0)$ theory. These two projections have a geometrical interpretation of doubling one radius of T^2 , in contrast to the $(-1)^{n^1}$ one. However, $(2, 2)$ perturbative modular invariance requires at the same time half-integer n^1 charges in the twisted sector. This implies also half-integer \tilde{m}_2 charges in the twisted sector of the dual $(2, 2)$ theory, which should presumably be accompanied by a $(-1)^{\tilde{n}^2}$ under some “non-perturbative modular invariance” requirement. This in turn would imply that the correct projection on the $(2, 2)$ side is $(-1)^{m_1 + \tilde{n}^2}$, which reduces to $(-1)^{m_1}$ in the perturbative spectrum. This ambiguity does not affect the perturbative evaluation of thresholds. As for non-perturbative corrections, the relevant instantons are a subset of the original ones, which have been shown to not contribute to R^2 couplings. Restricting to a projection on the electrical momenta only (cases I, II, III in Table C.1), we find from Eq. (6.6) the result:

$$\Pi_6^{(4,0)}(w_{\text{I,II,III}}) : \Delta_{\text{gr}}(S) = -12 \log \left(S_2 |\vartheta_4(S)|^4 \right) + \text{const.} \quad (6.12)$$

This exhibits the expected feature [7] that the S -duality symmetry is broken to a $\Gamma(2)_S$ subgroup of $SL(2, Z)_S$, namely the subgroup that leaves $\vartheta_4(S)$ invariant. The two theories are weakly coupled in the regime $T_2, S_2 \rightarrow \infty$. The $T_2 \rightarrow \infty$ decompactification limit of shifted $(2, 2)$ lattice sums was investigated in [21], with the result:

$$\Gamma_{2,2}^{w_{\text{I,II,III}}} \begin{bmatrix} h \\ g \end{bmatrix} \xrightarrow{T_2 \rightarrow \infty} \frac{T_2}{\tau_2} \delta_{h,0} \delta_{g,0} \quad (6.13)$$

up to exponentially suppressed corrections. This selects the untwisted unprojected sector of the two models (6.1), (6.7), thereby restoring $N = 8$ supersymmetry for both of them, in agreement with U -duality conjecture. Expanding Eq. (6.12) in the weak $(4, 0)$ coupling limit, we find

$$\Pi_6^{(4,0)}(w_{\text{I,II,III}}) : \Delta_{\text{gr}}(S) \xrightarrow{S_2 \rightarrow \infty} -12 \log S_2 + O(e^{-S_2}) . \quad (6.14)$$

The result exhibits the correct vanishing of perturbative $O(S_2^n)$ corrections, together with the already encountered non-perturbative logarithmic divergence.

Let us now turn to the strong-coupling behaviour of the $(4, 0)$ ground state. The $S_2 \rightarrow 0$ limit of $(4, 0)$ is mapped under duality to the $T_2 \rightarrow 0$ limit of the $(2, 2)$ ground state, for which we can again use the results of Ref. [21]:

$$\Gamma_{2,2}^{w_{\text{I,II,III}}} \begin{bmatrix} h \\ g \end{bmatrix} \xrightarrow{T_2 \rightarrow 0} \frac{1}{\tau_2 T_2} \forall h, g \quad (6.15)$$

up to exponentially suppressed corrections. The orbifold action does not affect the T^2 part any longer, thereby yielding the standard type II on $K3 \times T^2$ model of Section 5 at small radius. This is strictly true only in the perturbative regime of type II, because of the non-perturbative ambiguities mentioned before. This is further mapped to the HET_{22} model at large coupling $S_2 \rightarrow 0$ and large radius $T_2 \rightarrow \infty$. We therefore conclude that the $(4, 0)$ model and the standard heterotic model on T^6 are equivalent in the strong-coupling large-radius limit⁷. This can also be checked on the explicit R^2 coupling

$$\Pi_6^{(4,0)}(w_{\text{I,II,III}}) : \Delta_{\text{gr}}(S) \xrightarrow{S_2 \rightarrow 0} -12 \log S_2 + 12\pi S_2 + O(e^{-S_2}) , \quad (6.16)$$

which reproduces the correct heterotic on T^4 tree-level coupling (5.19). This set of relations is depicted on the upper and rear faces of the cube in Fig. 1.

6.2 $\text{II}_6^{(2,2)}$ free orbifold of type II on $K3$ and its heterotic dual HET_6

We now turn to another example of $N = 4$ four-dimensional duality, which this time descends from six-dimensional string-string duality by a freely-acting orbifold, namely a half-lattice shift on T^2 together with a minus sign on the twisted sector of $K3$. The adiabatic argument [10] guarantees that the heterotic model obtained by translating this action in heterotic string on T^4 is still dual to the type II orbifold. To be explicit, the resulting partition function for this type II model, denoted by $\Pi_6^{(2,2)}(w)$, is given by

$$\begin{aligned} \Pi_6^{(2,2)}(w) : Z &= \frac{1}{\tau_2 |\eta|^{24}} \frac{1}{2} \sum_{a,b=0}^1 (-1)^{a+b+ab} \vartheta^2 \begin{bmatrix} a \\ b \end{bmatrix} \frac{1}{2} \sum_{\bar{a},\bar{b}=0}^1 (-1)^{\bar{a}+\bar{b}+\mu\bar{a}\bar{b}} \bar{\vartheta}^2 \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} \\ &\times \frac{1}{2} \sum_{h,g=0}^1 \frac{1}{2} \sum_{h',g'=0}^1 (-1)^{hg'+gh'} \vartheta \begin{bmatrix} a+h \\ b+g \end{bmatrix} \vartheta \begin{bmatrix} a-h \\ b-g \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a}+h \\ \bar{b}+g \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a}-h \\ \bar{b}-g \end{bmatrix} \end{aligned}$$

⁷One may ask whether the two limits commute. The correct prescription is to first take $T_2 \rightarrow \infty$ and then only $S_2 \rightarrow 0$ in $(4, 0)$ variables, since we needed the $(2, 2)$ dual to be weakly coupled before we could conclude anything about its small-radius limit.

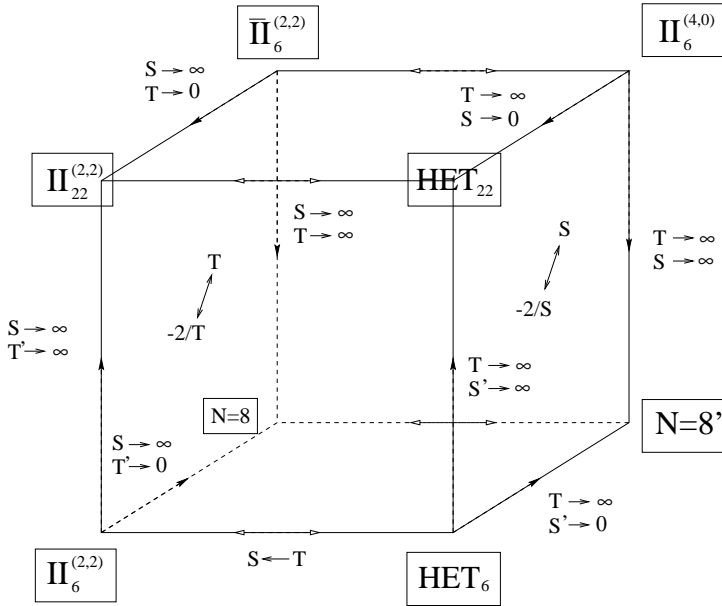


Figure 1: The cube of duality, decompactification and strong/weak coupling relations

$$\times \Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix} \Gamma_{2,2}^w \begin{bmatrix} h' \\ g' \end{bmatrix}. \quad (6.17)$$

Again, the shift vector w has to satisfy $w^2 = 0$ for modular invariance. The (h, g) projections are associated with the T^4/Z_2 orbifold, while the freely-acting transformations correspond to the (h', g') projection.

The massless spectrum is most easily obtained from the results at the beginning of Section 5, by noting that the $(-1)^h$ orbifold projects out the twisted states, so that we are left with the following untwisted four-dimensional $N = 4$ multiplets:

$$1 \text{ supergravity multiplet, } 6 \text{ vector multiplets.} \quad (6.18)$$

The relevant helicity supertraces are (we use again the results (B.19) and (B.20))

$$B_4 = 6 \left(3\Gamma_{2,2} - \sum'_{(h,g)} \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \right) \sim 12, \quad (6.19a)$$

$$B_6 = 15 \left(3\Gamma_{2,2} - \sum'_{(h,g)} \left(1 - \text{Re} H \begin{bmatrix} h \\ g \end{bmatrix} \right) \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \right) \sim 60, \quad (6.19b)$$

where functions $H \begin{bmatrix} h \\ g \end{bmatrix}$ are given in Eq. (B.21). We deduce the type IIA gravitational thresholds:

$$\Pi_6^{(2,2)}(w) : \Delta_{\text{gr}}(T) = -12 \log \left(T_2 \frac{|\eta(T)|^6}{|\vartheta_i(T)|^2} \right) + \text{const.}, \quad (6.20)$$

where $i = 2, 3, 4$, depending on the shift vector w (see Table C.1). As advocated in the previous section, we shall restrict our discussion to shift vectors leading to $i = 4$, for which the resulting T -duality group is $\Gamma^+(2)_T$.

We now want to discuss the heterotic dual for this model. From six-dimensional string duality, the Z_2 symmetry acting as -1 on all twisted states of $K3$ at the orbifold point has to have an equivalent in the dual heterotic string for the corresponding values of the $SO(6, 22)$ heterotic moduli. At present, there remains a puzzle as to what these values are [40]. Nevertheless, this symmetry can in principle be used to construct a freely-acting orbifold of heterotic string on T^4 , and the adiabatic argument guarantees that the resulting model will be dual to the present $\text{II}_6^{(2,2)}(w)$ model. Henceforth we shall refer to this model as $\text{HET}_6(w)$. The heterotic coupling is given by the area of the type II torus, which, owing to the free action, is $T/2$. We therefore deduce the non-perturbative threshold for $\text{HET}_6(w)$:

$$\begin{aligned} \text{HET}_6(w_{\text{I,II,III}}) : \Delta_{\text{gr}}(S) &= -12 \log \left(2S_2 \frac{|\eta(2S)|^6}{|\vartheta_4(2S)|^2} \right) + \text{const.} \\ &\xrightarrow{S_2 \rightarrow \infty} -12 \log S_2 + 12\pi S_2 + O(e^{-S_2}) . \end{aligned} \quad (6.21)$$

In particular, we observe that the tree-level contribution matches the one of the HET_{22} model (5.19), as it should, since the tree-level effective action is universal for all heterotic ground states. The cases corresponding to $i = 2, 3$ in the threshold (6.20) are obtained by applying T -duality on the type II side, yielding $T \rightarrow -1/2S, T \rightarrow 2S - 1$, respectively.

The (large-radius) weak-coupling limit of $\text{HET}_6(w)$ is mapped to the (weak-coupling) large-radius limit of $\text{II}_6^{(2,2)}(w)$, which by the same techniques as in the previous section turns out to be the standard $\text{II}_{22}^{(2,2)}$ model. The latter being dual to the standard HET_{22} , we conclude that $\text{HET}_6(w)$ and HET_{22} are the same in the (large-radius) perturbative regime. The relation between the quartet of theories that we have been discussing can be seen on the front side of the cube in Fig. 1.

The (large-radius) strong-coupling limit of $\text{HET}_6(w)$ can be discussed in the same way as for the $\text{II}_6^{(4,0)}(w)$ model: it corresponds to the (weak-coupling) small-radius limit of $\text{II}_6^{(2,2)}(w)$, which from the partition function (6.17) and from Eq. (6.15) appears to restore $N = 8$ supersymmetry. In fact, $\text{II}_6^{(2,2)}(w)$ and $\overline{\text{II}}_6^{(2,2)}(w)$ are identical under transformation of the moduli, thanks to the relation (C.18)

$$\frac{1}{2} \sum_{h', g'=0}^1 (-1)^{hg'+gh'} \Gamma_{2,2}^w \begin{bmatrix} h' \\ g' \end{bmatrix} (T', U') = \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} (T, U) . \quad (6.22)$$

The precise mapping $(T, U) \rightarrow (T', U')$ is shown in Table C.1. for the various lattice shifts, $T \rightarrow -2/T$ for the cases I, II, III at hand, leading to $i = 4$ in the above formula (6.20). The $N = 8$ (weak-coupling) large-radius limit of $\text{II}_6^{(2,2)}(w)$ therefore coincides with the $N = 8$ (weak-coupling) small-radius limit of $\overline{\text{II}}_6^{(2,2)}(w)$, and is dual to the $N = 8$ (large-radius) strong-coupling limit of $\text{HET}_6(w)$. Furthermore, this implies that $\text{HET}_6(w)$ and $\text{II}_6^{(4,0)}(w)$ are mapped to each other under $S \rightarrow -2/S$.

The various relations among the octet of theories that has been discussed in this and the previous section are summarized by the duality cube in Fig. 1. In this figure, the horizontal connections correspond to $S \leftrightarrow T$ duality and the various connections on the sides of the cube are limits.

6.3 $II_{14}^{(2,2)}$ free orbifold of type II on K3 and its heterotic dual HET_{14}

We now turn to another example of $N = 4$ four-dimensional duality, which this time descends from six-dimensional string-string duality. We now wish to construct models with an intermediate gauge-group rank. To achieve that we need to project out part of the 16 twisted states of T^4/Z_2 . This can be done by using a Z_2 subgroup of the $(D_4)^4$ discrete symmetry of the orbifold T^4/Z_2 [41], generated by

$$D : |+\rangle \leftrightarrow |-\rangle, |m, n\rangle \rightarrow (-1)^m |m, n\rangle \quad (6.23a)$$

$$\tilde{D} : |+\rangle \rightarrow -|+\rangle, |-\rangle \rightarrow |-\rangle, |m, n\rangle \rightarrow (-1)^n |m, n\rangle \quad (6.23b)$$

on each circle, where $|\pm\rangle$ denote the two twisted states and $|m, n\rangle$ the untwisted momentum-winding states corresponding to the chosen circle. The operation D can be interpreted as the remnant of a Z_2 translation on the original circle, carrying one fixed point onto the other.

As a first step we will examine the possibility of projecting out one half of the twisted states and obtain an $SO(6, 14)$ model. Starting from the $T^4/Z_2 \times T^2$ orbifold blocks, $\frac{\Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix} \Gamma_{2,2}}{|\eta|^{12}}$, we mod out a further Z_2 , which acts as a shift on the two-torus, and as the D -operation described above on the T^4/Z_2 . The $(6, 6)$ conformal blocks entering the partition function (4.3) now read:

$$\Pi_{14}^{(2,2)}(w) : Z_{6,6} \begin{bmatrix} h \\ g \end{bmatrix} = \frac{1}{2} \sum_{h', g'=0}^1 \frac{\Gamma_{4,4} \begin{bmatrix} h; h' \\ g; g' \end{bmatrix} \Gamma_{2,2}^w \begin{bmatrix} h' \\ g' \end{bmatrix}}{|\eta|^{12}}, \quad \forall h, g. \quad (6.24)$$

In this expression, (h, g) refer to the original twist while (h', g') refer to the D -shift. According to the definition of the latter (see Eq. (6.23)), the $(4, 4)$ orbifold blocks possess the following properties: for $(h, g) \neq (0, 0)$, $\Gamma_{4,4} \begin{bmatrix} h; 0 \\ g; 0 \end{bmatrix} = \Gamma_{4,4} \begin{bmatrix} h; h \\ g; g \end{bmatrix} = \Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix}$ (ordinary twist); $\Gamma_{4,4} \begin{bmatrix} 0; h \\ 0; g \end{bmatrix}$ is a $(4, 4)$ lattice sum with one shifted momentum (or winding if \tilde{D} is used instead of D), analogous to the $(2, 2)$ constructions of Appendix C; finally, $(4, 4)$ orbifold blocks with $(h, g) \neq (0, 0)$, $(h', g') \neq (0, 0)$ and $h \neq h'$ or $g \neq g'$ vanish because the trace is performed over the original twisted states with the insertion of an operator under which half of the states have eigenvalue $+1$ and the others -1 .

We can now proceed to the computation of the helicity supertraces. For $(h, g) \neq (0, 0)$ our orbifold blocks are of the form (B.18). We therefore use the results (B.19) and (B.20) and find:

$$B_4 = 6 \left(3\Gamma_{2,2} + \sum'_{(h,g)} \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \right) \sim 24, \quad (6.25a)$$

$$B_6 = 15 \left(3\Gamma_{2,2} + \sum'_{(h,g)} \left(1 + \frac{1}{2} \text{Re } H \begin{bmatrix} h \\ g \end{bmatrix} \right) \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \right) \sim 75. \quad (6.25b)$$

We note again that the infrared behaviours of B_4 and B_6 are in agreement with the massless content of the model, namely 1 supergravity multiplet and 14 vector multiplets (of which 8 are twisted).

The gravitational threshold corrections follow from Eq. (6.5),

$$\Pi_{14}^{(2,2)}(w) : \Delta_{\text{gr}}(T) = -24 \log \left(T_2 |\vartheta_i(T)| |\eta(T)|^3 \right) + \text{const.}; \quad (6.26)$$

here the index i depends on the choice of shift vector w (see Table C.1).

As in Subsection 6.2, this model is guaranteed to have a heterotic dual obtained by translating the $(D_4)^4$ action on the heterotic side, at the corresponding point in moduli space. This symmetry is likely to be non-perturbative again. However, in this case it is possible to construct a perturbative heterotic dual with the correct rank 14, which we will denote by $\text{HET}_{14}(w)$. We consider the decomposition of the $\Gamma_{6,22}$ lattice according to

$$\Gamma_{6,22} = \Gamma_{5,5} \oplus \Gamma_{1,1} \oplus \Gamma_{0,8} \oplus \Gamma_{0,8}, \quad (6.27)$$

where the last two terms give $E_8 \times E_8$. The operation that reduces the rank acts as an exchange of the two $\Gamma_{0,8}$ lattices coupled with a translation in $\Gamma_{1,1}$, thereby reducing $E_8 \times E_8$ to its diagonal level-2 subgroup⁸.

Again, the heterotic non-perturbative threshold is obtained by exchanging T with $2S$ (for lattice shift corresponding to $i = 4$ in Eq. (6.26)), and reads:

$$\text{HET}_{14}(w_{\text{I,II,III}}) : \Delta_{\text{gr}}(S) = -24 \log \left(2S_2 |\vartheta_4(2S)| |\eta(2S)|^3 \right) + \text{const.} \quad (6.28)$$

The above expression exhibits the correct tree-level heterotic contribution and the breaking of S -duality by instanton effects.

6.4 $\text{II}_{10}^{(2,2)}$ free orbifold of type II on K3 and its heterotic dual HET_{10}

The method presented in the previous section can be slightly modified so that the original twisted sector of the T^4/Z_2 is left with one quarter of the states only. The model obtained in this way will have rank 16 and $SO(6, 10)/(SO(6) \times SO(10))$ moduli space.

Starting from the orbifold blocks (6.24) of the $SO(6, 14)$ model, we perform an extra Z_2 , which acts on the $(4, 4)$ part as a D -operation along another circle (see Eq. (6.23)), while it amounts to a further shift on the $(2, 2)$ with respect to some momentum-winding direction. In other words, we perform a $Z_2 \times Z_2$ on the original $T^4/Z_2 \times T^2$ construction. The result for the $(6, 6)$ blocks is

$$\Pi_{10}^{(2,2)}(w) : Z_{6,6} \begin{bmatrix} h \\ g \end{bmatrix} = \frac{1}{2} \sum_{h_1, g_1=0}^1 \frac{1}{2} \sum_{h_2, g_2=0}^1 \frac{\Gamma_{4,4} \begin{bmatrix} h; h_1, h_2 \\ g; g_1, g_2 \end{bmatrix} \Gamma_{2,2}^{w_1, w_2} \begin{bmatrix} h_1, h_2 \\ g_1, g_2 \end{bmatrix}}{|\eta|^{12}}, \quad \forall h, g, \quad (6.29)$$

where $\Gamma_{2,2}^{w_1, w_2} \begin{bmatrix} h_1, h_2 \\ g_1, g_2 \end{bmatrix}$ are the $Z_2 \times Z_2$ freely-acting constructions explained in Appendix C and $\Gamma_{4,4} \begin{bmatrix} h; h_1, h_2 \\ g; g_1, g_2 \end{bmatrix}$ are orbifold blocks whose non-vanishing components are the following: for

⁸A rank-14 heterotic model has also been constructed in [39].

$(h, g) \neq (0, 0)$ $\Gamma_{4,4} \begin{bmatrix} h; 0, 0 \\ g; 0, 0 \end{bmatrix} = \Gamma_{4,4} \begin{bmatrix} h; h, 0 \\ g; g, 0 \end{bmatrix} = \Gamma_{4,4} \begin{bmatrix} h; 0, h \\ g; 0, g \end{bmatrix} = \Gamma_{4,4} \begin{bmatrix} h; h, h \\ g; g, g \end{bmatrix} = \Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix}$ (ordinary twist); $\Gamma_{4,4} \begin{bmatrix} 0; h_1, h_2 \\ 0; g_1, g_2 \end{bmatrix}$, which is an ordinary $(4, 4)$ shifted lattice sum corresponding to a freely-acting $Z_2 \times Z_2$, analogous to the ones studied in Appendix C for the $(2, 2)$ lattices. The precise structure of the latter plays no role for the computation of helicity supertraces, since only the $(h, g) \neq (0, 0)$ blocks contribute to gravitational thresholds. By using the results (C.23), these blocks are recast as:

$$\Pi_{10}^{(2,2)}(w) : Z_{6,6} \begin{bmatrix} h \\ g \end{bmatrix} = \frac{1}{4} \frac{\Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix}}{|\eta|^8} \left(\frac{\Gamma_{2,2}}{|\eta|^4} + \sum_{w \in \{w_1, w_2, w_{12}\}} \frac{\Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}}{|\eta|^4} \right), \text{ for } (h, g) \neq (0, 0). \quad (6.30)$$

This expression, combined with Eqs. (B.18), (B.19) and (B.20), therefore leads to the following helicity supertraces:

$$B_4 = 3 \left(3\Gamma_{2,2} + \sum_{w \in \{w_1, w_2, w_{12}\}} \sum'_{(h,g)} \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \right) \sim 18, \quad (6.31)$$

$$B_6 = 15 \left(3\Gamma_{2,2} + \sum_{w \in \{w_1, w_2, w_{12}\}} \sum'_{(h,g)} \left(1 + \frac{1}{2} \text{Re} H \begin{bmatrix} h \\ g \end{bmatrix} \right) \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \right) \sim \frac{135}{2}. \quad (6.32)$$

The leading infrared behaviours reflect the presence of 10 vector multiplets, 6 untwisted and 4 twisted, as expected by construction. The gravitational thresholds are determined as usual:

$$\Pi_{10}^{(2,2)}(w_{(i),(ii),(iii)}) : \Delta_{\text{gr}}(T) = -18 \log \left(T_2 |\vartheta_i(T)|^2 |\eta(T)|^2 \right) + \text{const.}, \quad (6.33)$$

where $i = 4, 2, 3$ respectively for the $Z_2 \times Z_2$ shifted models (i), (ii) and (iii) in Table C.2. Models (iv), (v) and (vi) lead, on the other hand, to the result:

$$\Pi_{10}^{(2,2)}(w_{(iv),(v),(vi)}) : \Delta_{\text{gr}}(T) = -18 \log \left(T_2 |\eta(T)|^4 \right) + \text{const.} \quad (6.34)$$

It is remarkable that this threshold is invariant under the full $SL(2, Z)_T$ duality, but one should refrain from concluding that the $SL(2, Z)_T$ symmetry is restored, since the breaking may appear in quantities other than R^2 thresholds.

In order to construct the heterotic dual with rank 10, we consider the $SO(8) \times SO(8)$ decomposition of each E_8 and the decomposition of $\Gamma_{6,6}$ into $\Gamma_{4,4} \oplus \Gamma_{1,1}^{(1)} \oplus \Gamma_{1,1}^{(2)}$. This lattice has an enhanced $SO(8) \times SO(8)' \times SO(8)'' \times SO(8)'''$ symmetry point⁹, from which we can switch on two discrete Wilson lines, which act independently with exchange and shift as for rank 14. We then perform two Z_2 orbifolds, the first one exchanging $SO(8) \times SO(8)'$ with $SO(8)'' \times SO(8)'''$ while shifting the $\Gamma_{1,1}^{(1)}$, and the second one exchanging $SO(8) \times SO(8)''$ with $SO(8)'' \times SO(8)'''$ while shifting the $\Gamma_{1,1}^{(2)}$. The remaining gauge symmetry is $SO(8)$ at level 4. Again, identifying the precise heterotic dual would require knowing the point in

⁹In four dimensions it is possible to build a model with such gauge group by switching on appropriate discrete Wilson lines, which act by breaking $E_8 \times E_8$ and shifting the mass of the spinors, preventing the reconstruction of E_8 .

heterotic moduli space corresponding to the $K3$ orbifold point, a piece of information that is lacking at present [40].

Finally, for a lattice shift corresponding to $i = 4$ in the threshold (6.33), we find that the duality maps T to $4S$. The other cases $i = 2, 3$ are obtained by applying T -duality on the type II side, yielding $T \rightarrow -1/4S, T \rightarrow 4S - 1$, respectively. We therefore conclude that the exact gravitational threshold in heterotic variables reads:

$$\text{HET}_{10} \left(w_{(i),(ii),(iii)} \right) : \Delta_{\text{gr}}(S) = -18 \log \left(4S_2 |\vartheta_4(4S)|^2 |\eta(4S)|^2 \right) + \text{const.} \quad (6.35)$$

These results indeed are in agreement with the fact that the heterotic dilaton should correspond to the volume form of the base of the $K3$ -fibration, which in this case is $T^2/(Z_2 \times Z_2)$.

On the other hand, in the case of models (iv), (v), (vi) leading to (6.34), the correct tree-level term on the heterotic side is only obtained by substituting $T \rightarrow 2S$, in apparent contradiction with the fact that we have a $Z_2 \times Z_2$ orbifold. This is due to the particular translation on T^2 used to obtain this models: one Z_2 acts as a translation on the electric momenta, which are mapped under type II–heterotic duality to the electric momenta on the heterotic side [22]. The second Z_2 acts instead on the electric windings, which are mapped to the magnetic momenta on the heterotic side, so it is not visible in the heterotic perturbative theory: from the heterotic point of view there is only one Z_2 . The correct map is therefore $T \rightarrow 2S$, and we obtain the threshold:

$$\text{HET}_{10} \left(w_{(iv),(v),(vi)} \right) : \Delta_{\text{gr}}(S) = -18 \log \left(2S_2 |\eta(2S)|^4 \right) + \text{const.} \quad (6.36)$$

7 Conclusions

We have considered the threshold corrections to low-energy R^2 and other four-derivative couplings in heterotic and type II ground states with 16 unbroken supercharges. In particular, we have discussed the ordinary $K3$ compactification and a family of type II vacua that have spontaneously broken $N = 8 \rightarrow 4$ supersymmetry and 4 massive gravitinos in the perturbative spectrum. Those are special cases of more general models with spontaneously broken supersymmetry studied in [21, 22, 23].

We have argued that there are no perturbative or non-perturbative corrections to the R^2 couplings in heterotic ground states in dimension higher than four. In four dimensions, instanton corrections are expected from the heterotic Euclidean 5-brane, and they depend on the S field only. In type II ground states with $(2, 2)$ supersymmetry we have argued that there are no non-perturbative corrections to the R^2 couplings in four dimensions or more. The full result arises from one loop. We have first analysed this threshold in six dimensions, which provides a guide on what to expect in lower dimensions. We have subsequently evaluated this one-loop threshold for several $(2, 2)$ four-dimensional models with various numbers of massless vector multiplets. All such ground states have heterotic duals, and the type II result translates into 5-brane instanton corrections from the heterotic point of view. Most reduced-rank models have an Olive–Montonen duality group that is a subgroup of $SL(2, Z)_S$, namely $\Gamma(2)_S$, which is reflected in the behaviour of the non-perturbative corrections.

The above non-perturbative results should provide a guideline towards the determination of the rules for calculating instanton corrections in string theory. Several steps in this

direction were recently taken [8, 25, 33, 34, 43, 44, 45, 46, 47, 48]. The ultimate goal is to be able to handle non-perturbative effects with less supersymmetry or in its absence.

We have also analysed the CP-odd R^2 four-dimensional couplings, and resolved an apparent puzzle: the type II result implies, via duality, a CP-even coupling at one loop on the heterotic side between the antisymmetric tensor and the gravitational Chern–Simons form. We have shown that this is compatible with heterotic perturbation theory since such a coupling is invisible in on-shell amplitudes.

Finally we have considered type II dual pairs with 16 supercharges and $(2, 2)$ or $(4, 0)$ supersymmetry. The situation there is analogous to the type II–heterotic case. By using some additional perturbative relationships, we find quartets of dual models, one of which is a heterotic ground state, with $N = 4$ in four dimensions, and which, at strong coupling, exhibits enhanced $N = 8$ supersymmetry! The interpretation of this ground state is a spontaneously broken $N = 8 \rightarrow 4$ theory, with 4 solitonic massive gravitinos that become massless at strong coupling, enhancing the supersymmetry to $N = 8$. We believe this possibility to be valuable for constructing interesting models with less supersymmetry and an $N = 8$ high-energy behaviour.

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Appendix A: Kinematics and on-shell field theory vertices

Throughout this paper, we use a d -dimensional metric $g_{\mu\nu}$ with signature $(+, -, -, \dots)$. We evaluate the leading fourth order in momenta scattering amplitudes of gravitational particles in six dimensions, together with moduli in four dimensions. Particles are characterized by their light-like momentum p_i and (except for the moduli) their transverse (i.e. $p_i \epsilon_i = 0$) polarization tensors ϵ_i . The latter are symmetric for gravitons h , antisymmetric for antisymmetric tensors b and pure trace for dilatons Φ . By the latter we mean a polarization $\epsilon_{\mu\nu} = (g_{\mu\nu} - p_\mu k_\nu - k_\mu p_\nu)$, where k is an auxiliary vector such that $k \cdot p = 1$. We let $\rho_i = \pm 1$ according to whether ϵ_i is symmetric (h, Φ) or antisymmetric (b). All amplitudes exhibit the gauge invariance $\epsilon_i \rightarrow \epsilon_i + p_i \otimes \zeta_i + \rho_i \zeta_i \otimes p_i$, where ζ_i is the transverse (i.e. $p_i \cdot \zeta_i = 0$) infinitesimal gauge transformation parameter in momentum space (different for each particle). These gauge symmetries correspond to general covariance for gravitons, gauge invariance for antisymmetric tensors, and k -arbitrariness for dilatons. Therefore, k drops out of all amplitudes involving dilatons, and can safely be set to zero so long as one imposes the correct $\text{Tr} \epsilon = 2$ for the dilaton polarization tensor (as is obvious in light-cone gauge).

Whenever possible, we omit Lorentz indices and implicitly contract indices from left to right, for example

$$p_1 \epsilon_2 \epsilon_1 p_2 \equiv p_1^\mu \epsilon_{2\mu\nu} \epsilon_1^{\nu\rho} p_{2\rho}, \quad (\text{A.1a})$$

$$p_1 \wedge p_2 \wedge p_1 \epsilon_2 \wedge p_2 \epsilon_1 \wedge \epsilon_3 \equiv \epsilon_{\lambda\mu\nu\rho\sigma\tau} p_1^\lambda p_2^\mu p_{1\kappa} \epsilon_2^{\kappa\nu} p_{2\xi} \epsilon_1^{\xi\rho} \epsilon_3^{\sigma\tau}, \quad (\text{A.1b})$$

where we define the CP-odd antisymmetric Levi–Civita tensor ϵ such that $\epsilon_{0123} = +\sqrt{-g}$ and $\epsilon_{012345} = +\sqrt{-g}$ in four and six dimensions, respectively. Our convention for n -forms is such that $A = A_{\mu\nu\dots\rho}dX^\mu \wedge dX^\nu \wedge \dots \wedge dX^\rho$. The exterior derivative acts as $dA = \partial_\alpha A_{\mu\dots\rho}dX^\alpha \wedge dX^\mu \wedge \dots \wedge dX^\rho$.

First quantized string perturbation theory forces us to restrict to on-shell amplitudes, and we systematically impose, in the three-particle scattering case:

$$p_i \cdot p_j = p_i \epsilon_i = p_1 + p_2 + p_3 = 0. \quad (\text{A.2})$$

This drastically reduces the number of independent kinematic structures, for instance

$$p_1 \epsilon_2 p_3 = -p_1 \epsilon_2 p_1 = -p_3 \epsilon_2 p_3 = p_3 \epsilon_2 p_1 = \rho_2 p_1 \epsilon_2 p_3. \quad (\text{A.3})$$

In several instances we reduce the product of two CP-odd Levi–Civita tensors in CP-even terms using the Minkowskian identity

$$\epsilon^{\alpha_1 \alpha_2 \dots \alpha_d} \epsilon^{\beta_1 \beta_2 \dots \beta_d} = - \sum_{S_d} \tau(\sigma) \left(g^{\alpha_1 \beta_{\sigma(1)}} g^{\alpha_2 \beta_{\sigma(2)}} \dots g^{\alpha_d \beta_{\sigma(d)}} \right), \quad (\text{A.4})$$

where the sum runs over the $d!$ permutations σ of d elements with signature $\tau(\sigma) = \pm 1$. When some indices are already contracted on the left-hand side, one can significantly reduce the number of terms in the sum by using the Minkowskian¹⁰ identities:

$$\begin{aligned} \epsilon^{\alpha_1 \alpha_2 \dots \alpha_d} \epsilon^{\beta_1 \beta_2 \dots \beta_d} g_{\alpha_d \beta_d} &= -1! \sum_{S_{d-1}} \tau(\sigma) \left(g^{\alpha_1 \beta_{\sigma(1)}} g^{\alpha_2 \beta_{\sigma(2)}} \dots g^{\alpha_{d-1} \beta_{\sigma(d-1)}} \right) \\ \epsilon^{\alpha_1 \alpha_2 \dots \alpha_d} \epsilon^{\beta_1 \beta_2 \dots \beta_d} g_{\alpha_{d-1} \beta_{d-1}} g_{\alpha_d \beta_d} &= -2! \sum_{S_{d-2}} \tau(\sigma) \left(g^{\alpha_1 \beta_{\sigma(1)}} g^{\alpha_2 \beta_{\sigma(2)}} \dots g^{\alpha_{d-2} \beta_{\sigma(d-2)}} \right) \\ &\vdots \\ \epsilon^{\alpha_1 \alpha_2 \dots \alpha_d} \epsilon^{\beta_1 \beta_2 \dots \beta_d} g_{\alpha_1 \beta_1} \dots g_{\alpha_d \beta_d} &= -d! \end{aligned} \quad (\text{A.5})$$

The four-derivative low-energy effective action is obtained by finding Lorentz invariants that will induce the same interactions of the massless spectrum as those given by the string amplitude. Three-particle interactions have the simplification that no field-theory subtraction is required, and the field-theory vertex has to match the precise string amplitude. This is also the case in the four-particle interactions we are considering.

The field theory vertices are obtained by expanding the Lorentz invariant around flat backgrounds $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $B_{\mu\nu} = 0 + b_{\mu\nu}$, $\Phi = \Phi_0 + \delta\Phi$, going to momentum space variables

$$h_{\mu\nu}(x) = \int \frac{d^4 p}{(2\pi)^4} h_{\mu\nu}(p) e^{ipx} \quad (\text{A.6})$$

and imposing on-shell conditions.

To order h^2 , the Riemann tensor with covariant indices becomes

$$R_{\alpha\beta\gamma\delta} = \left(\frac{1}{2} h_{\alpha\delta, \beta\gamma} + \frac{1}{8} \left(h_{\alpha\delta, \mu} h_{\beta\gamma}{}^{\prime\mu} + \left(h^\mu{}_{\alpha, \delta} + h^\mu{}_{\delta, \alpha} - 2h_{\alpha\delta, \mu} \right) \left(h^\mu{}_{\beta, \gamma} + h^\mu{}_{\gamma, \beta} \right) \right) \right)$$

¹⁰The minus sign has to be omitted in Euclidean space.

$$-(\alpha \leftrightarrow \beta) - (\gamma \leftrightarrow \delta) + ((\alpha, \beta) \leftrightarrow (\gamma, \delta)) \quad (\text{A.7})$$

so that

$$\begin{aligned} \int d^6x \sqrt{-g} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \quad \text{on shell} \quad & \int \frac{d^6p_1 d^6p_2}{(2\pi)^6} \delta^{(6)}(p_1 + p_2) (p_1 h(p_2) p_1) (p_2 h(p_1) p_2) \\ & + \int \frac{d^6p_1 d^6p_2 d^6p_3}{(2\pi)^{12}} \delta^{(6)}(p_1 + p_2 + p_3) (p_1 h(p_2) p_3) (p_2 h(p_1) h(p_3) p_2) \end{aligned} \quad (\text{A.8})$$

The first term vanishes on shell because of momentum conservation, but becomes relevant for two-graviton–one-modulus scattering when the coefficient of R^2 is moduli-dependent as in Eq. (4.2). The second term induces a three-graviton amplitude:

$$\sum_{6 \text{ perm}} (p_1 \epsilon_2 p_3) (p_2 \epsilon_1 \epsilon_3 p_2), \quad (\text{A.9})$$

reproducing Eq. (D.8). The same kind of manipulations yield the other vertices in Eqs. (3.12) and (4.2).

Appendix B: Helicity supertraces and \mathfrak{g} -function identities

Helicity supertraces are defined as

$$B_{2n} \equiv \text{Str } \lambda^{2n}, \quad (\text{B.1})$$

where λ stands for the physical four-dimensional helicity. In models with $N = 4$ supersymmetry, B_2 vanishes (this is responsible for the vanishing of the one-loop corrections to two-derivative terms in the effective action), B_4 receives contributions from short representations only, while B_6 receives also contributions from intermediate ones. This property can be proved by computing supertraces for individual supermultiplets:

$$B_4(\text{supergravity}) = 3, \quad B_4(\text{vector}) = \frac{3}{2}, \quad (\text{B.2a})$$

$$B_4(S^j) = \frac{3}{2}(2j+1)(-1)^{2j}, \quad B_4(I^j) = 0, \quad B_4(L^j) = 0; \quad (\text{B.2b})$$

$$B_6(\text{supergravity}) = \frac{195}{4}, \quad B_6(\text{vector}) = \frac{15}{8}, \quad (\text{B.3a})$$

$$B_6(S^j) = \frac{15}{8}(2j+1)^3(-1)^{2j}, \quad B_6(I^j) = \frac{45}{4}(2j+1)(-1)^{2j+1}, \quad B_6(L^j) = 0, \quad (\text{B.3b})$$

where S^j, I^j, L^j are the short, intermediate and long representations, respectively.

In the framework of string theory, the physical four-dimensional helicity is $\lambda = \lambda_L + \lambda_R$, where $\lambda_{L,R}$ are the contributions to the helicity from the left- (right-) movers. We introduce the helicity-generating function as

$$Z(v, \bar{v}) = \text{Tr}' q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} e^{2\pi i(v\lambda_L - \bar{v}\lambda_R)}, \quad (\text{B.4})$$

where the prime over the trace excludes the zero-modes related to the space-time coordinates (consequently $Z(v, \bar{v})|_{v=\bar{v}=0} = \tau_2 Z$). At the perturbative level, helicity supertraces are obtained by taking appropriate derivatives of (B.4), using

$$\lambda_L = \frac{1}{2\pi i} \partial_v, \quad \lambda_R = -\frac{1}{2\pi i} \partial_{\bar{v}}. \quad (\text{B.5})$$

In this paper we are mostly interested in $N = 4$ type II four-dimensional models of the Z_2 -orbifold type, for which the partition function (4.3) results into a helicity-generating function¹¹

$$\begin{aligned} Z(v, \bar{v}) &= \frac{1}{|\eta|^{12}} \frac{1}{2} \sum_{a,b=0}^1 (-1)^{a+b+ab} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (v) \vartheta \begin{bmatrix} a \\ b \end{bmatrix} \frac{1}{2} \sum_{\bar{a}, \bar{b}=0}^1 (-1)^{\bar{a}+\bar{b}+\mu\bar{a}\bar{b}} \bar{\vartheta} \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} (\bar{v}) \bar{\vartheta} \begin{bmatrix} \bar{a} \\ \bar{b} \end{bmatrix} \\ &\times \frac{1}{2} \sum_{h,g=0}^1 \vartheta \begin{bmatrix} a+h \\ b+g \end{bmatrix} \vartheta \begin{bmatrix} a-h \\ b-g \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a}+h \\ \bar{b}+g \end{bmatrix} \bar{\vartheta} \begin{bmatrix} \bar{a}-h \\ \bar{b}-g \end{bmatrix} Z_{6,6} \begin{bmatrix} h \\ g \end{bmatrix} \xi(v) \bar{\xi}(\bar{v}), \end{aligned} \quad (\text{B.6})$$

where

$$\xi(v) = \prod_{n=1}^{\infty} \frac{(1-q^n)^2}{(1-q^n e^{2\pi i v})(1-q^n e^{-2\pi i v})} = \frac{\sin \pi v}{\pi} \frac{\vartheta_1'(0)}{\vartheta_1(v)} \quad (\text{B.7})$$

counts the helicity contributions of the space-time bosonic oscillators.

Owing to the (2, 2) supersymmetry of our models¹², the first non-trivial helicity supertraces can be computed by using the following formulas:

$$B_4 = \left\langle (\lambda_L + \lambda_R)^4 \right\rangle = 6 \left\langle \lambda_L^2 \lambda_R^2 \right\rangle = \frac{6}{16\pi^4} \partial_v^2 \partial_{\bar{v}}^2 Z(v, \bar{v})|_{v=\bar{v}=0}, \quad (\text{B.8a})$$

$$B_6 = \left\langle (\lambda_L + \lambda_R)^6 \right\rangle = 15 \left\langle \lambda_L^4 \lambda_R^2 + \lambda_R^4 \lambda_L^2 \right\rangle = -\frac{15}{64\pi^6} (\partial_v^4 \partial_{\bar{v}}^2 + \partial_{\bar{v}}^4 \partial_v^2) Z(v, \bar{v})|_{v=\bar{v}=0}. \quad (\text{B.8b})$$

In the rest of this appendix, we collect some of the identities involving ϑ -functions, which are useful for these computations.

Our conventions for the ϑ -functions are

$$\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (v) = \sum_{p \in \mathbb{Z}} e^{\pi i \tau (p + \frac{a}{2})^2 + 2\pi i (v + \frac{b}{2})(p + \frac{a}{2})} \quad (\text{B.9})$$

so that

$$\vartheta_1 = \vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vartheta_2 = \vartheta \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vartheta_3 = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vartheta_4 = \vartheta \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (\text{B.10})$$

¹¹We use the short-hand notation $\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (v)$ for $\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (v|\tau)$.

¹²In situations where $N = 4$ supersymmetry is realized as (4, 0) (see e.g. model $\Pi_6^{(4,0)}$ with vacuum amplitude given in (6.7)), formulas (B.8) get modified as follows:

$$\begin{aligned} B_4 &= \langle \lambda_L^4 \rangle = \frac{1}{16\pi^4} \partial_v^4 Z(v, \bar{v})|_{v=\bar{v}=0}, \\ B_6 &= \langle \lambda_L^6 \rangle + 15 \langle \lambda_L^4 \lambda_R^2 \rangle = -\frac{1}{64\pi^6} (\partial_v^6 + 15\partial_v^4 \partial_{\bar{v}}^2) Z(v, \bar{v})|_{v=\bar{v}=0}. \end{aligned}$$

We also recall that

$$\partial_v^2 \vartheta \begin{bmatrix} a \\ b \end{bmatrix} = 4\pi i \partial_\tau \vartheta \begin{bmatrix} a \\ b \end{bmatrix} \quad (\text{B.11})$$

and

$$\vartheta'_1(0) = -2\pi\eta^3 = -\pi\vartheta_2\vartheta_3\vartheta_4. \quad (\text{B.12})$$

A very useful identity is the Riemann identity:

$$\begin{aligned} & \frac{1}{2} \sum_{a,b=0}^1 (-1)^{a+b+\mu ab} \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (v) \vartheta \begin{bmatrix} a \\ b \end{bmatrix} (0) \vartheta \begin{bmatrix} a+h \\ b+g \end{bmatrix} (0) \vartheta \begin{bmatrix} a-h \\ b-g \end{bmatrix} (0) \\ &= \vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{v}{2}\right) \vartheta \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{v}{2}\right) \vartheta \begin{bmatrix} 1+h \\ 1+g \end{bmatrix} \left(\frac{v}{2}\right) \vartheta \begin{bmatrix} 1-h \\ 1-g \end{bmatrix} \left(\frac{v}{2}\right). \end{aligned} \quad (\text{B.13})$$

Taking the second derivative of Eq. (B.13) with respect to v at $v = 0$ and using (B.11) and (B.12) leads to

$$\sum_{a,b=0}^1 (-1)^{a+b+\mu ab} \vartheta^2 \begin{bmatrix} a \\ b \end{bmatrix} \vartheta \begin{bmatrix} a+h \\ b+g \end{bmatrix} \vartheta \begin{bmatrix} a-h \\ b-g \end{bmatrix} i \partial_\tau \log \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\eta} = \pi\eta^6 \vartheta \begin{bmatrix} 1+h \\ 1+g \end{bmatrix} \vartheta \begin{bmatrix} 1-h \\ 1-g \end{bmatrix}. \quad (\text{B.14})$$

Finally, we present some properties involving the bosonic helicity factor $\xi(v) \equiv \xi(-v)$ defined in (B.7):

$$\xi(0) = 1, \quad \xi'(0) = 0, \quad \xi''(0) = \frac{\pi^2}{3} (E_2 - 1), \quad E_2 = \frac{12}{\pi i} \partial_\tau \log \eta, \quad (\text{B.15})$$

as well as the following relations:

$$\partial_v^2 \left(\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (v) \xi(v) \right) \Big|_{v=0} = \left(4\pi i \partial_\tau \log \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix}}{\eta} - \frac{\pi^2}{3} \right) \vartheta \begin{bmatrix} a \\ b \end{bmatrix}, \quad (\text{B.16})$$

$$\begin{aligned} & \partial_v^4 \left(\vartheta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{v}{2}\right) \vartheta \begin{bmatrix} 1+h \\ 1+g \end{bmatrix} \left(\frac{v}{2}\right) \vartheta \begin{bmatrix} 1-h \\ 1-g \end{bmatrix} \left(\frac{v}{2}\right) \xi(v) \right) \Big|_{v=0} \\ &= 2\pi^2 \eta^6 \left(12\pi i \partial_\tau \log \frac{\vartheta \begin{bmatrix} 1-h \\ 1-g \end{bmatrix}}{\eta} - 2\pi^2 \right) \vartheta \begin{bmatrix} 1+h \\ 1+g \end{bmatrix} \vartheta \begin{bmatrix} 1-h \\ 1-g \end{bmatrix}. \end{aligned} \quad (\text{B.17})$$

We will now focus on a specific class of $(2, 2)$ models that appear in the text, which share the following property: the corresponding orbifold blocks are of the form

$$Z_{6,6} \begin{bmatrix} h \\ g \end{bmatrix} = \frac{\Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix}}{|\eta|^8} Z_{2,2} \begin{bmatrix} h \\ g \end{bmatrix}, \quad \text{for } (h, g) \neq (0, 0). \quad (\text{B.18})$$

Here $\Gamma_{4,4} \begin{bmatrix} h \\ g \end{bmatrix}$ are the ordinary Z_2 -twisted $(4, 4)$ lattice sums (see (3.2)), whereas $Z_{2,2} \begin{bmatrix} h \\ g \end{bmatrix}$ are generic blocks. For these models, the above identities (B.16) and (B.17) can be used together with the definitions (B.8) and the helicity-generating function (B.6) to obtain finally:

$$B_4 = 12|\eta|^4 \sum'_{(h,g)} Z_{2,2} \begin{bmatrix} h \\ g \end{bmatrix} \quad (\text{B.19})$$

and

$$B_6 = 30|\eta|^4 \sum'_{(h,g)} \left(1 + \frac{1}{2} \operatorname{Re} H \begin{bmatrix} h \\ g \end{bmatrix} \right) Z_{2,2} \begin{bmatrix} h \\ g \end{bmatrix}, \quad (\text{B.20})$$

where the functions $H \begin{bmatrix} h \\ g \end{bmatrix}$ are given by

$$H \begin{bmatrix} h \\ g \end{bmatrix} = \frac{12}{\pi i} \partial_\tau \log \frac{\vartheta \begin{bmatrix} 1-h \\ 1-g \end{bmatrix}}{\eta} = \begin{cases} \vartheta_3^4 + \vartheta_4^4, & (h, g) = (0, 1) \\ -\vartheta_2^4 - \vartheta_3^4, & (h, g) = (1, 0) \\ \vartheta_2^4 - \vartheta_4^4, & (h, g) = (1, 1). \end{cases} \quad (\text{B.21})$$

Notice also the property

$$\sum'_{(h,g)} H \begin{bmatrix} h \\ g \end{bmatrix} = 0. \quad (\text{B.22})$$

Appendix C: $\Gamma_{2,2}$ lattice sums and fundamental-domain integrals

In this appendix we give our notation and conventions for the usual (2, 2) and shifted (2, 2) lattice sums used in the text. We also give the explicit results for the relevant fundamental-domain integrals of these lattice sums.

The (2, 2) lattice sum is given by

$$\begin{aligned} \Gamma_{2,2}(T, U) &= \sum_{\{p_L, p_R\} \in \Gamma_{2,2}} q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} \\ &= \frac{T_2}{\tau_2} \sum_{A \in GL(2, Z)} \exp \left(-2\pi i T \det A - \frac{\pi T_2}{\tau_2 U_2} \left| \begin{pmatrix} 1 & U \\ & 1 \end{pmatrix} A \begin{pmatrix} \tau \\ 1 \end{pmatrix} \right|^2 \right), \end{aligned} \quad (\text{C.1})$$

where

$$p_L^2 = \frac{|Um_1 - m_2 + Tn^1 + TUn^2|^2}{2T_2U_2}, \quad p_L^2 - p_R^2 = 2\vec{m}\vec{n} \quad (\text{C.2})$$

($\vec{m}\vec{n}$ stands for $m_I n^I$). In terms of the background fields G_{IJ} and B_{IJ} , the left and right momenta can be written as

$$p_L^I = \frac{1}{\sqrt{2}}(N + n)^I, \quad p_R^I = \frac{1}{\sqrt{2}}(N - n)^I, \quad (\text{C.3})$$

where

$$N^I = G^{IJ} (m_J - B_{JK} n^K), \quad (\text{C.4})$$

so that

$$p_L^2 = p_L^I G_{IJ} p_L^J, \quad p_R^2 = p_R^I G_{IJ} p_R^J. \quad (\text{C.5})$$

The matrix identities (D.32a), (D.32b) follow, after some algebra, using the parametrization of G_{IJ} and B_{IJ} in terms of the moduli T and U ¹³:

$$G = \frac{T_2}{U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}, \quad B = T_1 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (\text{C.6})$$

¹³When $T_1 = U_1 = 0$, the usual parametrization is $T_2 = R_1 R_2$, $U_2 = R_2/R_1$, where R_i are the radii of compactification.

The relation (D.32c) follows from the definition of the lattice sum and the identity,

$$\frac{\partial p_L^2}{\partial E_{IJ}} = \frac{\partial p_R^2}{\partial E_{IJ}} = \frac{1}{2}(N+n)^I(N-n)^J = p_L^I p_R^J, \quad E \equiv G+B, \quad (\text{C.7})$$

which may be derived from (C.5). Finally, the relevant fundamental-domain integral is [49]

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} (\Gamma_{2,2}(T,U) - 1) = -\log \left(T_2 |\eta(T)|^4 U_2 |\eta(U)|^4 \right) - \log \frac{8\pi e^{1-\gamma}}{3\sqrt{3}}. \quad (\text{C.8})$$

The subtraction of the massless-states contribution in this integral is necessary for regularizing the logarithmic divergence, and results in a non-harmonic dependence on T, U .

The Z_2 -shifted $(2, 2)$ lattice sums are

$$\Gamma_{2,2}^w(T,U) \begin{bmatrix} h \\ g \end{bmatrix} = \sum_{\{p_L, p_R\} \in \Gamma_{2,2} + w\frac{h}{2}} e^{-\pi i g \ell \cdot w} q^{\frac{1}{2} p_L^2} \bar{q}^{\frac{1}{2} p_R^2}, \quad (\text{C.9})$$

where the shifts h and projections g take the values 0 or 1. Here, w denotes the shift vector with components (a_1, a_2, b^1, b^2) , and $\ell \equiv (m_1, m_2, n^1, n^2)$. We have also introduced the inner product¹⁴

$$\ell \cdot w = \vec{m}\vec{b} + \vec{a}\vec{n}, \quad w^2 = 2\vec{a}\vec{b}, \quad (\text{C.10})$$

so that a_I generates a winding shift in the I direction, whereas b^I shifts the I th momentum. The vector ℓ is associated with the $\Gamma_{2,2}$ lattice and therefore the vector associated with the shifted lattice will be

$$p \equiv \ell + w\frac{h}{2}. \quad (\text{C.11})$$

With these conventions, left and right momenta read:

$$p_L^2 = \frac{\left| U \left(m_1 + a_1 \frac{h}{2} \right) - \left(m_2 + a_2 \frac{h}{2} \right) + T \left(n^1 + b^1 \frac{h}{2} \right) + TU \left(n^2 + b^2 \frac{h}{2} \right) \right|^2}{2T_2 U_2}, \quad (\text{C.12a})$$

$$p_L^2 - p_R^2 = 2 \left(m_I + a_I \frac{h}{2} \right) \left(n^I + b^I \frac{h}{2} \right). \quad (\text{C.12b})$$

It is easy to check the periodicity properties (h, g integers)

$$Z_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} = Z_{2,2}^w \begin{bmatrix} h+2 \\ g \end{bmatrix} = Z_{2,2}^w \begin{bmatrix} h \\ g+2 \end{bmatrix} = Z_{2,2}^w \begin{bmatrix} -h \\ -g \end{bmatrix} \quad (\text{C.13})$$

as well as the modular transformations that expression

$$Z_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} = \frac{\Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}}{|\eta|^4} \quad (\text{C.14})$$

¹⁴For $w_1 = (\vec{a}_1, \vec{b}_1)$ and $w_2 = (\vec{a}_2, \vec{b}_2)$, the inner product is defined as $w_1 \cdot w_2 = \vec{a}_1 \vec{b}_2 + \vec{a}_2 \vec{b}_1$.

obeys:

$$\tau \rightarrow \tau + 1 : Z_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \rightarrow e^{\pi i \frac{w^2}{2} \frac{h^2}{2}} Z_{2,2}^w \begin{bmatrix} h \\ h+g \end{bmatrix} \quad (\text{C.15a})$$

$$\tau \rightarrow -\frac{1}{\tau} : Z_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} \rightarrow e^{-\pi i \frac{w^2}{2} hg} Z_{2,2}^w \begin{bmatrix} g \\ -h \end{bmatrix}. \quad (\text{C.15b})$$

The relevant parameter for these transformations is

$$\lambda \equiv \frac{w^2}{2} = \vec{a}\vec{b}. \quad (\text{C.16})$$

From expressions (C.9) we learn that the integers a_I and b^I are defined modulo 2, in the sense that adding 2 to any one of them amounts at most to a change of sign in $Z_{2,2}^w \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Such a modification is necessarily compensated by an appropriate one in the rest of the partition function in order to ensure modular invariance; we are thus left with the same model. On the other hand, adding 2 to a_I or b^I translates into adding a multiple of 2 to λ . Therefore, although λ can be any integer, only $\lambda = 0$ and $\lambda = 1$ correspond to truly different situations.

We now would like to discuss the issue of target-space duality in these models, where the Z_2 orbifold acts as a translation in one complex plane. The moduli dependence of the two-torus shifted sectors (see Eq. (C.9)) reduces in general the duality group to some subgroup¹⁵ of $SL(2, Z)_T \times SL(2, Z)_U \times Z_2^{T \leftrightarrow U}$. Transformations that do not belong to this subgroup map a model w to some other model w' leaving invariant, however, $\lambda = \frac{w^2}{2} = \frac{w'^2}{2}$. This means in particular that for a given model, decompactification limits that are related by transformations that do not belong to the actual duality group are no longer equivalent.

To be more specific, by using expression (C.9), we can determine the transformation properties of $\Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix}$ under the full group $SL(2, Z)_T \times SL(2, Z)_U \times Z_2^{T \leftrightarrow U}$:

$$SL(2, Z)_T : \begin{pmatrix} a_1 \\ a_2 \\ b^1 \\ b^2 \end{pmatrix} \rightarrow \begin{pmatrix} d & 0 & 0 & b \\ 0 & d & -b & 0 \\ 0 & -c & a & 0 \\ c & 0 & 0 & a \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b^1 \\ b^2 \end{pmatrix}, \quad ad - bc = 1, \quad (\text{C.17a})$$

$$SL(2, Z)_U : \begin{pmatrix} a_1 \\ a_2 \\ b^1 \\ b^2 \end{pmatrix} \rightarrow \begin{pmatrix} a' & -c' & 0 & 0 \\ -b' & d' & 0 & 0 \\ 0 & 0 & d' & b' \\ 0 & 0 & c' & a' \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b^1 \\ b^2 \end{pmatrix}, \quad a'd' - b'c' = 1 \quad (\text{C.17b})$$

and

$$Z_2^{T \leftrightarrow U} : \begin{pmatrix} a_1 \\ a_2 \\ b^1 \\ b^2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ b^1 \\ b^2 \end{pmatrix}. \quad (\text{C.17c})$$

¹⁵The subgroups of $SL(2, Z)$ that will actually appear in the sequel are $\Gamma^\pm(2)$ and $\Gamma(2)$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents an element of the modular group, $\Gamma^+(2)$ is defined by a, d odd and b even, while for $\Gamma^-(2)$ we have a, d odd and c even. Their intersection is $\Gamma(2)$.

Thus, we can determine the duality group for a given model by demanding that the components of the vectors \vec{a} and \vec{b} remain invariant modulo 2. For example, in the $\lambda = 0$ situation defined by $\vec{a} = (0, 0)$ and $\vec{b} = (1, 0)$, the target-space duality group turns out to be $\Gamma^+(2)_T \times \Gamma^-(2)_U$, whereas for the case with $\lambda = 1$ and $\vec{a} = (1, 0)$, $\vec{b} = (1, 0)$, we find $\Gamma(2)_T \times \Gamma(2)_U \times Z_2^{T \leftrightarrow U}$.

At this point, we would like to mention a remarkable identity, which plays a role in the computation of fundamental-domain integrals, as well as in the identification of several type II constructions (see Subsection 6.2). Starting from the definition (C.9) of shifted lattice sums, one checks easily that

$$\frac{1}{2} \sum_{h', g'=0}^1 (-1)^{hg'+gh'} \Gamma_{2,2}^w \begin{bmatrix} h' \\ g' \end{bmatrix} (T', U') = \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} (T, U) \quad (\text{C.18})$$

for any shift vector such that $w^2 = 0$. The precise relation between (T', U') and (T, U) depends on the specific shift vector w , and is presented in the Table C.1 for all distinct $\lambda = 0$ situations.

case	\vec{a}	\vec{b}	T'	U'	i	j
I	(0, 0)	(1, 0)	$-\frac{2}{T}$	$-\frac{1}{2U}$	4	2
II	(0, 0)	(0, 1)	$-\frac{2}{T}$	$-\frac{2}{U}$	4	4
III	(0, 0)	(1, 1)	$-\frac{2}{T}$	$\frac{1+U}{1-U}$	4	3
IV	(1, 0)	(0, 0)	$-\frac{1}{2T}$	$-\frac{2}{U}$	2	4
V	(0, 1)	(0, 0)	$-\frac{1}{2T}$	$-\frac{1}{U}$	2	2
VI	(1, 1)	(0, 0)	$-\frac{1}{2T}$	$\frac{1+U}{1-U}$	2	3
VII	(1, 0)	(0, 1)	$\frac{1+T}{1-T}$	$-\frac{2}{U}$	3	4
VIII	(0, 1)	(1, 0)	$\frac{1+T}{1-T}$	$-\frac{1}{2U}$	3	2
IX	(1, -1)	(1, 1)	$\frac{1+T}{1-T}$	$\frac{1+U}{1-U}$	3	3

Table C.1: The nine physically distinct models with $\lambda = 0$.

After Poisson resummation in m_1, m_2 , the shifted lattice sum (C.9) takes the alternative form

$$\Gamma_{2,2}^w(T, U) \begin{bmatrix} h \\ g \end{bmatrix} = \frac{T_2}{\tau_2} \sum_A \exp -\pi i \left(\frac{w^2}{4} hg - \vec{a} (g\vec{n} - h\vec{m}) \right) \exp \left(-2\pi i T \det A - \frac{\pi T_2}{\tau_2 U_2} \left| \begin{pmatrix} 1 & U \\ & 1 \end{pmatrix} A \begin{pmatrix} \tau \\ 1 \end{pmatrix} \right|^2 \right), \quad (\text{C.19})$$

where the summation is performed over the set of matrices of the form

$$A = \begin{pmatrix} n^1 + b^1 \frac{h}{2} & m_1 + b^1 \frac{g}{2} \\ n^2 + b^2 \frac{h}{2} & m_2 + b^2 \frac{g}{2} \end{pmatrix}. \quad (\text{C.20})$$

Modular-invariant combinations of blocks $\Gamma_{2,2}^w(T, U) \begin{bmatrix} h \\ g \end{bmatrix}$ can be integrated over the fundamental domain by decomposing the set of matrices A with respect to orbits of the modular group. In this paper, we are mainly interested in the case $\lambda = 0$ ¹⁶, for which the relevant

¹⁶Heterotic constructions with $\lambda = 1$ can be found in [23].

integrals can be obtained from (C.8) by using (C.18) together with

$$\vartheta_2(\tau) = 2 \frac{\eta^2(2\tau)}{\eta(\tau)}, \quad \vartheta_4(\tau) = \frac{\eta^2\left(\frac{\tau}{2}\right)}{\eta(\tau)}, \quad \vartheta_3(\tau) = \frac{2e^{-\frac{i\pi}{3}} \eta^2\left(\frac{1+\tau}{1-\tau}\right)}{1-\tau} \frac{1}{\eta(\tau)}. \quad (\text{C.21})$$

As a result,

$$\int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \left(\sum'_{(h,g)} \Gamma_{2,2}^w \begin{bmatrix} h \\ g \end{bmatrix} (T, U) - 1 \right) = -\log \left(T_2 |\vartheta_i(T)|^4 U_2 |\vartheta_j(U)|^4 \right) - \log \frac{\pi e^{1-\gamma}}{6\sqrt{3}}, \quad (\text{C.22})$$

where the relation between the shift vector $w = (\vec{a}, \vec{b})$ and the pairs (i, j) is taken from Table C.1.

In the construction of reduced-rank models of Section 6, we introduce shifted $(2, 2)$ lattices where the free action is of the type $Z_2 \times Z_2$. Each of the Z_2 's acts according to the above analysis on a given set of momenta and windings. Consistency of the $Z_2 \times Z_2$ action demands that the intersection of these two sets be empty. In other words, the corresponding shift vectors w_1 and w_2 must satisfy $w_1 \cdot w_2 = 0$. Notice that the union of these sets corresponds to the action of the diagonal Z_2 . The lattice sum will be denoted $\Gamma_{2,2}^{w_1, w_2} \begin{bmatrix} h_1, h_2 \\ g_1, g_2 \end{bmatrix}$ and we have in particular

$$\Gamma_{2,2}^{w_1, w_2} \begin{bmatrix} h, 0 \\ g, 0 \end{bmatrix} = \Gamma_{2,2}^{w_1} \begin{bmatrix} h \\ g \end{bmatrix}, \quad \Gamma_{2,2}^{w_1, w_2} \begin{bmatrix} 0, h \\ 0, g \end{bmatrix} = \Gamma_{2,2}^{w_2} \begin{bmatrix} h \\ g \end{bmatrix}, \quad \Gamma_{2,2}^{w_1, w_2} \begin{bmatrix} h, h \\ g, g \end{bmatrix} = \Gamma_{2,2}^{w_{12}} \begin{bmatrix} h \\ g \end{bmatrix}, \quad (\text{C.23})$$

where $w_{12} \equiv w_1 + w_2$ reflects the action of the diagonal Z_2 .

As an example, consider the situation where $w_1 = (0, 0, 1, 0)$, $w_2 = (0, 0, 0, 1)$ and therefore $w_{12} = (0, 0, 1, 1)$. In that case, the first (resp. second) Z_2 shifts the momenta of the first (resp. second) plane (insertion of $(-1)^{m_1}$ (resp. $(-1)^{m_2}$)), while the diagonal Z_2 amounts to inserting $(-1)^{m_1+m_2}$. The lattice sum now reads:

$$\Gamma_{2,2} \begin{bmatrix} h_1, h_2 \\ g_1, g_2 \end{bmatrix} = \sum_{\vec{m}, \vec{n} \in Z} (-1)^{m_1 g_1 + m_2 g_2} \exp \left(2\pi i \bar{\tau} \left(m_1 \left(n^1 + \frac{h_1}{2} \right) + m_2 \left(n^2 + \frac{h_2}{2} \right) \right) - \frac{\pi \tau_2}{T_2 U_2} \left| T \left(n^1 + \frac{h_1}{2} \right) + T U \left(n^2 + \frac{h_2}{2} \right) + U m_1 - m_2 \right|^2 \right), \quad (\text{C.24})$$

from which Eqs. (C.23) are immediatly checked.

In the framework of Subsection 6.4, the requirement of modular invariance implies that $w_1^2 = w_2^2 = w_{12}^2 = 0$. This reduces the number of distinct possibilities to the six listed in Table C.2. The first of these corresponds to the example whose lattice sum is given in Eq. (C.24).

case	w_1	w_2
(i)	$(0, 0, 1, 0)$	$(0, 0, 0, 1)$
(ii)	$(1, 0, 0, 0)$	$(0, 1, 0, 0)$
(iii)	$(1, 0, 0, 1)$	$(0, -1, 1, 0)$
(iv)	$(1, 0, 0, 0)$	$(0, 0, 0, 1)$
(v)	$(0, 0, 1, 0)$	$(0, 1, 0, 0)$
(vi)	$(0, 0, 1, 1)$	$(1, -1, 0, 0)$

Table C.2: The six physically distinct models with $w_i \cdot w_j = 0 \forall i, j = 1, 2$.

Appendix D: Details of string amplitude calculations

In this section we compute in great detail the stringy scattering amplitude (3.5) of three gravitons (or two-forms) for type II superstring on $K3$, and subsequently the scattering amplitude (4.4) of two gravitons (or two-forms or dilatons) with moduli of T^2 for type II on $K3 \times T^2$.

D.1 String amplitude toolbox

For these computations we use the following contraction formulae:

$$\langle X^\mu(\bar{z}, z) X^\nu(0) \rangle = g^{\mu\nu} \Delta(\bar{z}, z) \equiv -g^{\mu\nu} \log \left(e^{-2\pi \frac{z\bar{z}}{\tau_2}} \left| \frac{\vartheta_1(z)}{\vartheta_1(0)} \right|^2 \right) \quad (\text{D.1a})$$

$$\langle \bar{\partial} X^\mu(\bar{z}, z) \partial X^\nu(0) \rangle = -g^{\mu\nu} \bar{\partial} \partial \Delta(\bar{z}, z) = -\frac{\pi}{\tau_2} g^{\mu\nu} \quad (\text{D.1b})$$

$$\begin{aligned} \langle \partial_1 X^\mu(\bar{z}_1, z_1) p_2 \cdot X(\bar{z}_2, z_2) \rangle \langle \partial_2 X^\lambda(\bar{z}_2, z_2) p_1 \cdot X(\bar{z}_1, z_1) \rangle &= \\ = -p_2^\mu p_1^\lambda \langle \partial_1 X(\bar{z}_1, z_1) X(\bar{z}_2, z_2) \rangle^2 & \quad (\text{D.1c}) \end{aligned}$$

$$\langle \bar{\partial} X^I(\bar{z}, z) \partial X^J(0) \rangle = p_{\text{R}}^I p_{\text{L}}^J + G^{IJ} \frac{\pi}{\tau_2} - G^{IJ} \partial \bar{\partial} \Delta(\bar{z}, z) \equiv p_{\text{R}}^I p_{\text{L}}^J \quad (\text{D.1d})$$

$$\langle \psi(z_1)^\mu \psi(z_2)^\nu \rangle \begin{bmatrix} a \\ b \end{bmatrix} = g^{\mu\nu} \langle \psi(z_1) \psi(z_2) \rangle \begin{bmatrix} a \\ b \end{bmatrix} = g^{\mu\nu} \frac{\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (z_{12}) \vartheta_1'(0)}{\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (0) \vartheta_1(z_{12})} \quad (\text{D.1e})$$

$$\langle p_1 \cdot \psi(z_1) \psi^\lambda(z_2) \rangle \langle \psi^\mu(z_1) p_2 \cdot \psi(z_2) \rangle = p_1^\lambda p_2^\mu \langle \psi(z_1) \psi(z_2) \rangle^2, \quad (\text{D.1f})$$

where the Greek space-time indices run from 0 to 5 (resp. 3) in the six- (resp. four-) dimensional case, and the indices I, J run on the two compactified directions of T^2 . As in Appendix C, $p_{\text{L,R}}$ denote the left- and right-moving momenta of T^2 .

A few remarks about these equations are in order. Equation (D.1a) gives the propagator of a non-compact boson on the space of non-zero-modes. Equation (D.1b) omits a delta function singularity, which has to be subtracted for tree-level factorization. The first term in (D.1d) is the contribution of the winding zero-modes of a compact boson written in Hamiltonian representation, to be added to the non-zero-mode contribution (D.1b). Equation (D.1e) holds only for even spin structures where the world-sheet fermions do not have any zero-modes. In the odd spin structure, there is one zero-mode for each space-time or T^2 fermion, a total of six in both the six- and four-dimensional cases. These zero-modes have to be saturated in order to give a non-vanishing result, and we normalize them as

$$\langle \psi^\mu \psi^\nu \psi^\kappa \psi^\lambda \psi^\rho \psi^\sigma \rangle = \epsilon^{\mu\nu\kappa\lambda\rho\sigma} \quad \text{in six dimensions,} \quad (\text{D.2a})$$

$$\langle \psi^\mu \psi^\nu \psi^\kappa \psi^\lambda \psi^I \psi^J \rangle = \epsilon^{\mu\nu\kappa\lambda} \epsilon^{IJ} \quad \text{in four dimensions,} \quad (\text{D.2b})$$

where $\epsilon^{12} = -\epsilon^{21} = 1/\sqrt{G}$. Saturation of the zero-modes at the same time induces the replacement

$$\vartheta^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \left(\vartheta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} (0) \right)^2 = 4\pi^2 \eta^6 \quad (\text{D.3})$$

in the partition function.

We also need the integrated propagators on the torus:

$$\int \frac{d^2 z}{\tau_2} (\partial \Delta(\bar{z}, z))^2 = -4\pi i \partial_\tau \log \left(\eta(\tau) \tau_2^{1/2} \right), \quad (\text{D.4a})$$

$$\int \frac{d^2 z}{\tau_2} \langle \psi(z) \psi(0) \rangle^2 \begin{bmatrix} a \\ b \end{bmatrix} = 4\pi i \partial_\tau \log \left(\vartheta \begin{bmatrix} a \\ b \end{bmatrix} (\tau) \tau_2^{1/2} \right), \quad (\text{D.4b})$$

where (a, b) is an even spin structure. We normalize the measure of integration on vertex positions as $\int \frac{d^2 z}{\tau_2} = 1$. Expressions analogous to (D.1f,c) and (D.4b) for the left side follow by complex conjugation. Useful Riemann identities for the summing of spin structures are assembled in Appendix B.

D.2 Three-graviton scattering in six dimensions

Here we wish to evaluate the amplitude (3.5) and derive the corresponding four-derivative terms in the effective action. We need to distinguish according to the spin structures on both sides.

A. *CP-even* $\bar{e}-e$. In this sector we need to compute the correlation function

$$\begin{aligned} A^{\bar{e}-e} = & \left\langle \left(\bar{\partial}_1 X^\mu(\bar{z}_1, z_1) + ip_1 \cdot \bar{\psi}(\bar{z}_1) \bar{\psi}^\mu(\bar{z}_1) \right) \left(\partial_1 X^\nu(\bar{z}_1, z_1) + ip_1 \cdot \psi(z_1) \psi^\nu(z_1) \right) e^{ip_1 \cdot X(\bar{z}_1, z_1)} \right. \\ & \times \left(\bar{\partial}_2 X^\kappa(\bar{z}_2, z_2) + ip_2 \cdot \bar{\psi}(\bar{z}_2) \bar{\psi}^\kappa(\bar{z}_2) \right) \left(\partial_2 X^\lambda(\bar{z}_2, z_2) + ip_2 \cdot \psi(z_2) \psi^\lambda(z_2) \right) e^{ip_2 \cdot X(\bar{z}_2, z_2)} \\ & \left. \times \left(\bar{\partial}_3 X^\rho(\bar{z}_3, z_3) + ip_3 \cdot \bar{\psi}(\bar{z}_3) \bar{\psi}^\rho(\bar{z}_3) \right) \left(\partial_3 X^\sigma(\bar{z}_3, z_3) + ip_3 \cdot \psi(z_3) \psi^\sigma(z_3) \right) e^{ip_3 \cdot X(\bar{z}_3, z_3)} \right\rangle. \end{aligned} \quad (\text{D.5})$$

The Riemann identity (B.13) shows that at least two pairs of fermions must be contracted together on both sides, since contributions with less fermionic contractions vanish after a sum on even spin structures. Each fermion pair comes with one power of momentum; we therefore need precisely two such contractions¹⁷. The two pairs of fermions have to be chosen in two different vertices on both sides, since the polarizations are traceless:

$$\begin{aligned} A_{\text{four deriv}}^{\bar{e}-e} = & (i)^4 \left\langle \bar{\partial}_1 X^\mu \partial_3 X^\sigma \right\rangle \left\langle p_2 \cdot \bar{\psi}(\bar{z}_2) \bar{\psi}^\rho(\bar{z}_3) \right\rangle \left\langle p_3 \cdot \bar{\psi}(\bar{z}_3) \bar{\psi}^\kappa(\bar{z}_2) \right\rangle \\ & \times \left\langle p_1 \cdot \psi(z_1) \psi^\lambda(z_2) \right\rangle \left\langle p_2 \cdot \psi(z_2) \bar{\psi}^\nu(z_1) \right\rangle + \text{perm.} \end{aligned} \quad (\text{D.6})$$

¹⁷It is known that those singularities arising when two vertices come together can yield poles $O(1/(p_i p_j))$ that can cancel against six-derivative terms to yield $O(p^4)$ terms [50]. We evaluated these contributions and found a precise cancellation of the corresponding terms, in agreement with the expectation that these terms reproduce field-theory subtractions that are absent in the case at hand.

Making use of Eq. (D.1f) and of the Riemann identities in Appendix B, it can be shown that the integrand, after summation over spin structures, no longer depends on the position of the vertices, so that we can apply Eq. (D.4) to obtain:

$$\mathcal{I}^{\bar{e}-e} = \mathcal{T}^{\bar{e}-e} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{\tau_2^3}{\pi^3} Z_{\text{II}} \frac{\pi}{\tau_2} (4\pi i) \partial_\tau \log \left(\frac{\vartheta \left[\begin{smallmatrix} a \\ b \end{smallmatrix} \right]}{\eta} \right) (-4\pi i) \partial_{\bar{\tau}} \log \left(\frac{\bar{\vartheta} \left[\begin{smallmatrix} \bar{a} \\ \bar{b} \end{smallmatrix} \right]}{\bar{\eta}} \right). \quad (\text{D.7})$$

Here Z_{II} stands for the unintegrated partition function in Eq. (3.1). We also defined the kinematic structure

$$\mathcal{T}^{\bar{e}-e} = (p_1 \epsilon_2 p_3)(p_2 \epsilon_1 \epsilon_3 p_2) + 5 \text{ perm.} \quad (\text{D.8})$$

It can easily be shown that

$$\mathcal{T}^{\bar{e}-e} = \rho_1 \rho_2 \rho_3 \mathcal{T}^{\bar{e}-e}, \quad (\text{D.9})$$

so that the amplitude is non-vanishing only for three gravitons or for two antisymmetric tensors and one graviton. Identity (B.14) shows that the untwisted sector $(h, g) = (0, 0)$ does not contribute, and we use the explicit expression (3.2) for the twisted $\Gamma_{4,4} \left[\begin{smallmatrix} h \\ g \end{smallmatrix} \right]$ to obtain:

$$\mathcal{I}_{\text{reg}}^{\bar{e}-e} = \mathcal{T}^{\bar{e}-e} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{\tau_2^2}{\pi^2} \frac{1}{8} \frac{1}{\tau_2^2} \times 4\pi^2 \times 4\pi^2 \times 16 \times 3 = 32\pi^3 \mathcal{T}^{\bar{e}-e}, \quad (\text{D.10})$$

where we also used the standard modular-invariant integral $\int_{\mathcal{F}} d^2\tau / \tau_2^2 = \pi/3$.

Comparing Eq. (D.8) with Eq. (3.12) we find that the three-graviton- and one-graviton-two-two-forms in $\bar{e}-e$ spin structure can be described by the following vertex in the effective action:

$$\mathcal{I}_{\text{eff}}^{\bar{e}-e} = 32\pi^3 \int d^6x \sqrt{-g} \left(R^2 + \frac{1}{6} \nabla H \nabla H \right). \quad (\text{D.11})$$

B. CP-even $\bar{o}-o$. In this sector the correlation function we have to compute is modified to

$$\begin{aligned} A^{\bar{o}-o} = & \left\langle \left(\bar{\partial}_1 X^\mu(\bar{z}_1, z_1) + ip_1 \cdot \bar{\psi}(\bar{z}_1) \bar{\psi}^\mu(\bar{z}_1) \right) \left(\partial_1 X^\nu(\bar{z}_1, z_1) + ip_1 \cdot \psi(z_1) \psi^\nu(z_1) \right) e^{ip_1 \cdot X(\bar{z}_1, z_1)} \right. \\ & \times \left(\bar{\partial}_2 X^\kappa(\bar{z}_2, z_2) + ip_2 \cdot \bar{\psi}(\bar{z}_2) \bar{\psi}^\kappa(\bar{z}_2) \right) \left(\partial_2 X^\lambda(\bar{z}_2, z_2) + ip_2 \cdot \psi(z_2) \psi^\lambda(z_2) \right) e^{ip_2 \cdot X(\bar{z}_2, z_2)} \\ & \left. \times \bar{\psi}^\rho(\bar{z}_3) \psi^\sigma(z_3) e^{ip_3 \cdot X(\bar{z}_3, z_3)} \partial X^\alpha(0) \psi_\alpha(0) \bar{\partial} X^\beta(0) \bar{\psi}_\beta(0) \right\rangle, \end{aligned} \quad (\text{D.12})$$

where we have used for the third vertex operator the -1 -picture on both the left and the right sides, and inserted the left- and right-moving supercurrents. All fermions have to be contracted in order to saturate zero-modes, and the remaining $\bar{\partial} X^\alpha \partial X^\beta$ from the two supercurrents does not yield any singular contribution because of the antisymmetry of Levi-Civita tensors.

We are therefore left with the following term

$$A_{\text{four deriv}}^{\bar{o}-o} = (i)^4 p_{1\alpha_1} p_{2\alpha_2} \epsilon^{\alpha_1 \mu \alpha_2 \kappa \rho \alpha} p_{1\beta_1} p_{2\beta_2} \epsilon^{\beta_1 \nu \beta_2 \lambda \sigma \beta} \frac{\pi}{\tau_2} g_{\alpha\beta} \quad (\text{D.13})$$

obtained by using Eq. (D.2a). Then using Eq. (D.3) on both the left and the right sides, the integrated three-point amplitude becomes, after some algebra:

$$\mathcal{I}^{\bar{o}-o} = \mathcal{T}^{\bar{o}-o} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{\tau_2^3}{\pi^3} \frac{1}{8} \frac{1}{\tau_2^2} \times (-4\pi^2) \times (-1)^\mu 4\pi^2 \times \frac{\pi}{\tau_2} \times 16 \times 3 = -\varepsilon 32\pi^3 \mathcal{T}^{\bar{o}-o}, \quad (\text{D.14})$$

where we have defined the tensor structure

$$\mathcal{T}^{\bar{o}-o} = \epsilon_{1\mu\nu}\epsilon_{2\kappa\lambda}\epsilon_{3\rho\sigma}p_{1\alpha_1}p_{2\alpha_2}\epsilon^{\alpha_1\alpha_2\mu\kappa\rho\alpha}p_{1\beta_1}p_{2\beta_2}\epsilon^{\beta_1\beta_2\nu\lambda\sigma\beta}g_{\alpha\beta}. \quad (\text{D.15})$$

Expanding the product of the two CP-odd Levi–Civita tensors in terms of the metric in Eq. (A.1) and comparing Eqs. (D.8) and (D.15), we can show that, without any assumption on the symmetry properties of the polarization tensors,

$$\mathcal{T}^{\bar{o}-o} = -(p_1\epsilon_2p_3)(p_2\epsilon_1\epsilon_3p_2) + 5 \text{ perm.} \quad (\text{D.16})$$

Therefore, the $\bar{o}-o$ spin structure yields exactly the same interactions as the $\bar{e}-e$ one, but with a sign depending on whether we are in type IIA or IIB.

Hence, we record for the corresponding term in the effective action

$$\mathcal{I}_{\text{eff}}^{\bar{o}-o} = 32\pi^3\epsilon \int d^6x\sqrt{-g} \left(R^2 + \frac{1}{6}\nabla H\nabla H \right). \quad (\text{D.17})$$

C. CP-odd. We first work out the result in the sector $\bar{e}-o$, in which we need to compute the correlator

$$\begin{aligned} A^{\bar{e}-o} = & \left\langle \left(\bar{\partial}_1 X^\mu(\bar{z}_1, z_1) + ip_1 \cdot \bar{\psi}(\bar{z}_1)\bar{\psi}^\mu(\bar{z}_1) \right) \left(\partial_1 X^\nu(\bar{z}_1, z_1) + ip_1 \cdot \psi(z_1)\psi^\nu(z_1) \right) e^{ip_1 \cdot X(\bar{z}_1, z_1)} \right. \\ & \times \left(\bar{\partial}_2 X^\kappa(\bar{z}_2, z_2) + ip_2 \cdot \bar{\psi}(\bar{z}_2)\bar{\psi}^\kappa(\bar{z}_2) \right) \left(\partial_2 X^\lambda(\bar{z}_2, z_2) + ip_2 \cdot \psi(z_2)\psi^\lambda(z_2) \right) e^{ip_2 \cdot X(\bar{z}_2, z_2)} \\ & \left. \times \left(\bar{\partial}_3 X^\rho(\bar{z}_3, z_3) + ip_3 \cdot \bar{\psi}(\bar{z}_3)\bar{\psi}^\rho(\bar{z}_3) \right) \psi^\sigma(z_3) e^{ip_3 \cdot X(\bar{z}_3, z_3)} \partial X^\gamma(0)\psi_\gamma(0) \right\rangle, \end{aligned} \quad (\text{D.18})$$

where we have used the -1 -picture on the right for the third vertex operator and inserted the right-moving supercurrent (3.9). Again, no contact terms are involved and the relevant four-derivative term is

$$\begin{aligned} A_{\text{four deriv}}^{\bar{e}-o} &= (i)^4 \left(\left\langle p_1 \cdot \bar{\psi}(\bar{z}_1)\bar{\psi}^\kappa(\bar{z}_2) \right\rangle \left\langle \bar{\psi}^\mu(\bar{z}_1)p_2 \cdot \bar{\psi}(\bar{z}_2) \right\rangle \left\langle \bar{\partial} X^\rho \partial X^\gamma \right\rangle \right. \\ &+ \left\langle p_1 \cdot \bar{\psi}(\bar{z}_1)\bar{\psi}^\rho(\bar{z}_3) \right\rangle \left\langle \bar{\psi}^\mu(\bar{z}_1)p_3 \cdot \bar{\psi}(\bar{z}_3) \right\rangle \left\langle \bar{\partial} X^\kappa \partial X^\gamma \right\rangle \\ &+ \left. \left\langle p_2 \cdot \bar{\psi}(\bar{z}_2)\bar{\psi}^\rho(\bar{z}_3) \right\rangle \left\langle \bar{\psi}^\kappa(\bar{z}_2)p_3 \cdot \bar{\psi}(\bar{z}_3) \right\rangle \left\langle \bar{\partial} X^\mu \partial X^\gamma \right\rangle \right) p_{1\alpha}p_{1\beta}\epsilon^{\alpha\nu\beta\lambda\sigma\delta}g_{\gamma\delta} \\ &= \frac{\pi}{\tau_2} \left(\epsilon^{\alpha\beta\nu\lambda\sigma\rho}p_1^\kappa p_2^\mu + \epsilon^{\alpha\beta\nu\lambda\sigma\kappa}p_1^\rho p_3^\mu + \epsilon^{\alpha\beta\nu\lambda\sigma\mu}p_2^\rho p_3^\kappa \right) p_{1\alpha}p_{2\beta} \\ &\times \left\langle \bar{\psi}(\bar{z})\bar{\psi}(0) \right\rangle^2, \end{aligned} \quad (\text{D.19})$$

where in the second step we used Eq. (D.1f) and the fact that each of the three fermion correlators will contribute the same amount thanks to translation invariance.

Using the result (D.19) in (3.5) along with the partition function (3.1), the integrated correlator (D.4b) and the replacement (D.3) on the right side, we obtain for the integrated three-point function in this sector

$$\mathcal{I}^{\bar{e}-o} = \mathcal{T}^{\bar{e}-o} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \frac{\tau_2^3}{\pi^3} \frac{1}{8} \frac{1}{\tau_2^2} \times (-4\pi^2) \times 4\pi^2 \times \frac{\pi}{\tau_2} \times 16 \times 3 = -32\pi^3 \mathcal{T}^{\bar{e}-o}, \quad (\text{D.20})$$

where we have defined the tensor structure

$$\begin{aligned}\mathcal{T}^{\bar{e}-o} &= \epsilon_{1\mu\nu}\epsilon_{2\kappa\lambda}\epsilon_{3\rho\sigma} \left(\epsilon^{\alpha\beta\nu\lambda\sigma\rho} p_1^\kappa p_2^\mu + \epsilon^{\alpha\beta\nu\lambda\sigma\kappa} p_1^\rho p_3^\mu + \epsilon^{\alpha\beta\nu\lambda\sigma\mu} p_2^\rho p_3^\kappa \right) p_{1\alpha} p_{2\beta} \\ &= \frac{1}{2} p_1 \wedge p_2 \wedge p_1 \epsilon_2 \wedge p_2 \epsilon_1 \wedge \epsilon_3 + \text{perm.}\end{aligned}\quad (\text{D.21})$$

Following the same steps, and using Eq. (3.7), it is not difficult to show that in the other CP-odd sector the result is

$$\mathcal{I}^{\bar{o}-e} = -\epsilon \rho_1 \rho_2 \rho_3 \mathcal{I}^{\bar{e}-o}, \quad (\text{D.22})$$

so that the total result for the CP-odd part of the three-point function (3.5) is

$$\mathcal{I}^{\text{CP-odd}} = -(1 - \epsilon \rho_1 \rho_2 \rho_3) 32\pi^3 \mathcal{T}^{\bar{e}-o}. \quad (\text{D.23})$$

This implies that we need, for the non-zero couplings:

$$\text{type IIA : } \rho_1 \rho_2 \rho_3 = -1, \quad (\text{D.24a})$$

$$\text{type IIB : } \rho_1 \rho_2 \rho_3 = 1, \quad (\text{D.24b})$$

so that in type IIA we need an odd number of antisymmetric tensors and in type IIB an even number. Note also that one cannot construct any CP-odd four-derivative on-shell coupling between three gravitons.

Comparing Eq. (D.21) with (3.12), we conclude that the one-loop correction to CP-odd four-derivative gravitational couplings is

$$\mathcal{I}_{\text{eff, IIA}}^{\text{CP-odd}} = 32\pi^3 \int d^4x \sqrt{-g} \frac{1}{2} (B \wedge R \wedge R + B \wedge \nabla H \wedge \nabla H), \quad (\text{D.25a})$$

$$\mathcal{I}_{\text{eff, IIB}}^{\text{CP-odd}} = -32\pi^3 \int d^4x \sqrt{-g} \left(\frac{1}{6} H \wedge H \wedge R \right). \quad (\text{D.25b})$$

D.3 Two-graviton– N -moduli scattering in four dimensions

Here we evaluate the (leading) four-momentum piece of the $(N+2)$ -point amplitude in Eq. (4.4). We first define a set of signs specifying the nature of the moduli:

$$\chi_\phi = \begin{cases} 1, & \phi = T, U \\ -1, & \phi = \bar{T}, \bar{U} \end{cases} \quad \sigma_\phi = \begin{cases} 1, & \phi = T, \bar{T} \\ -1, & \phi = U, \bar{U} \end{cases}. \quad (\text{D.26})$$

With these notations, the selection rules read:

$$\begin{aligned}v(\phi_i)_{IJ} G^{JK} v(\phi_j)_{LK} &= 0, \quad \text{if } \sigma_i = \sigma_j, \\ v(\phi_i)_{JI} G^{JK} v(\phi_j)_{KL} &= 0, \quad \text{if } \sigma_i \chi_i = \sigma_j \chi_j.\end{aligned}\quad (\text{D.27})$$

Let us first focus on the $\bar{e}-e$ case, and first on the left side. If the moduli are chiral, they all have the same vertex $\partial X + ip \cdot \psi \Psi$ on the left side; therefore, they can only contribute

through the zero-mode p_L of ∂X . If on the other hand modulus i is chiral and modulus j is antichiral, there can a priori be a contraction $\partial X(\bar{z}_i, z_i)\partial\bar{X}(\bar{z}_j, z_j)$, but this will be a total derivative with respect to z_i , unless there is also a contraction $\bar{\partial}X(\bar{z}_i, z_i)\bar{\partial}X(\bar{z}_j, z_j)$ on the right side. But this can only occur if ϕ_i and ϕ_j have also opposite vertices on the right side, that is $\phi_i = \bar{\phi}_j$, a case that we excluded. Therefore, only the zero-modes $p_R^I p_L^J$ of $\partial X^I \partial X^J$ contribute. Moreover, we must contract the fermionic parts of the graviton–two-form vertices together, since other contractions vanish after the sum over even spin structures, thereby providing four powers of momenta. All in all,

$$A^{\bar{e}-e} = \left(p_1^\kappa p_2^\mu - p_1 \cdot p_2 g^{\mu\kappa}\right) \left(p_1^\lambda p_2^\nu - p_1 \cdot p_2 g^{\nu\lambda}\right) \langle\psi\psi\rangle^2 \langle\bar{\psi}\bar{\psi}\rangle^2 \prod_{j=3}^{N+2} v_{IJ}(\phi_j) p_R^I p_L^J. \quad (\text{D.28})$$

A similar reasoning applies when one of the spin structures is odd and shows that the 2 fermionic zero-modes on T^2 have to come from the vertex in the -1 -picture together with the T^2 piece of the supercurrent, while all other vertices contribute through the bosonic zero-modes. The space-time fermionic zero-modes are then provided by the graviton or two-form vertex operators. We find:

$$\begin{aligned} A^{\bar{o}-o} &= \epsilon^{\kappa\mu\alpha\beta} \epsilon^{\lambda\nu\rho\sigma} p_{1\alpha} p_{2\beta} p_{1\rho} p_{2\sigma} \left(G\epsilon v(\phi)\epsilon G\right)_{IJ} p_R^I p_L^J \\ A^{\bar{e}-o} &= \left(p_1^\kappa p_2^\mu - p_1 \cdot p_2 g^{\mu\kappa}\right) \epsilon^{\lambda\nu\alpha\beta} p_{1\alpha} p_{2\beta} \left(\langle\bar{\psi}\bar{\psi}\rangle^2 - \langle\bar{\partial}X X\rangle^2\right) \left(v(\phi)\epsilon G\right)_{IJ} p_R^I p_L^J \\ A^{\bar{o}-e} &= \epsilon^{\kappa\mu\alpha\beta} p_{1\alpha} p_{2\beta} \left(p_1^\lambda p_2^\nu - p_1 \cdot p_2 g^{\nu\lambda}\right) \left(\langle\psi\psi\rangle^2 - \langle\partial X X\rangle^2\right) \left(G\epsilon v(\phi)\right)_{IJ} p_R^I p_L^J. \end{aligned} \quad (\text{D.29})$$

The Riemann identity (B.13) allows us to carry out the spin structure summation and shows that the integrand is in fact independent of the position of the vertices. In the odd spin structure, the saturation of zero-modes induces the replacement (D.3).

We can simplify the kinematic structures by making use of Eq. (A.4), and rewrite them as (recall that $p_1 \cdot p_2$ is not restricted to vanish anymore):

$$\begin{aligned} \mathcal{T}^{\bar{e}-e} &= (p_1 \epsilon_2 p_1)(p_2 \epsilon_1 p_2) - (p_1 p_2) \left((p_2 \epsilon_1^T \epsilon_2 p_1) + (p_2 \epsilon_1 \epsilon_2^T p_1) \right) + (p_1 p_2)^2 (\epsilon_1^T \epsilon_2) \\ \mathcal{T}^{\bar{o}-o} &= - (p_1 \epsilon_2 p_1)(p_2 \epsilon_1 p_2) + (p_1 p_2) \left((p_2 \epsilon_1 \epsilon_2 p_1) + (p_2 \epsilon_1 \epsilon_2 p_1) \right) \\ &\quad + (p_1 p_2)^2 \left((\epsilon_1)(\epsilon_2) - (\epsilon_1 \epsilon_2) \right) \\ \mathcal{T}^{\bar{e}-o} &= p_1 \epsilon_2 \wedge p_2 \epsilon_1 \wedge p_1 \wedge p_2 + (p_1 p_2) p_1 \wedge p_2 \wedge \epsilon_1^T \epsilon_2 \\ \mathcal{T}^{\bar{o}-e} &= \epsilon_2 p_1 \wedge \epsilon_1 p_2 \wedge p_1 \wedge p_2 + (p_1 p_2) p_1 \wedge p_2 \wedge \epsilon_1 \epsilon_2^T. \end{aligned} \quad (\text{D.30})$$

If particle 1 is a dilaton, this can be further reduced to

$$\mathcal{T}^{\bar{e}-e} = \mathcal{T}^{\bar{o}-o} = (p_1 p_2)^2 (\epsilon_2), \quad \mathcal{T}^{\bar{e}-o} = -\mathcal{T}^{\bar{o}-e} = -(p_1 p_2) p_1 \wedge p_2 \wedge \epsilon_2, \quad (\text{D.31})$$

so that one dilaton only couples to another dilaton (CP-even) or to an antisymmetric tensor (CP-odd) at this order. One also notes that the $\bar{o}-o$ contribution is opposite to the $\bar{e}-e$ contribution in the two-graviton case, equal in the b^2 and Φ^2 cases.

We can then make use of the identities

$$v(\phi)\epsilon G = i\chi_\phi v(\phi), \quad \chi_\phi = \begin{cases} 1, & \phi = T, U \\ -1, & \phi = \bar{T}, \bar{U} \end{cases} \quad (\text{D.32a})$$

$$G\epsilon v(\phi) = i\sigma_\phi \chi_\phi v(\phi), \quad \sigma_\phi = \begin{cases} 1, & \phi = T, \bar{T} \\ -1, & \phi = U, \bar{U} \end{cases} \quad (\text{D.32b})$$

$$\sum_{p_L, p_R} v_{IJ}(\phi) p_R^I p_L^J q^{\frac{1}{2}p_L^2} \bar{q}^{\frac{1}{2}p_R^2} = \frac{1}{\pi\tau_2} \partial_\phi \Gamma_{2,2} \quad (\text{D.32c})$$

and find the general result:

$$\mathcal{I}_\phi^{\bar{i}j} = \mathcal{T}^{\bar{i}j} \kappa^{\bar{i}j} \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \left(\frac{\tau_2}{\pi}\right)^3 \frac{1}{\pi\tau_2} \partial_\phi Z^{\bar{i}j}, \quad (\text{D.33})$$

where i, j run over the even and odd spin structures. The quantities $\kappa^{\bar{i}j}$ and $Z^{\bar{i}j}$ are defined in Eqs. (4.12) and (4.13). In the last equation, ∂_ϕ stands for the product of derivatives with respect to the moduli ϕ_i . For $N > 1$, these derivatives are actually promoted to modular covariant derivatives due to reducible diagrams that we disregarded. The signs $\kappa^{\bar{i}j}$ make reference to the modulus in the -1 -ghost-picture. The various coefficients in Eq. (4.11) are then obtained by comparing the kinematical factors (D.30) to the vertices (4.2) and taking proper account of symmetry weights.

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