

# Wilson line in high temperature particle physics

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The physics of the Wilson line leads to new developments in high temperature particle physics. The main tool is the effective action for a given fixed value of the phase of the Wilson line. It furnishes a gauge invariant infrared cut off, and yields for small values of the phases a systematic procedure for obtaining a power series in the coupling  $g$  and  $\log(1/g)$ . It breaks the centergroup symmetry of the gauge group only at high temperature so leads to domain walls disappearing at low temperatures. It shows long lived metastable states in the standard model, SU(5), SO(10) and its SUSY partners, with possibilities for CP violation and thermal inflation.

## 1 Introduction

Phase transitions have been a recurrent theme at this meeting. We have heard about the quantitative progress in understanding the electro-weak transition<sup>1</sup> and along similar lines the SU(5) GUT phase transition<sup>2</sup> and to some extent the deconfinement transition<sup>3</sup>.

Despite their differences there is a common feature: all of them have an order parameter, the Wilson line. The Wilson line is strictly speaking an order parameter, when there are no complex representations of the gauge group present in the particle content. Only then it will be strictly zero in one phase and non-zero in the other.

Usually the Wilson line is mentioned in discussions of the quark gluon transition (apart from the chiral condensate), not of the other transitions, like the electro-weak and GUT transitions where the Higgs fields are the protagonists in the transition.

In this paper we will discuss the role of the loop; in particular its phase, and the dependence of the free energy on that phase. The latter is called the effective action and is the main mathematical tool.

For small values of the phases this effective action is a gauge invariant infrared cut off version of the free energy and the Debye mass. As such it is a nice

starting point for the perturbative evaluation of these quantities<sup>4</sup>. Infra red singularities upset the usual series in  $g^2$ , and their computation necessitates a cutoff. This is briefly discussed in section 3.

For values of the phase in the centergroup one finds metastable minima. They have one common property: they are very long lived on any cosmological scale. This is true for the standard model and for GUT models like SU(5) and SO(10) and their SUSY versions. A rather detailed description of where the minima are and how they decay is found in section 4. Then we turn to the other order parameters, the Higgs fields. The potential has qualitatively different behaviour due to the Wilson line condensate, in case the Higgs field carries Z(N) charge. This is the subject of section 5.

In section 5.3 we turn to the physics of these metastable states. As the CP violating properties of some of the metastable states have been analysed elsewhere<sup>5</sup>, we limit ourselves to thermal inflation. This happens in the SO(10) model and its SUSY analogue.

The question of how the universe can arrive in such a metastable state arises naturally<sup>5</sup>. There has been debate on this<sup>6</sup>, as well as on the thermodynamical properties of the metastable states<sup>7</sup> and the physical relevance of the phase of the loop<sup>8</sup>. On the latter two the reader can find partial satisfaction in section 2.

## 2 Wilson line as order parameter

In this section we will resume some well known facts about gauge systems at high temperature in equilibrium. First the academic but instructive case of pure SU(N) theory is reviewed. Then we introduce the Wilson line as the order parameter that describes walls, and the effective action that controls its behaviour. In the last subsection we discuss the boundary conditions that trigger localised walls.

### 2.1 Pure gauge theory

Let us consider a pure SU(N) gauge theory<sup>9</sup>.

Take an elongated box in 3D with size  $L_{tr}^2 L_z$ . The size in the z-direction is by far the largest in order to have walls that separate vacuum states. Boundary conditions on the gauge potentials are periodic.

Physical states are by definition invariant under periodic gauge transformations. This means Gauss' law is satisfied everywhere. Consider now a gauge transformation periodic in the transverse directions, but periodic modulo a centergroup element  $\exp i\frac{2k\pi}{N}$  in the z-direction. Such transformations  $U_k$  differ from one another by a periodic transformation, so have *all* the same effect

on a physical state. Like a periodic transformation they do commute with the Hamiltonian. So we can diagonalise them simultaneously. Acting on the physical states they have eigenvalues all of which are again  $Z(N)$  phases<sup>a</sup>

We can easily get a physical picture of what these states are: start from a state on which *all* operators  $U_k$  have eigenvalue one. This state has no electric flux, only glueballs are thermally excited. Now take a string running from  $z = 0$  to  $z = L_z$  given by the path-ordered operator  $TrP \exp i \int dz A_z(x, y, z)$  acting on the no flux state. This state has one electric flux, since the eigenvalue of  $U_k$  is  $\exp ik \frac{2\pi}{N}$ . If we add another string  $k$  will be replaced by  $2k$  in the phase. Note that a state with a given flux  $e$  can have  $e \bmod N$  strings. We can create strings independently in all directions. This means that a state will be given by a flux vector  $\vec{e}$ . Such a flux state is obtained by acting with a projector  $P_{\vec{e}}$  on a physical state. This projector is related to the operators  $U_{\vec{k}}$  by a  $Z(N)$  Fourier transform:

$$P_{\vec{e}} = \frac{1}{N^3} \sum_{\vec{k}} \exp -i\vec{k} \cdot \vec{e} \frac{2\pi}{N} U_{\vec{k}} \quad (1)$$

Since flux is conserved (the Hamiltonian commutes with the  $U_{\vec{k}}$ ), we can define a flux free energy  $F_{\vec{e}}$  by the Gibbs trace over flux  $e$  states:

$$\exp \frac{-F_{\vec{e}}}{T} \equiv \sum \langle \vec{e} | \exp \frac{-H}{T} | \vec{e} \rangle \quad (2)$$

These flux free energies should tell us how the strings behave. Now we expect at low  $T$  confinement, hence a string tension  $\rho$ . Let us take the case of one flux in the  $z$ -direction. Then at low  $T$  ( $\ll \rho$ ):

$$F_1 - F_0 = \rho L_z \quad (3)$$

For high enough  $T(T_c)$  this behaviour changes into:

$$F_1 - F_0 = L_z \exp \frac{-\alpha(T)}{TL_{tr}^2} \quad (4)$$

The physics of these equations is simple. At low  $T$  creation of one string has a small probability  $\sim \exp -\frac{\rho}{T} L_z$ . At the same time the difference between the free energies as the length of the box increases. The unique groundstate is given by  $F_0$ .

As the temperature goes up so goes the probability for exciting strings. The number of strings present in the Gibbs sum grows and so does the entropy. At  $T_c$  the entropy overtakes the energy  $\rho$  and above  $T_c$  the free energies

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<sup>a</sup>That's because  $U_{\vec{k}}^N$  is periodic, hence does not change the physical state.

become exponentially degenerate. This means that the  $Z(N)$  symmetry is spontaneously broken.<sup>b</sup>

The question is now: does the parameter  $\alpha$  that controls the decay of the flux free energy correspond to a surface tension between two regions degenerate in energy? If so, there must be an order parameter telling the difference between the two degenerate states.

## 2.2 Domainwalls as Wilson line profiles

To study domain walls one has to introduce the duals  $Z_{\vec{k}}$  of the flux free energies. They are called “twisted” transition elements and are defined by

$$Z_{\vec{k}} = Tr_{phys} \exp(-H/T) U_{\vec{k}} \quad (5)$$

The  $Z_{\vec{k}}$  are related to the  $F_{\vec{e}}$  by the formula 1 for the electric projectors, that is, by a  $Z(N)$  Fourier transform.

They can be rewritten into 4D “twisted” path integrals. To see how this works, take  $U_{\vec{k}} = 1$ . This corresponds to the well-known pathintegral with an integration over periodic  $A_0$  coming from the physicality constraints on the states.<sup>c</sup>

The presence of  $U_{\vec{k}}$  *does* change the state the path integral starts with at time  $\tau = 0$ : it creates a center group discontinuity when going from one side ( $z = 0$ ) to the other ( $z = L_z$ ). Then we go in the time direction to the point  $\tau = 1/T, z = L_z$ , keeping this discontinuity. Going first in the time- and then in the  $z$  direction one meets no discontinuity, so we have a vortex in the  $z - \tau$  planes with strength  $\exp ik \frac{2\pi}{N}$ . Consider a Wilson line

$$p(\vec{x}) = Tr P \exp i \int_0^{\frac{1}{T}} A_0(\tau, \vec{x}) d\tau \quad (6)$$

at  $z = 0$ . When we push the line to  $z = L_z$  it will pick up the phase of the vortex, as soon as it crosses the center of the vortex.<sup>d</sup>

This is an important property of the line: at high T it will have a non-zero expectation value and change its phase going from one side of the box to the

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<sup>b</sup>This behaviour has been found analytically in gauge Potts models<sup>10</sup> and in gauge theory by Monte Carlo simulations<sup>11</sup>. For other mechanisms of symmetry breaking at high temperature see ref.<sup>12</sup>

<sup>c</sup>Strictly speaking,  $A_0$  need not be periodic in the Euclidean time direction. Constraining it to be periodic does not affect the thermodynamical properties

<sup>d</sup>Clearly such a line in the  $z$ -direction will have this property too. However, it will not have a VEV at high T, contrary to the Wilson line. For a more complete discussion, see ref.<sup>13</sup>

other. This answers the question we posed at the end of the last paragraph: the parameter  $\alpha$  corresponds to a surface tension, the surface separating two phases where the Wilson line takes different phases. Note that we have not as yet a three dimensional interpretation of the Wilson line. As it stands it is a Euclidean path integral object.

The twisted transition element has been analysed numerically in four dimensions.<sup>11</sup> In three dimensions also the profiles  $p(z)$  of the Wilsonlines have been measured, clearly indicating domain walls above  $T_c$ .<sup>13</sup>

### 2.3 *Wilsonline and heavy quarksource*

We would like to associate the Wilsonline in the periodic system with the presence of a heavy quark. This in order to have a three dimensional interpretation of the line and of the twisted transition element.

However in a periodic volume with Gauss' law everywhere imposed a single heavy quark source cannot exist. So we have to drop the periodicity and search for convenient boundary conditions, that replace our twisted box and introduce a localised wall. In this article we will not explain this in detail, but the strategy can be read off from a simple  $Z(2)$  lattice gauge model.<sup>14</sup> The model is the limit of an  $SU(2)$  gauge model where all particles are made very massive by adjoint Higgs multiplets. This leaves only  $Z(2)$  centergroup transformations, as they still commute with the adjoint Higgses. There one can show that at high temperature a domain wall is created with a profile of positive energy density. The rigour is the same as used in proving that the Ising model can produce walls.

## 3 **Effective action, perturbative expansion, and infra red cut off**

The surface tension  $\alpha$  was computed in perturbation theory.<sup>15</sup> Also the profile of the Wilson line. The computation leads in a natural way to the realisation that the loop serves as a gauge invariant infra red cut-off<sup>4</sup> in the effective action. This is the subject of this section.

The effective action  $U$  for the Wilson line is defined for any given profile  $p(z)$  as follows:

$$\exp -\frac{U(p)}{T} L_{tr}^2 = \int DA_\mu \delta(p - \bar{P}(A_0)) \exp -\frac{1}{g^2} S(A) \quad (7)$$

To avoid clutter in eq. 7 we left out the  $z$  dependence.  $\bar{P}(A_0)$  stands for the normalised average over the transverse directions of the Wilson line. Hence it need not be unitary.

The effective action gives the twisted transition element (eq. 5) by integrating over all profile configurations with boundary conditions appropriate to the twist. In the time direction the statistics of the boson fields imposes periodicity, for fermions anti periodicity.

Let us note that the Wilson line transforms under periodic gauge transformations as an adjoint. So only the phases are gauge invariant. Hence the loop  $P$  in eq. 7 stands for  $TrP$ , and to get all eigenvalues one should admit winding the loop several times (from 1 to  $N-1$ ) before taking the trace.

$U(p)$  has a symmetry under gauge transformations that are periodic in the time direction modulo a centergroup element. Such a transformation indeed leaves the action and the measure invariant, but changes the Wilson line by that same phase. Hence we have:

$$U(p \exp ik \frac{2\pi}{N}) = U(p) \quad (8)$$

The same stays true when we admit fields with no  $Z(N)$  charge. But fields that do carry  $Z(N)$  charge will see their boundary conditions in the time direction changed by the phase. Hence the relation (8) is no longer true; it only holds approximately, the better the larger the masses of those fields are.

In what follows we consider the perturbative expansion of  $U$ . This implies we work at very high  $T$ , where the VEV of the Wilsonline has modulus one, so only phases:

$$p(z) = \exp i2\pi C(z) \quad (9)$$

The matrix  $C$  is in the Cartan subspace of the Lie algebra of  $SU(N)$  and contains all the eigenvalues  $C_i, i = 1 \dots N$ , with the constraint that they add up to zero. So our perturbative expansion will be around  $C$  as background.

$U(C)$ , the effective action, will take the general form of a kinetic and a potential part:

$$U(C) = \int dz \left( K(C) Tr(2\pi T \frac{\partial}{\partial z} C)^2 + \pi^2 T^4 V(C) \right) \quad (10)$$

Both  $K$  and  $V$  are  $Z(N)$  periodic just as  $U(C)$  in eq. 8, and will loose that property when there  $Z(N)$  charged fields.

At large values of  $L_{tr}$  the pathintegral over all profiles is dominated by the extremum of  $U(p)$ . At large values of  $|z|$  the wall will be determined by the behaviour of kinetic and potential terms at  $C=0$ . The behaviour of the wall in that region is determined by the large distance behaviour of the correlation of the phase of the loop. Hence it will be dominated by the Debye mass, not by the magnetic glueball mass, since the correlation is odd under  $CT$ <sup>16</sup>.

### 3.1 Background field expansion

The expansion introduces fluctuation fields  $Q_\mu$  around the background  $C$ :

$$A_\mu = 2\pi TC\delta_{\mu,0} + gQ_\mu \quad (11)$$

Now we expand the Wilson line around  $C$ . This gives:

$$\begin{aligned} P(A_0) = & \exp i2\pi C + \int_0^{\frac{1}{T}} d\tau \exp(i2\pi TC\tau) igQ_0(\tau) \exp(i2\pi TC(1/T - \tau)) \\ & + \int_0^{\frac{1}{T}} d\tau_1 \int_{\tau_1}^{\frac{1}{T}} d\tau_2 \exp(i2\pi TC\tau_1) igQ_0(\tau_1) \exp(i2\pi TC(\tau_2 - \tau_1)) \\ & igQ_0(\tau_2) \exp(i2\pi TC(1/T - \tau_2)) + O(g^3) \quad (12) \end{aligned}$$

This expansion satisfies the constraint in the effective action up to terms of order  $g$ . The delta function is then expanded around these terms. Since  $C$  is diagonal the trace of the order  $g$  term tells us that static diagonal  $\bar{Q}_0$  should not be integrated over. On the other hand expanding the action gives a linear term containing *only* static diagonal  $\bar{Q}_0$ . So  $C$  is the correct minimum for the expansion.

We have to choose gauge fixing and will take it of the form<sup>17</sup>:

$$S_{gf} = \frac{1}{\xi} Tr \left( \frac{1}{\xi'} D_0 Q_0 + \partial_i Q_i \right)^2 \quad (13)$$

which reduces to usual background field gauge fixing for  $\xi' = 1$ .  $\xi' \rightarrow 0$  gives us static background gauge, in which  $Q_0$  decouples from the loop in eq. 12.

So in general thermal fluctuations of the loop expectation value will be  $O(g^2)$ , and in static background gauge absent.<sup>17</sup> This is important for the tunneling results of next section. The fluctuations will contribute to the effective action, and render it gauge independent.<sup>18</sup>

Another remark concerns the propagators of the quantum fields  $Q$ : with our gauge choice the background field will enter the quadratic part of the action through the covariant time derivative:  $Tr Q_\mu \cdot (-D_0^2 - \vec{\partial}^2) \cdot Q_\mu$ . The background enters only through commutators. Let us denote the field with all components zero, except the one on the row  $i$  and column  $j$  by  $Q^{ij}$ . Then the inverse propagator will look for *all* polarisations like:

$$((2\pi T)^2 (n + C_i - C_j)^2 + \vec{p})^2 \quad (14)$$

This shows that the static configurations ( $n = 0$ ) are screened by the phases, except where the differences are integer valued. This happens precisely

in centergroup values  $\exp i2\pi C = \exp ik\frac{2\pi}{N}$ . Fields diagonal in colour are not screened. From those diagonal fields the constraint in eq. 7 eliminates part. In other words the  $U(1)^{N-1}$  subgroup (including for the SM QED) keeps a infra red problem.

The same is true for any particle species that has no  $Z(N)$  charge, e.g. is in the adjoint representation. Actually for those species all centergroup elements look the same. This confirms on the perturbative level what we found quite generally in the previous subsection.

For particles in the fundamental  $N$ -dimensional representation the inverse propagators look like:

$$((2\pi T)^2(n + C_i)^2 + \vec{p})^2 \quad (15)$$

for the  $i$ -th component. Now there is still screening in the centergroup elements.

### 3.2 Walls and profiles in perturbation theory

The effective action is the key quantity to compute. In the limit of very large  $L_{tr}$  its extremum is going to dominate the path integral over profiles. For the one dimensional action in eq. 7 this extremum is simply given by the equations of motion for the profile:

$$\sqrt{K(C)}2\pi T\frac{\partial}{\partial z}C - \pi T^2\sqrt{V(C)} = 0 \quad (16)$$

The eigenvalues  $C_1 = -C_2$  in  $SU(2)$  are parametrised by  $2C = C_1$ ,  $V(C)$  is periodic mod 1 in  $C$ , as is obvious from the discussion on  $Z(N)$  symmetry in the previous sections. It is normalised to zero for  $C=0$ . Solving this equation with the boundary conditions  $C=0$  and  $C=1$  ( $p=1$  and  $p=-1$ ) at  $z = \pm\infty$  gives a profile, that is controlled at large  $|z|$  according to the general arguments just above subsection 3.1 by the Debye mass

$$m_D = \frac{1}{2} \lim_{C \rightarrow 0} \sqrt{\frac{V(C)}{K(C)}} \quad (17)$$

Let us see how this works in perturbation theory.

One can readily do this in the background expansion and one finds to two loop order<sup>15</sup>:

$$\begin{aligned} K(C) &= 1/g^2(1 + g^2 K_2(C)) \\ V(C) &= C^2(1 - |C|)^2 - 5\frac{g^2}{8\pi^2}C^2(1 - |C|)^2 \end{aligned} \quad (18)$$



The kinetic term  $K_2(C)$  behaves well for all  $C$ , except at small values, where it develops a pole. This is in contrast with our general expectation of the large  $z$  behaviour. It should be governed by the Debye mass in eq. 17. This equation, together with eq. 18, tells us that the correction to the lowest order result  $\sqrt{N/3}gT$  for  $m_D$  is  $(1 + O(\frac{g^2}{C}))!$  However the work of Rebhan<sup>19</sup> shows that this correction is *not*  $O(g^2)$ . It is *larger*, of order  $g \log(1/g)$  !

### 3.3 Postmortem

Our seemingly disastrous result, the pole in the kinetic term, just indicates, that for small  $C$ , naive perturbation theory breaks down. For small values of  $C$ , say  $O(g^2)$ , the screening of the propagators in eq. 14 becomes ineffective. For those values of the background one has to do first the integration over all non-static configurations, and obtain an effective 3D action  $S_3$ . This integration will induce terms of order  $g^2$  (the one containing the lowest order Debye mass term for the static fourth component of the quantum field) and higher.

The new action will only depend on the static quantum fields, the background  $C$  and inherits the gauge fixing in eq. 13. It will receive contributions from the non-static part of the constraint. The latter is gauge dependent and absent in static background gauge. Like in the 4D calculation they are the guarantee for gauge choice independence.

We have calculated the Debye mass from this effective action.<sup>4</sup> In Feynman background gauge there is only one 3D diagram contributing. The result coincides with that of Rebhan.

## 4 Effective potential for the Standard Model and beyond

In this section the one and two loop potentials for the Standard model and beyond are analysed.

There are two interesting types of minima: the absolute minimum and the metastable minima. It will turn out that the latter are very long lived on cosmological scales.

Let us first analyse the contribution to the potential of a generic particle species coupled through the covariant time derivative to its Wilson line phase  $x_s$ :

$$D_0(x) = 2\pi T(n + x) \tag{19}$$

The one loop result for a complex boson neglecting its mass is then:

$$V_b(x) = Tr_T D^2(x) - Tr_{T=0} D^2(x)$$

$$= \frac{-\pi^2}{45}T^4 + \frac{2}{3}\pi^2T^4x^2(1-|x|)^2 \quad (20)$$

The trace at temperature  $T$  stands for the sum over Matsubara frequencies and a regulated integral over  $3 - 2\epsilon$  dimensions. We subtracted the  $T=0$  contribution, and get, because of eq. 19 a result which is periodic in  $x \bmod 1$ . The contribution of a Dirac helicity doublet is simply:

$$V_f(x) = -V_b(x + \frac{1}{2}) \quad (21)$$

The minus sign comes from the fermion determinant and the shift from the anti-periodicity for fermions used in eq. 19.

Note that the sum of the two is antisymmetric when  $x$  is shifted over  $\frac{1}{2}$ . So SUSY theories, though broken at high temperature, have this discrete symmetry for every species.

Now we have to specify what  $x$  is in terms of the various Cartan group charges of our gauge group.

For the SM and  $SU(5)$  we are working in a four dimensional space of phases. So our matrix  $C$  (eq. 9) can be conveniently described by an  $SU(5)$  matrix in which the phases for colour  $SU(3)$  are given by  $q$  and  $r$ , of weak  $SU_L(2)$  by  $s$ , and of weak hypercharge  $U(1)$  by  $t$ :

$$diag(\frac{q}{3} + \frac{r}{2} - \frac{t}{3}, \frac{q}{3} - \frac{r}{2} - \frac{t}{3}, -\frac{2q}{3} - \frac{t}{3}, \frac{s}{2} + \frac{t}{2}, -\frac{s}{2} + \frac{t}{2}) \quad (22)$$

So e.g. for the righthanded electron  $x$  equals  $-\frac{1}{2}t$ .

All what is left to do is to sum over all particle species. We then normalise to zero by subtracting out the Planck free energies in eq. 20 and eq. 21. The result is plotted for  $SU(3)$  in fig.1, and for the SM and  $SU(5)$  against the weak hypercharge phase in fig.2. The Higgs content of  $SU(5)$  was taken to be the 5 and the 24. For  $SO(10)$  the 16 and 45. For SUSY  $SO(10)$  we took the complexified 45 *and* 54 to guarantee a good  $SU(5)$  limit.

For  $SO(10)$  the plot in fig.2 is in terms of a  $U(1)$  charge  $u$ , orthogonal to the  $SU(5)$  charges. The reason is that  $SO(10)$  contains  $SU(5) \times U(1)$ .

All minima are in the centergroup elements. The trivial centergroup element contains the absolute minimum. The other centergroup elements are metastable points ( though not all, as in the SM). There would have been degeneracy, had all the species been  $Z(N)$  neutral. But the fermions, the fundamental Higgs have  $Z(N)$  charge, and lift the degeneracy.

How can we be sure of seeing all the relevant minima? This will be explained in subsection 4.1.

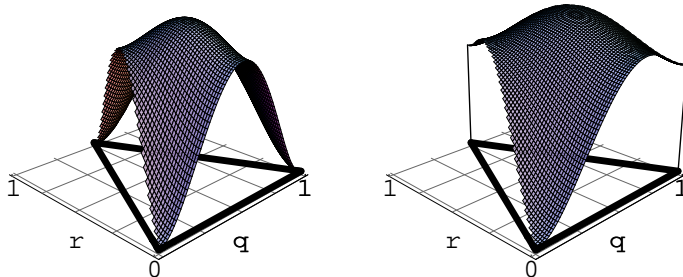


Figure 1:  $SU(3)$  potentials on the elementary cell (without and with fermions), fermions are explicitly breaking  $Z(3)$  symmetry.

Then there are the tunneling rates of these metastable states into the stable state. They will be discussed in subsection 4.2. Lastly the thermodynamics is discussed.

#### 4.1 Centergroup lattice of $SU(N)$ and $SO(2N)$ groups

We begin the lattice of points in the Cartan subspace, that contains all centergroup elements. Such a lattice is easy to generalise from the the case of  $SU(3)$  in fig.1.

Consider the set of  $N$  basis vectors  $e_1, e_2, \dots, e_N$  given by the diagonal  $N \times N$  matrices  $e_1 = \frac{1}{N} \text{diag}(1, 1, \dots, 1, 1 - n), e_2 = \frac{1}{N} \text{diag}(1 - N, 1, \dots, 1), \dots, e_N = \frac{1}{N} \text{diag}(1, 1, \dots, 1 - N, 1)$ . Their sum adds up to zero. Taking any linear combination  $L$  of the  $e_k$  with integer coefficients will give a centergroup element  $\exp i2\pi L$ . The inverse is true too: all centergroup elements are to be found on this lattice.

This lattice contains many elementary cells on which the potential is identical. This is because of the symmetries in eq. 19.

A convenient cell is starting in 0, and formed by the succession  $e_1, e_1 + e_2, \dots, e_1 + e_2 + \dots + e_{N-1}$ . Inside this cell there are conjugated points  $e_1$  and  $e_1 + e_2 + \dots + e_{N-1}$ ,  $e_1 + e_2$  and  $e_1 + e_2 + \dots + e_{N-2}$ , etc. related by charge (or CP) conjugation, and with complex conjugate values for the centergroup elements.

The cell of  $SO(10)$  is related to that of  $SU(5)$  because  $SO(10)$  or rather its covering group  $\text{Spin}(10)$  contains  $SU(5) \times U(1)$ . The centergroup is  $Z(4)$  as one can easily check from the 16 dimensional spin representation. We normalise the  $u$  variable by fixing  $P=-1$  at  $u=1$  and all other phases zero. Consider the

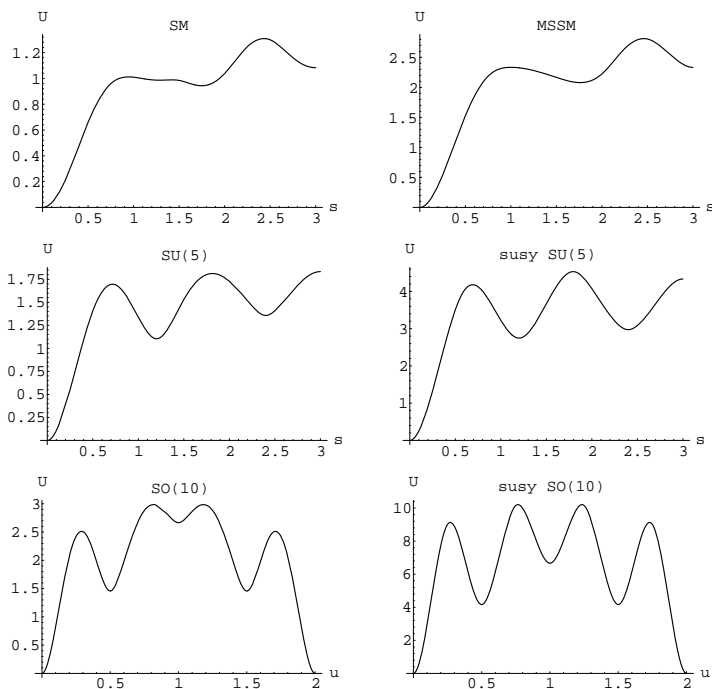


Figure 2: potentials in the weak hypercharge direction  $s$  ( $u$  for  $SO(10)$ ).

cell of  $SU(5)$ . Shift from  $e_1$  by  $-\frac{1}{5}$  in the  $u$  direction to get the vector  $u_1$  with  $P=-1$ . Shift  $e_1 + e_2$  by  $\frac{1}{10}$  in the  $u$  direction to get  $u_2$  with  $P=i$ . The complex conjugate of  $u_2$  is  $u_3$  and is obtained from  $e_1 + e_2 + e_3$  by shifting over  $-\frac{1}{10}$ , and that of  $u_1$  from  $-e_5$  by shifting over  $\frac{1}{5}$ .

Where are the minima of the potential? If only  $Z(N)$  neutral fields are around there is a proof to all orders in perturbation theory that the minima are in the centergroup elements<sup>17</sup>.

As discussed above, fermions and Higgs fields lift the degeneracy. We have verified numerically that no other metastable points develop. So we have a the following working hypothesis:

*Any metastable or stable minimum must be in the centergroup.*

A last question concerns how the minima of say  $SU(5)$  do give rise to the minima in the standard model, when decoupling the heavies in  $SU(5)$ .

Our working hypothesis will prove useful here. The heavies in  $SU(5)$  are given a common mass  $m$ , that we switch on from its value 0 in  $SU(5)$  to  $\infty$  in the SM. They are all multiplets of the SM gauge group, so switching on  $m$  will cause a movement of the minima of  $SU(5)$  along the centergroup elements of the SM, to wit  $Z(3)\times Z(2)\times U(1)$ . Since the process is continuous the minima can only slide along the six  $U(1)$  lines corresponding to the discrete  $Z(3)\times Z(2)$  group. This is shown in fig.3.

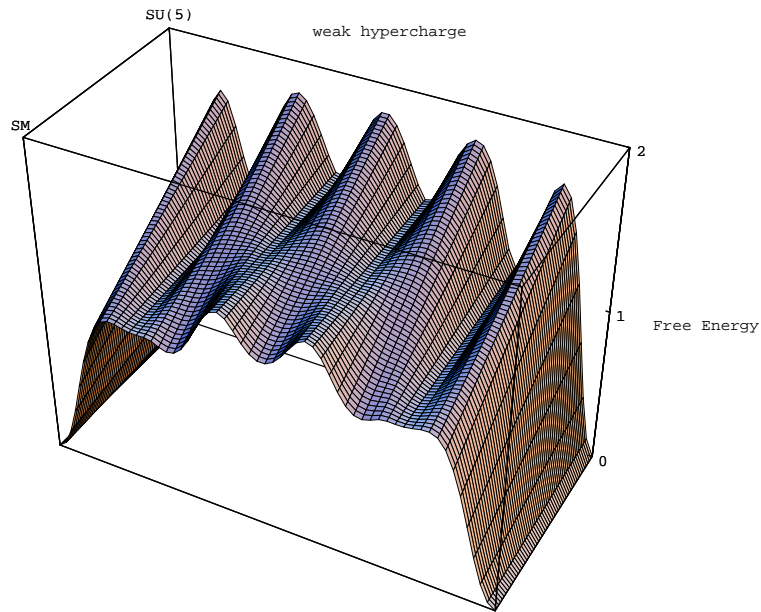


Figure 3: Potential interpolating between  $SU(5)$  and SM.

Similarly the evolution of the  $SO(10)$  minima into  $SU(5)$  minima occurs along the 5  $U(1)$  lines, that we used in the construction of the  $SO(10)$  cell out of the  $SU(5)$  cell.

In fig.4 we show how two loop corrections affect the  $SU(5)$  potential, for reasonable values of the couplings. Those are very small effects.

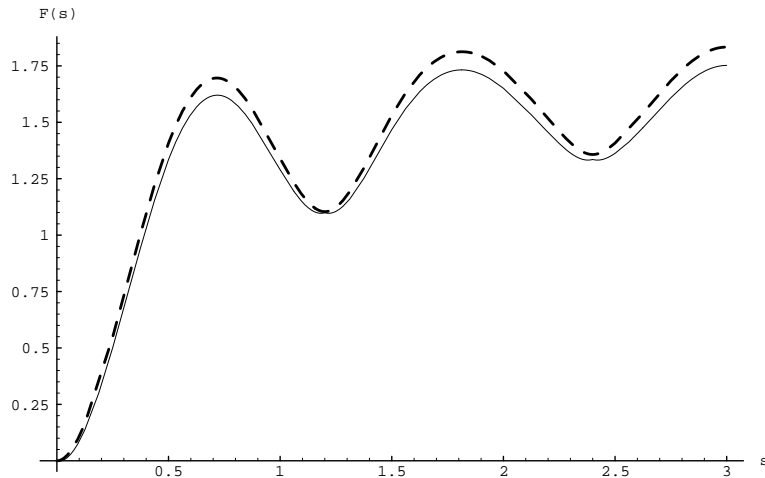


Figure 4:  $SU(5)$  potential: one-loop (dashed) and two-loop (solid).

#### 4.2 Tunneling paths and tunneling rates

Semi classical methods are appropriate, in our situation with small coupling, to compute the tunneling rate  $\Gamma$  per unit volume:

$$\Gamma = aT^4 \exp -S_b \quad (23)$$

The dimensionless quantity  $a$  derives from the determinant of the fluctuations around the classical bounce configuration  $B$  (for QCD and the SM this was done in ref. <sup>20</sup>)

The computation of the bounce consists of finding the solutions to the Euclidean equations of motion. The temperature  $T$  is much larger than the scale of the 4D Euclidean bounce, we can do the calculation in 3D <sup>22</sup> for a radially symmetric bounce:

$$\frac{d^2 B}{dr^2} + \frac{2}{r} \frac{dB}{dr} = -V'(B) \quad (24)$$

Boundary conditions are  $\lim_{r \rightarrow \infty} B = B_{ms}$  and  $\frac{dB}{dr} = 0$  at the starting point  $r_0$ . Our bounce is a multicomponent object (4 or 5) and  $B_{ms}$  is the metastable point in the elementary cell. We study the decay to the stable vacuum at the origin, so  $r_0$  is near zero, depending on the thickness of the wall. We have

developed a numerical method that solves the equations of motion and can handle the multicomponent field.

Luckily there is a lot of symmetry in our problem, that allows an educated guess of the actual tunneling path. So we can reduce the problem to a one component bounce, solvable with the undershoot-overshoot method.

The bounce path from a given metastable point in the cell is simply given by the straight path from that point. This was corroborated by our numerical method for the simple groups. For the SM the deviation from a straight path was small but significant<sup>e</sup>. The results for the bounce actions are presented in table 1.

Table 1: Numerical data on the bounces.

	metastable points	Bounce action ( $S_b/\frac{4\pi^2}{g^3}$ )	Critical radius ( $R/gT$ )
SM	$s = 1.28$	6.99	2.7
	$s = 1.74$	46.08	4.6
	$q = 1, s = -0.25$	2.64	2.0
MSSM	$s = 1.761$	37.60	3.25
SU(5)	$P_A = \exp(i4\pi/5)$	10.63	2.3
	$P_B = \exp(i2\pi/5)$	86.63	5.3
susy SU(5)	$P_A = \exp(i4\pi/5)$	9.92	1.75
	$P_B = \exp(i2\pi/5)$	50.47	3.3
SO(10)	$P = -1$	20.99	2.75
	$P = i$	308.8	7.4
susy SO(10)	$P = -1$	36.87	2.2
	$P = i$	286.07	5.1

What is striking is the large value of all bounce actions. Tunneling temperatures  $T_{tun}$  can be estimated by the following simple-minded argument. Tunneling occurs at a time  $t_c$  determined by  $\Gamma t_c^4 \sim 1$ . Matter in our metastable states has qualitatively the same free energy as in the stable vacuum. Only the pressure  $p = -\pi^2 T^4 V(C_{ms})$  is lower in the metastable states. So we will assume the relation  $t_c \sim M_{Planck}/T^2$ . Then  $T_{tun} \sim M_{Planck} \exp -S_b/4$ , using eq. 23 to eliminate  $t_c$ . Putting in the numbers from the table shows that the metastable states easily survive all known cosmological transitions.

It means that the behaviour of the potential as function of the Higgs fields

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<sup>e</sup>Not all metastable minima in fig. 2 lie in the same cell as one can easily check.

will be very relevant for how the metastable states will decay. This is the subject of section 5.

Fig. 4 shows the effect of the two loop contribution. the couplings were taken to be  $\alpha = \beta = \lambda_1 = \lambda_2 = 0.1$  and  $g = 0.4$ . The sign of the contribution is uniformly negative. That this was the case at  $C = 0$  was known since long<sup>23</sup>.

### 4.3 Thermodynamics

The calculation of the surface tension<sup>15</sup> motivated a renewed interest in the thermodynamics of the minima. In all the models considered in the previous subsections the energy and entropy densities of the minima are positive. But the thermal boundary conditions on fields with non-trivial  $Z(N)$  charge are changed in the metastable minima by a  $Z(N)$  phase, and as a consequence the occupation number is changed<sup>7</sup>:

$$n(E) = \frac{1}{\exp\left(\frac{E}{T} + ik\frac{2\pi}{N}\right) \pm 1} \quad (25)$$

So in the case of fermions one finds a purely imaginary value for the fermion number. States with complex conjugate  $Z(N)$  phase are related by CP conjugation. Our phases have the C and P transformation properties of a diagonal  $A_0$ , so are CP odd. This singles out the self conjugate states like the one in  $SO(10)$  with  $P = -1$ . There, fermion number is zero. So far no reasonable interpretation has been found for this behaviour.

Let us finally mention a case without any thermodynamic anomalies: large  $N$  pure gauge theory. To see this, consider the free energy without the Planck contribution:

$$V(C) = \frac{4}{3}\pi^2 T^4 \sum_{i \leq j} C_{ij}^2 (1 - |C_{ij}|)^2 \quad (26)$$

where  $C_{ij} = C_i - C_j$ . In that case due to the permutation symmetry of the variables  $C_i$ , the potential has an extremum in the barycenter of the elementary cell. in the one loop approximation this is an absolute maximum. Let us compute its value  $V_{max}$ . In the barycenter the value of  $C_{ij}$  is  $\frac{i-j}{N}$ . Substituting in eq. 26 gives:

$$V_{max} = \frac{\pi^2 T^4}{45} (N^2 - 1) \left( 1 + O\left(\frac{1}{N^2}\right) \right) \quad (27)$$

To leading order we find precisely the Planck free energy, but with the opposite sign! Hence adding the Planck free energy we have positive energy and entropy density, except in the maximum where both are zero. This result remains true in the next order.



For finite  $N$  there is a small cap around the maximum, where both are negative.

## 5 Combining Higgs and Wilson line potential

In this section we will compute the classical and part of the one loop contribution to the mass term in the Higgs-Wilson line potential. Already on the classical level a few subtleties merit attention.

### 5.1 Classical effective action

It is best to illustrate the problem by the specific examples of the adjoint Higgs field  $\Sigma$  and the fundamental Higgs field  $H$ .

The terms of interest in the original 4D action are the kinetic terms of the two Higgses, minimally coupled to the gauge fields:

$$S = \int d\vec{x}d\tau [Tr(D_\mu \Sigma)^2 + (D_\mu H)^\dagger D_\mu H + \dots] \quad (28)$$

To get the 3D effective action one would guess that the classical contribution would just be the 3D reduction of the kinetic terms. That would give us terms like  $Tr[C, \Sigma]^2$ .

This *cannot* be correct. We know on general grounds that the effective action for  $Z(N)$  neutral fields like  $\Sigma$  is identical in all centergroup elements. But this commutator term is not.

What couples in the 3D action to the Higgs is the phase of the loop  $p(C)$ . We define the matrix  $\log p(C)$  through the diagonal form of the argument, and under a gauge transform  $\Omega$  we have

$$\log p(C^\Omega) = \Omega p(C) \Omega^\dagger \quad (29)$$

Then the coupling of the Higgses to the Wilson line is given in a gauge invariant form in the effective action  $U(C)$  by:

$$U(C) = \int dz T^2 [Tr[\text{Im} \log p(C), \Sigma]^2 + (\text{Im} \log p(C) H)^\dagger (\text{Im} \log p(C) H) + \dots] \quad (30)$$

The VEV of the Wilson line induces mass terms that merit comment:

i) The adjoint mass term disappears in the centergroup, since the commutator vanishes there.

ii) The mass term for the fundamental Higgs does not vanishing the centergroup. Define  $|\Im \log p(C)| \leq \pi$ , then the mass term is the same for conjugate vacua.

iii) Outside the centergroup the induced Higgs masses are not necessarily the same, i.e. SU(5) invariant. In general, the potential is not anymore function of only  $Tr\Sigma^2$  etc., but contains other invariants involving  $p(C)$ .

## 5.2 Quantum corrections to the Higgs potential

The quantum corrections are crucial to the understanding of the temperature dependence of the Higgs potential and hence for the occurrence of a phase transition. These corrections have been calculated for the trivial Wilson line condensate.

In the presence of a non trivial condensate these corrections will be described below.

Let us take the concrete example of SU(5). The Higgs potential in the condensate characterised by  $\exp ik\frac{2\pi}{5}$  takes the form:

$$\begin{aligned} \pi^2 T^4 V(C) = & M^2(T, k) Tr\Sigma^2 + m^2(T, k) H^\dagger H + \lambda_1 (Tr\Sigma^2)^2 + \lambda_2 Tr\Sigma^4 \\ & + \alpha H^\dagger H Tr\Sigma^2 + \beta H^\dagger \Sigma^2 H \end{aligned} \quad (31)$$

The mass terms are temperature and k dependent through loop corrections. In the one loop calculation that we completed the  $\Sigma$  mass has the following corrections:

$$M^2(T, 0) = \left( 104\lambda_1 + \frac{188}{5}\lambda_2 + 60(1 - \xi)g^2 + 20\alpha + 4\beta \right) \frac{T^2}{24} \quad (32)$$

$$M^2(T, 1) = \left( 104\lambda_1 + \frac{188}{5}\lambda_2 + 60(1 - \xi)g^2 - \frac{36}{5}\alpha - \frac{36}{25}\beta \right) \frac{T^2}{24} \quad (33)$$

The mass term contains a dependence on the condensate uniquely through the couplings to the fundamental Higgs. This is a consequence of the  $\Sigma$  propagators not depending on the condensate. The induced terms will not change much the Higgs potential, so the SU(5) transition will not change essential features in a non-trivial condensate.

On the other hand the transition in SO(10) and the electro weak are changed by induced mass terms of order one:

$$m^2(T) = -m^2 + T^2(\text{Im } \log p(C))^2 + \text{O(couplings)} \quad (34)$$

### 5.3 Some physical consequences

Striking features of the potential are:

- i) CP conjugated states in the SM.
- ii) Occurrence of thermal terms of order unity for Higgs fields with non-trivial  $Z(N)$  charge.

The first point has been discussed<sup>5</sup> and we will not go into it. they involve *pe se* states with imaginary fermion number.

The second point has two important consequences. First, the transition will be at a *lower* temperature  $T_c(ms)$  than in the stable state at  $T_c$  by a factor  $\sqrt{\text{couplings}}$ . This follows by equating the temperature dependent mass term in eq. 34 to zero at the critical temperature.

Second, before the transition takes place, the system will undergo thermal inflation.<sup>21</sup> This happens in both states with complex and real values for the Wilson line in  $SO(10)$ . In the SM it is a negligible effect as will become clear below.

Let us denote the VEV of the  $SO(10)$  Higgs  $H$  in the 16 representation by  $v$ . The relevant terms in the effective action are then:

$$\pi^2 T^4 V = \frac{1}{2} \frac{m^2}{v^2} [H^\dagger H - v^2]^2 + T^2 (k\pi)^2 H^\dagger H + \dots \quad (35)$$

Here  $k = 1$ , or  $\pm \frac{1}{2}$  according to whether  $p = -1$ , or  $\pm i$ . As long as the temperature is above the value  $T_{out} = \frac{m}{k\pi}$  we have no VEV for the Higgs, and the energy density  $\epsilon$  of the vacuum equals  $\frac{1}{2}(mv)^2$ . This defines a second temperature scale  $T_{in} = (mv)^{\frac{1}{2}}$ . Below this scale the thermal contribution will become small with respect to the vacuum energy density till we reach the lower scale  $T_{out}$ . Below that scale the symmetry breaks, the Higgs gets the VEV  $v$  and the energy density becomes proportional to  $T^2$ , from eq. 35. So in between these temperatures we have inflation.

There is a third scale, appearing in the Friedman equation coupling the radius  $R$  of the universe to the energy density:

$$\frac{1}{R^2} \frac{dR^2}{dt} = \frac{\epsilon(T)}{M_{Pl}^2} \quad (36)$$

During the inflationary period the right hand side of this equation is approximately constant and equals the square of  $\frac{v}{M_{Pl}m}$ . This scale is the Hawking-Gibbons temperature  $T_{HG}$  and lies *below*  $T_{out}$ . That is to say, it will not play any role in the inflationary period. This is a good feature, because  $T_{HG}$  is a lower bound on the temperature during inflation.

## 6 Conclusions

We have given an overview of the use of the Wilson line in high temperature gauge field physics.

The effective potential of the Wilson line does not only give rise to symmetry breaking at high temperature. It provides in a natural way a gauge invariant infra red cut off and is useful in establishing the perturbative series for static quantities like free energy and Debye screening length. Many applications are possible, e.g. Callan-Symanzik type equations for the infra red behaviour of hot gauge theory<sup>25</sup>.

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## References

1. K.Kajantie, M. Laine, K. Rummukainen and M. Shaposhnikov, *Phys. Rev. Lett.* **77**, 2887 (1996).
2. A. Rajantie hep-ph/9702255.
3. K.Kajantie, M. Laine, K. Rummukainen and M. Shaposhnikov, hep-ph/9704416.
4. S. Bronoff et al., to be published.
5. C.P. Korthals Altes and N.J. Watson, *Phys. Rev. Lett.* **75**, 2799 (1995).
6. A. Abada, M.B. Gavela and O. Pene, hep-ph/9602388.
7. W. Chen, M.I. Dobroliubv and G.Semenoff, *Phys. Rev. D* **46**, 1223 (1992).
8. A.V. Smilga, *Ann. of Phys.* **234**, 1 (1994).
9. G.'t Hooft, *Nucl. Phys. B* **153**, 141 (1979).
10. Y. Goldschmidt and J. Shemitsu, *Nucl. Phys. B* **200**, 149 (1982)
11. K. Kajantie, L. Karkainen, R. Rummukainen, *Nucl. Phys. B* **357**, 693 (1991).
12. S. Weinberg, *Phys. Rev. D* **9**, 3357 (1974), A. Melfo and G. Senjanovic, hep-ph/9605284, J. Orloff, *Phys. Lett. B* **403**, 315 (1997).
13. S.T. West and J.F. Wheeler, *Nucl. Phys. B* **486**, 261 (1997).
14. S.Bronoff and C.P. Korthals Altes, Proceedings of the second workshop on Continuous Advances in QCD, ITP, University of Minnesota, Minneapolis, USA, March 28-31, 1996.

15. T. Bhattacharya, A. Gocksch, C.P. Korthals Altes, R. D. Pisarski, *Phys. Rev. Lett.* **66**, 998 (1991), T. Bhattacharya, A. Gocksch, C.P. Korthals Altes, R. D. Pisarski, *Nucl. Phys. B* **383**, 497 (1992).
16. P. Arnold and L.G. Yaffe, *Phys. Rev. D* **52**, 7208 (1995).
17. A. Gocksch and R.D.Pisarski, *Nucl. Phys. B* **402**, 657 (1993).
18. V. M. Belyaev, *Phys. Lett. B* **254**, 153 (1992).
19. A.K. Rebhan, *Phys. Rev. D* **48**, 3967 (1993).
20. J. Ignatius, K. Kajantie, K. Rummukainen, *Phys. Rev. Lett.* **68**, 737 (1992).
21. P. Binetruy and M.K. Gaillard, *Phys. Rev. D* **34**, 3069 (1986), D.H. Lyth and E.D. Stewart, *Phys. Rev. Lett.* **75**, 201 (1995).
22. A.D. Linde, *Nucl. Phys. B* **216**, 421 (1983), Erratum-ibid *Nucl. Phys. B* **223**, 544 (1983).
23. J.I. Kapusta, *Finite Temperature Field Theory* (Cambridge University Press, 1989).
24. C. P. Korthals Altes, *Nucl. Phys. B* **420**, 637 (1994).
25. S. Bronoff et al, to be published.
26. A. Roberge, N.Weiss, *Nucl. Phys. B* **275**, 734 (1986).