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Orbits of Exceptional Groups, Duality and BPS States in String Theory

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Abstract

We give an invariant classification of orbits of the fundamental representations of exceptional groups $E_{7(7)}$ and $E_{6(6)}$ which classify BPS states in string and M theories toroidally compactified to $d = 4$ and 5. The exceptional Jordan algebra and the exceptional Freudenthal triple system and their cubic and quartic invariants play a major role in this classification. The cubic and quartic invariants correspond to the black hole entropy in $d = 5$ and $d = 4$, respectively. The classification of BPS states preserving different numbers of supersymmetries is in close parallel to the classification of the little groups and the orbits of timelike, lightlike and space-like vectors in Minkowski space.

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1 Introduction

The exceptional groups $E_{7(7)}$ and $E_{6(6)}$ appear as duality symmetries [1, 2] of the low energy actions and their discrete subgroups as symmetries of the non-perturbative BPS spectrum of string and M theories in $d=4$ and 5 preserving $N=8$ supersymmetry [3]. The charges of the extremal BPS black holes can be assigned to the fundamental representations of the exceptional groups $E_{7(7)}$ and $E_{6(6)}$ which are 56 and 27 dimensional ,respectively. The entropy of these black holes in $d=5$ and $d=4$ is given by the square root of the cubic and quartic invariants of $E_{6(6)}$ and $E_{7(7)}$, respectively [4, 5]. However, the charge configurations must satisfy additional restrictions depending on the number of supersymmetries preserved. In fact, the eigenvalues of the central charge matrix must be degenerate when more than one supersymmetry is preserved by the black hole solution. These constraints were recently investigated in terms of a certain set of invariant conditions on the representation [6]. In this paper we give a classification of such BPS states in terms of orbits of $E_{6(6)}$ and $E_{7(7)}$ in the corresponding representation.

2 Jordan Algebras , Exceptional Groups and Their Orbits

The cubic invariant I_3 in the 27 dimensional representation of E_6 can be identified with the cubic norm of the exceptional Jordan algebra $J_3^{\mathbf{O}}$ of 3×3 hermitian matrices over the composition algebra of octonions $\mathbf{O}$ with the symmetric Jordan product

$$j_1 \circ j_2 = j_2 \circ j_1 \tag{2 - 1}$$

that satisfies the Jordan identity [7, 8, 9, 10, 11, 12]

$$j_1 \circ (j_2 \circ j_1^2) = (j_1 \circ j_2) \circ j_1^2 \tag{2 - 2}$$

A generic element j of $J_3^{\mathbf{O}}$ has the form

$$j = \begin{pmatrix} \alpha_1 & o_3 & o_2^* \\ o_3^* & \alpha_2 & o_1 \\ o_2 & o_1^* & \alpha_3 \end{pmatrix} \tag{2 - 3}$$

where α_i take values over the underlying field which we take to be real numbers $\mathbf{R}$ and o_i $(i=1,2,3)$ are elements of $\mathbf{O}$. The norm of an octonion

o is defined as

$$N(o) = oo^* = o^*o \qquad (2 - 4)$$

where $*$ denotes octonion conjugation. There are different forms of the composition algebra of octonions. For the division algebra of real octonions the norm is invariant under $O(8)$ and for split octonions the norm is invariant under $O(4,4)$. For $N = 8$ supergravity the relevant form of $J_3^{\mathbf{O}}$ is the one defined over the split octonions and for the exceptional $N = 2$ Maxwell-Einstein supergravity [11] it is the one defined over real octonions. In the rest of this paper we shall restrict ourselves to $J_3^{\mathbf{O}}$ defined over the split octonions and refer to it as the split exceptional Jordan algebra. [3] The automorphism group of the split exceptional Jordan algebra is the noncompact $F_{4(4)}$ with maximal compact subgroup $USp(6) \times USp(2)$ [9]. Note that $F_{4(4)}$ is also the isometry group of the quaternionic manifold of a maximal $N = 2$ matter-Einstein supergravity one can obtain by truncation of the $N = 8$ supergravity in $d = 5$ [11, 13]. The cubic norm I_3 of $J_3^{\mathbf{O}}$ is given by

$$I_3 = \alpha_1\alpha_2\alpha_3 - \alpha_1(o_1o_1^*) - \alpha_2(o_2o_2^*) - \alpha_3(o_3o_3^*) + 2Re(o_1o_2o_3) \qquad (2 - 5)$$

where Re represents the real part of an octonion and satisfies

$$Re(o_1o_2)o_3 = Reo_1(o_2o_3) \qquad (2 - 6)$$

The invariance group of the norm form of a Jordan algebra J is referred to as the reduced structure group [8] and denoted as $St_0(J)$. For the split exceptional Jordan algebra the reduced structure group is the exceptional group $E_{6(6)}$ with a maximal compact subgroup $USp(8)$. We should note that $USp(8)$ is the automorphism group of the $N = 8$ Poincare supersymmetry algebra in $d = 5$ [2]. An element of $J_3^{\mathbf{O}}$ can be brought to a diagonal form by an $F_{4(4)}$ rotation [10] and if we denote the eigenvalues of a generic element j as λ_i $(i = 1, 2, 3)$ the cubic norm is simply

$$I_3(j) = \lambda_1\lambda_2\lambda_3 \qquad (2 - 7)$$

To make the analysis that follows clearer from a physics perspective we shall make an analogy with Minkowski space M_4 and its symmetries following [9, 14, 15]. A four vector in M_4 can be represented by 2×2 matrices $x = x_\mu\sigma^\mu$ where $\sigma^0 = 1_2$ and σ^i $(i = 1, 2, 3)$ are the Pauli matrices. As 2×2

[3] We should note that the split exceptional Jordan algebra and its associated symmetries first appeared in physics literature in attempts to find octonionic realizations of space-time supersymmetry [9].

matrices the coordinates x can be considered as elements of the Jordan algebra $J_2^{\mathbf{C}}$ of Hermitian matrices over $\mathbf{C}$ with the symmetric Jordan product which preserves hermiticity. The automorphism group of $J_2^{\mathbf{C}}$ is the covering group $SU(2)$ of the rotation group which is the analog of $F_{4(4)}$ for $J_3^{\mathbf{O}}$. The norm form of $J_2^{\mathbf{C}}$ is quadratic and is given by the ordinary determinant. The invariance group the quadratic norm of $J_2^{\mathbf{C}}$ is the covering group $Sl(2,\mathbf{C})$ of the Lorentz group $SO(3,1)$ which is the analog of $E_{6(6)}$ for $J_3^{\mathbf{O}}$. In Minkowski space a vector is characterized by its norm and the parameters of the corresponding orbits. Time-like, space-like and light-like vectors corresponding to positive, negative and vanishing norms have orbits $\frac{Sl(2,\mathbf{C})}{SU(2)}$, $\frac{Sl(2,\mathbf{C})}{SU(1,1)}$ and $\frac{Sl(2,\mathbf{C})}{E_2}$, respectively. Similarly, we can characterize the elements of $J_3^{\mathbf{O}}$ by their norms and the parameters of their orbits. The generic orbit corresponding to a non-vanishing norm $I_3(j)$ has the 26 dimensional orbit

$$\frac{E_{6(6)}}{F_{4(4)}} \tag{2 - 8}$$

In contrast to the Minkowskian case the little groups of "space-like" and "time-like" vectors are the same in the case of $J_3^{\mathbf{O}}$ since its norm is cubic. [4] As for "light-like" elements j of $J_3^{\mathbf{O}}$ with $I_3(j) = 0$ there exist two distinct orbits depending on whether one or two of the eigenvalues of j vanish. The generic light-like orbit corresponding to a single vanishing eigenvalue is given by the 26 dimensional coset space

$$\frac{E_{6(6)}}{O(4,4) \odot (T_{8_v} \oplus T_{8_s} \oplus T_{8_c})} \tag{2 - 9}$$

where $\odot$ stands for semidirect product and T_{8_i} for $i = v, s, c$ are translations corresponding to three different eight dimensional representations of $O(4,4)$ that are in triality. Furthermore they satisfy

$$[T_{8_v}, T_{8_s}] = T_{8_c} \tag{2 - 10}$$

$$[T_{8_v}, T_{8_c}] = T_{8_s} \tag{2 - 11}$$

with all the other commutators among them vanishing. The critical light-like orbit corresponds to an element j with two vanishing eigenvalues and is given by the 17 dimensional space

$$\frac{E_{6(6)}}{O(5,5) \odot T_{16}} \tag{2 - 12}$$

[4] As we shall see later the orbits of "time-like" and "space-like" vectors are quite different in four dimensions where the invariant norm form is quartic!

where T_{16} is an Abelian subgroup that decomposes as $T_{8_s} \oplus T_{8_c}$ under the $O(4,4)$ subgroup. We should note that the distinction between generic and critical light-like orbits does not exist in the Minkowskian case since the norm is quadratic in that case.[5]

As is well-known the invariance group of the light-cone in Minkowski space M_4 is the conformal group $SO(4,2)$ which acts non-linearly. In fact the Minkowski space M_4 is simply the quotient space

$$\frac{SO(4,2)}{SO(3,1) \times O(1,1) \odot T_4} \tag{2 - 13}$$

When we represent the coordinates of M_4 in terms of hermitian 2×2 matrices the action of the conformal group can be represented as a linear fractional group. As such the conformal group can be interpreted as the linear fractional group of quaternions [16]. However, if we think of the 2×2 hermitian matrices as elements of the Jordan algebra $J_2^{\mathbf{C}}$ the conformal group becomes the linear fractional group of Jordan algebra $J_2^{\mathbf{C}}$ [17, 9, 14] which generalizes to all Jordan algebras and Jordan superalgebras [17, 14, 15]. The invariance group of the light-cone of $J_3^{\mathbf{O}}$ defined by the condition $I_3(j) = 0$ is the non-compact exceptional group $E_{7(7)}$ which acts as the linear fractional group of $J_3^{\mathbf{O}}$[17, 9, 15]. This implies that the 27 dimensional space of $J_3^{\mathbf{O}}$ can be regarded as the quotient space

$$\frac{E_{7(7)}}{E_{6(6)} \times O(1,1) \odot T_{27}} \tag{2 - 14}$$

The above examples of linear fractional group actions are particular cases of the general nonlinear actions of noncompact groups G whose Lie algebras g admit a three grading with respect to a maximal rank subalgebra g^0

$$g = g^{-1} \oplus g^0 \oplus g^{+1} \tag{2 - 15}$$

In such cases there exists a nonlinear action of G on the grade $+1$ space g^{+1} via fractional linear transformations [17, 18]. In the case of $E_{7(7)}$, g^0 is simply the Lie algebra of $E_{6(6)} \times O(1,1)$ and g^{+1} is the 27 dimensional subspace corresponding to $J_3^{\mathbf{O}}$. We will comment on the relevance of the "conformal" extensions of duality groups later.

We now consider the symmetries of superstring or M theories toroidally compactified to four dimensions with $N = 8$ supersymmetry. In this case

[5] In $d = 4$ where the cubic norm is replaced by a quartic norm an even richer structure exists as we shall see later.

the duality group is $E_{7(7)}$ with maximal compact subgroup $SU(8)$. The compact subgroup $SU(8)$ acts as the automorphism group of the $N=8$ supersymmetry algebra. The generic charged vector for a BPS state has 56 components with a quartic norm I_4. The 56 dimensional representation space of $E_{7(7)}$ can be represented as elements of the exceptional Freudenthal triple system [19] which can be realized as 2×2 "matrices"of the form [20]:

$$q = \begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix} \qquad (2 - 16)$$

where $\alpha, \beta \in \mathbf{R}$ and x, y are elements of $J_3^{\mathbf{O}}$. One can define a symmetric four-linear form over the exceptional Freudenthal system which induces a quartic norm. Up to an overall normalization the quartic norm can be written as [20]

$$I_4(q) = \{\alpha\beta - T(x,y)\}^2 + 6\{\alpha I_3(y) + \beta I_3(x) - T(x^{\#}, x^{\#})\} \qquad (2 - 17)$$

where $T(x,y) \equiv Trace(x \circ y)$ and $\#$ stands for the quadratic adjoint map of $J_3^{\mathbf{O}}$ which has the property [21]

$$x^{\#\#} = I_3(x)x \qquad (2 - 18)$$

The above quartic form $I_4(q)$ is invariant under the linear action of $E_{7(7)}$ on the exceptional Freudenthal triple system. The above realization of 56 of $E_{7(7)}$ corresponds to the decomposition

$$56 = 27^1 + \bar{27}^{-1} + 1^3 + \bar{1}^{-3} \qquad (2 - 19)$$

with respect to the $E_{6(6)} \times O(1,1)$ subgroup. We should also note that 56 can also be decomposed similarly with respect to the $E_{6(2)} \times U(1)$ subgroup of $E_{7(7)}$. In this case the two singlets are complex conjugates of each other carrying opposite charges with respect to $U(1)$. $E_{6(2)}$ has the maximal compact subgroup $SU(6)\times SU(2)$ and corresponds to the isometry group of the quaternionic manifold of a maximal $N=2$ matter-Einstein supergravity truncation of the $N=8$ supergravity in $d=4$. In contrast to the five dimensional case and in analogy with the Minkowskian case we have two different classes of generic orbits with non-vanishing quartic form I_4. They correspond to

$$\frac{E_{7(7)}}{E_{6(6)}} \qquad (2 - 20)$$

and to

$$\frac{E_{7(7)}}{E_{6(2)}} \tag{2 - 21}$$

As in [6] we choose the overall sign of the quartic invariant such that it corresponds to entropy of the BPS black holes. With this choice the orbit corresponding to $\frac{E_{7(7)}}{E_{6(6)}}$ has $I_4 < 0$ and the orbit corresponding to $\frac{E_{7(7)}}{E_{6(2)}}$ has $I_4 > 0$. This can be seen from the decomposition of 56 of $E_{7(7)}$ with respect to $SU(6) \times SU(2)$

$$56 = (15,1) + (\bar{15},1) + (6,2) + (\bar{6},\bar{2}) + 1 + \bar{1} \tag{2 - 22}$$

and retaining the singlets [13].

We now consider "light-like orbits" for which $I_4 = 0$. There are 3 distinct cases depending on the number of vanishing eigenvalues that lead to vanishing I_4. We define the generic light-like orbit to be one for which a single eigenvalue vanishes. The orbit in this case is given by

$$\frac{E_{7(7)}}{F_{4(4)} \odot T_{26}} \tag{2 - 23}$$

where T_{26} is a 26 dimensional Abelian subgroup of $E_{7(7)}$. The critical light-like orbit has two vanishing eigenvalues and correspond to the 45 dimensional orbit

$$\frac{E_{7(7)}}{O(5,5) \odot (T_{10} \oplus T_{16} \oplus T_{\bar{16}} \oplus T_1)} \tag{2 - 24}$$

The strange looking 88 dimensional triangular subgroup of $E_{7(7)}$ above can be seen from the better known triangular subgroup

$$O(6,6) \odot (T_{32} \oplus T_1) \tag{2 - 25}$$

of $E_{7(7)}$ [22, 24]. The doubly critical light-like orbit with three vanishing eigenvalues is given by the 28 dimensional quotient space

$$\frac{E_{7(7)}}{E_{6(6)} \odot T_{27}} \tag{2 - 26}$$

We should note that the determination of the little groups that appear in the denominators of the above quotient spaces follows directly from the various symmetry groups of $J_3^{\mathbf{O}}$ and their different gradings [22, 23, 24]. In the next section we shall obtain the counting of the dimensions of the orbits via a complementary procedure that follows from the normal form for the central charge matrix and which relates orbits to BPS states preserving different number of supersymmetries.

3 BPS States and Supersymmetry

Extremal BPS black holes of $N = 8$ supergravity correspond to massive representations of the $N = 8$ supersymmetry algebra that saturate the BPS bound. They fall into three categories depending on whether the black hole background preserves $1/2, 1/4$ or $1/8$ of the original supersymmetry [25]. BPS states preserving $1/8$ supersymmetry are the only ones with non-vanishing entropy and regular horizon. BPS states with $1/4$ and $1/2$ supersymmetry have vanishing entropy [5]. In this section we will relate the orbits of the fundamental representations of $E_{6(6)}$ and $E_{7(7)}$ to these different cases. To this end we will relate our classification to the analysis of [6]. The degeneracy of the eigenvalues of the central charge matrix was there related to U-duality invariant constraints on the central charge matrix. This analysis heavily depends on the so-called normal frame of the charge matrix which is generically obtained by making a rotation under the automorphism group of the supersymmetry algebra. The automorphism group of the supersymmetry algebra essentially coincides with the maximal compact subgroup of the duality group.

Let us first study the case of $d = 5$. The 27 dimensional representation of $E_{6(6)}$ corresponds to the symplectic traceless anti-symmetric tensor representation of $USp(8)$. It can be brought to a skew diagonal form via an $USp(8)$ transformation. In terms of the eigenvalues e_i of this matrix the cubic invariant takes the form [6]

$$I_3 = (e_1 + e_2)(e_1 + e_3)(e_2 + e_3) \tag{3 - 1}$$

We then see that the three different orbits described in the preceding section correspond to the following three cases [6]

a) $I_3 \neq 0$
b) $I_3 = 0, \qquad \frac{\partial I_3}{\partial e_i} \neq 0$
c) $\frac{\partial I_3}{\partial e_i} = 0$

They correspond to the cases of $1/8, 1/4$ and $1/2$ supersymmetry since in case a) all eigenvalues are different from zero ; in case b) two eigenvalues coincide and in case c) all three eigenvalues coincide. Let us now count the parameters of these 3 different orbits. We first note that the subgroup of $F_{4(4)}$ that preserves the normal form is $O(4,4)$ with maximal compact subgroup $SU(2)^4$. Thus the generic case of $I_3 \neq 0$ involves 3 eigenvalues [26] plus 24 angles corresponding to

$$\frac{USp(8)}{SU(2)^4} \tag{3 - 2}$$

In case b) the little group is the inhomogeneous $ISO(4,4)$. This is a subgroup of the triangular subgroup $O(4,4) \odot (T_{8_v} \oplus T_{8_s} \oplus T_{8_c})$ of $E_{6(6)}$ and again we have $2+24=26$ parameters. In case c) corresponding to the critical orbit the little group of the normal form is $O(5,5)$ with maximal compact subgroup $O(5) \times O(5) = USp(4) \times USp(4)$. Thus the number of parameters is one eigenvalue plus 16 angles of $\frac{USp(8)}{USp(4)\times USp(4)}$. Note that $O(5,5)$ is a subgroup of the triangular little group of the 17 dimensional orbit.

In $d=4$ the 56 dimensional representation of $E_{7(7)}$ can also be described by a complex 8×8 matrix and its complex conjugate. This 8×8 matrix can be brought to a skew diagonal form by an $SU(8)$ rotation. The skew diagonal form has 5 parameters [26], an overall phase and four real positive skew diagonal eigenvalues. The skew diagonal form is invariant under $O(4,4)$ which is the subgroup of $E_{6(2)}$ that preserves the normal form. The generic orbit then can be parametrized by five "normal" coordinates plus 51 angles in $\frac{SU(8)}{USp(2)^4}$. In four dimensions the extra condition for a generic state to be 1/8 BPS with non-vanishing entropy is that $I_4 > 0$ [6]. This is a consequence of the fact that at the horizon all central charge eigenvalues but the BPS mass vanish. Thus one has [4]

$$I_4 = I_{4Horizon} = M^4_{BPS} \tag{3 - 3}$$

This selects the "time-like" orbit $\frac{E_{7(7)}}{E_{6(2)}}$.

Let us now consider the 3 light-like orbits. The generic (55 dimensional) light-like orbit has four different eigenvalues of the 8×8 matrix and still preserves 1/8 supersymmetry. The critical light-like orbit corresponding to 1/4 supersymmetry has eigenvalues that coincide in pairs and zero overall phase [6]. The simple part of the little group in this case is $O(5,5)$ and the number of parameters is given by the two normal parameters plus the 43 angles of $\frac{SU(8)}{USp(4)^2}$. The double critical orbit corresponds to four coinciding eigenvalues in the normal form , zero phase and 1/2 supersymmetry. The little group preserving this form is $E_{6(6)}$ with maximal compact subgroup $USp(8)$. The total number of parameters of the double critical orbit is one normal parameter and 27 angles of $\frac{SU(8)}{USp(8)}$ which agrees with the results of the previous section.

4 Conclusions

Above we have determined the orbits of the exceptional groups corresponding to duality symmetries of toroidally compactified string or M theories to 4 and 5 dimensions. our analysis is classical and for the quantum theory the relevant U–duality groups become discrete [3]. We expect our results can be extended to the discrete cases as well.

An intriguing aspect of our results is the appearance of larger symmetries acting non-linearly on the generalized light-cones defined by vanishing cubic and quartic forms. In $d = 5$ this turns out to be $E_{7(7)}$ that acts via linear fractional transformations on $J_3^{\mathbf{O}}$. There is a a discrete subgroup of $E_{7(7)}$ that acts via discrete linear fractional transformations on $J_3^{\mathbf{O}}$. This makes it tempting to speculate that the generalized conformal group acts as spectrum generating symmetry of the string or M theory toroidally compactified to $d = 5$. In four dimensions we expect the analog of this generalized conformal group to be $E_{8(8)}$. However, $E_{8(8)}$ does not admit 3-grading with respect to any maximal rank subalgebra and hence does not act via linear fractional transformations on 56 of $E_{7(7)}$. However, it has a non-linear action on a 57 dimensional space which splits as $56 + 1$ under $E_{7(7)}$[27]. The physical meaning of this extra singlet is not clear. This problem may be related to the difficulty in extending the results on BPS black holes to 3 dimensions. We should also note that our results can be extended to theories with less supersymmetry such as heterotic strings and to dualities in space-time dimensions greater than five. The results for the higher dimensional theories and further details on the four and five dimensional theories will be given elsewhere [28].

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