

CERN-TH/97-171

Saclay T97/084

hep-th/9707226

Geometrical approach to duality in $N = 1$ supersymmetric theories

Ph. BRAX ^a, C. GROJEAN ^{a,b} and C.A. SAVOY ^{a,b}^a CEA-SACLAY, Service de Physique Théorique
F-91191 Gif-sur-Yvette Cedex, FRANCE^b CERN-TH
CH-1211 Geneva 23, SWITZERLAND

Abstract

We investigate the geometry of the moduli spaces of dual electric and magnetic $N = 1$ supersymmetric field theories. Using the $SU(N_c)$ gauge group as a guideline we show that the electric and magnetic moduli spaces coincide for a suitable choice of the Kähler potential of the magnetic theory. We analyse the Kähler structure of the dual moduli spaces.

CERN-TH/97-171

July 1997

e-mail addresses:

brax, grojean, savoy@spht.saclay.cea.fr

The long-distance behaviour of $N = 1$ supersymmetric field theories is believed to unravel a new kind of duality [1]. Two theories described in the ultraviolet by different gauge contents and gauge groups flow in the infrared towards the same fixed point of the renormalization group. They effectively describe the same long-range physics. The non-trivial feature of this duality is that the gauge groups of the dual pairs are different. The gist of this relationship is better understood in terms of the dual pair of moduli spaces. In general the scalar potential of a supersymmetric gauge theory comprises two terms known as the D - and F - terms. The possible supersymmetric vacua are labelled by the vanishing of both the D - and F - terms. The locus of the scalar fields where the scalar potential vanishes is the moduli space. In general the gauge group is completely broken by these vacua. The moduli space is obtained when the spurious degrees of freedom associated to the Higgs mechanism are eliminated. The structure of the classical moduli space is preserved to all orders in perturbation by the non-renormalization theorem. In fact non-perturbative effects do not modify the moduli space of dual pairs in all the known examples. This implies that the low-energy supersymmetric field theory is a supersymmetric non-linear sigma model whose target space is the (classical) moduli space.

When there is no superpotential, the moduli space reduces to the equations $D^A = 0$, where A labels the generators of the gauge group G . The moduli space is the quotient $(D^A = 0)/G$ after going to the unitary gauge. It is a subset of the (flat) vector space V spanned by the scalar fields of the ultraviolet theory.

The key idea for the construction of the dual magnetic theory stems from a deep result in geometric-invariant theory. The moduli space is homeomorphic to the space of closed orbits in V under the action of the complexified gauge group G^c . This allows the identification of the chiral ring of the electric theory with the ring of G -invariant polynomials [1]. The dual magnetic theory corresponds to the same superconformal field theory and is characterized by the same chiral ring as the electric theory. From a geometric point of view, the equality between the chiral rings of the electric and magnetic theories implies that the electric and magnetic moduli spaces are isomorphic as algebraic varieties.

In the present paper we shall study the isomorphism of the electric and magnetic moduli spaces from the point of view of complex Kählerian geometry. In particular the Kählerian equivalence between the electric and magnetic moduli space entails that the corresponding low-energy non-linear sigma models are identical.

In supersymmetric theories, holomorphy is probably one of the most important principles [2]–[4]. Usually, the scalar potential is a sum of F - and D -terms: the F -terms are the norm of the gradient of a holomorphic function, the superpotential W . For the D -terms, the relation with holomorphy is a bit more subtle and appears when we are interested in supersymmetric vacua [5]. Consider for instance a compact Lie group G with Hermitian generators T^A acting on a representation containing some scalar fields z ; then

$$D^A \equiv \text{tr} \left(\frac{\partial \mathcal{K}}{\partial z} T^A z \right) \quad (1)$$

where $\frac{\partial \mathcal{K}}{\partial z}$ denotes the gradient of the Kähler potential \mathcal{K} with respect to the scalar fields z .

A sufficient and necessary condition for the vanishing of the D -terms was given in [5] in terms of holomorphic gauge invariants: for any holomorphic gauge-invariant polynomial I in the scalar fields, each vev ξ verifying

$$\frac{\partial I}{\partial z^a} \Big|_{z=\xi} = C \frac{\partial \mathcal{K}}{\partial z^a} \Big|_{z=\xi}, \quad (2)$$

where C is a complex constant, is a solution of the set of equations $D^A = 0$ (of course, the whole G -orbit associated to such a vev is also a solution), and to any solution of $D^A = 0$ one can associate a holomorphic gauge invariant satisfying (2). The proof of this result [6] was obtained by studying the closed orbits of the complexified gauge group G^c and the ring of G -invariant polynomials. This ring is finitely generated: one can find an integrity basis *i.e.* a set of G -invariant holomorphic polynomials $\{I^a(z)\}_{a=1\dots d}$ such that every G -invariant polynomial in z can be written as a polynomial in the $I^a(z)$. However the elements of an integrity basis are not always algebraically independent, *i.e.* there exist algebraic relations (called *syzigies*) satisfied by the fundamental invariants¹. Now every G -invariant holomorphic polynomial I is automatically G^c -invariant, so that to each G^c -orbit corresponds a vector in \mathbb{C}^d made out of the values of fundamental invariants along this orbit. Conversely, it can be shown that to each vector in \mathbb{C}^d satisfying the syzigies is associated a unique closed G^c -orbit. In that sense the algebraic subset of \mathbb{C}^d defined by the syzigies is identified with the set of closed G^c -orbits.

¹A trivial example is provided by the $SU(N_c)$ gauge theory with N_c fields in the fundamental and antifundamental representations; the fundamental invariants are the mesons $M = z\tilde{z}$ and the baryons $B = \det z$ and $\tilde{B} = \det \tilde{z}$; classically, they are constrained by $\det M - B\tilde{B} = 0$.

Equations (2) can be seen as a condition for the points of a closed G^c -orbit to extremize the Kähler potential, *i.e.* the Euclidean length in flat space, the constant C^{-1} being a Lagrange multiplier. A result of geometric invariant theory [7] states that the points extremizing the Kähler potential on a G^c -orbit form a unique G -orbit. Identifying the points on a same G -orbit, there is a one-to-one correspondence between the solutions of $D^A = 0$ and the closed G^c -orbits.

We can also interpret (2) in another way: keep \mathcal{K} constant and extremize an invariant I (now C itself appears as a Lagrange multiplier). Here we work on an iso-Kähler surface and we analyse how the value of the invariant changes from one G^c -orbit to another.

Yet, the description of the moduli space in terms of polynomial invariants is not well adapted to the calculation of the Kähler potential induced from a flat Kähler potential on V . In some simple examples [8], [9], the moduli space has been described in terms of polynomial invariants and the corresponding Kähler potential induced from the flat Kähler potential to the flat directions. However, in these few examples, there are no syzigies to constrain the integrity basis, which is then very simple. The syzigies are an obvious obstacle to any attempt to derive the low-energy effective theory in terms of the analytic invariants I^a , as they have to be solved to define a set of independent analytic coordinates. This corroborates the fact that the 't Hooft anomaly matching conditions cannot be verified by using the polynomial invariants. For all these reasons we do not approach the question in terms of these invariants.

To identify and to count the moduli fields among the scalars in a given theory is quite easy. Correspondingly to the D - and F -terms in the potential one obtains mass matrices M_D^2 and M_F^2 for the scalars. If the gauge group G is broken down to H , the scalars along the $\dim(G/H)$ complex directions $\langle D_i^A \rangle z^i$ complete the massive vector multiplets corresponding to the broken generators, so that (for CP -conserving vacua) CP -even scalar components of these supermultiplets are the only eigenstates of M_D^2 with non-zero eigenvalues. Their CP -odd partners are unphysical degrees of freedom associated to the Goldstone zero eigenstates of M_D^2 . This eliminates $2 \dim(G/H)$ scalars that are certainly not moduli.

The gauge invariance of the superpotential entails, at a supersymmetric vacuum, $F_{ij} D_i^A = 0$, which explicitly shows that the massive eigenstates of M_F^2 are orthogonal to the scalars absorbed in the massive vector multiplets. Hence one has to eliminate $\text{rank}(F_{ij})$ complex scalars, leaving $\dim V - 2 \dim(G/H) - 2 \text{rank}(F_{ij})$ massless states. Then one has to remove

all massless states with gauge interactions (non-singlet under the unbroken H subgroup) and superpotential interactions, involving only the massless states. Of course this identification gives no further information on the geometry of the moduli manifold.

In order to identify this manifold, one has to determine the symmetries of the flat potential condition $V = 0$, namely the simultaneous zeros of the D^A 's and F_i 's :

$$\begin{aligned} D^A(\xi, \xi^*) &\equiv \xi^\dagger T^A \xi = 0 ; \\ F_i(\xi) &= 0 . \end{aligned} \tag{3}$$

These symmetries include of course non-compact transformations. For instance, under a dilation plus overall phase transformation, $\delta\xi = \epsilon\xi$, the invariance of the moduli space requires $F_{i,\xi}\xi = 0$. The homogeneity of the superpotential in the fields is a sufficient, not necessary in general, condition for the dilation invariance of the moduli space. The identification of the complete symmetries of the vacuum degeneracy defines the geometry of the moduli space. In particular, one must find the same moduli spaces for dual supersymmetric theories.

We shall carry out our analysis in the explicit context of the $SU(N_c)$ gauge theory with N_f flavours of quarks and antiquarks. The classical moduli space is not modified non-perturbatively when $N_f > N_c$. The construction of the duality between the electric and the magnetic theories has been obtained in the conformal range $\frac{3}{2}N_c < N_f < 3N_c$ in which the electric and magnetic theories are both asymptotically free.

Let us first provide a geometrical description of the so-called electric theory [1]. This theory possesses gauge invariance $SU(N_c)$ with N_f flavours of quarks and antiquarks transforming as the fundamental and antifundamental representations. We denote by Q_α^i and $\tilde{Q}_{\tilde{\alpha}}^{\tilde{i}}$ the corresponding scalar fields. The Kähler potential is

$$\mathcal{K}(Q, Q^\dagger, \tilde{Q}, \tilde{Q}^\dagger) = \text{tr} (Q^\dagger Q + \tilde{Q}^\dagger \tilde{Q}) . \tag{4}$$

The anomaly-free global symmetries of the theory are $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R$, with Q_α^i transforming as $(N_f, 1, 1, (N_f - N_c)/N_f)$ and $\tilde{Q}_{\tilde{\alpha}}^{\tilde{i}}$ as $(1, \bar{N}_f, -1, (N_f - N_c)/N_f)$.

The moduli space parametrizes the variety of degenerate supersymmetric vacuum states, *i.e.* states with zero energy. Here these vacua are solutions of the D -flatness equations:

$$D^A \equiv (Q_\alpha^i)^* T_{\alpha\beta}^A Q_\beta^i - \tilde{Q}_{\tilde{\alpha}}^{\tilde{i}} T_{\tilde{\alpha}\tilde{\beta}}^A (\tilde{Q}_{\tilde{\beta}}^{\tilde{i}})^* = 0 , \tag{5}$$

where A runs over all the gauge indices. The previous equations are equivalent to the single relation:

$$(Q_\alpha^i)^* Q_\beta^i - \tilde{Q}_{\tilde{\alpha}}^i (\tilde{Q}_{\tilde{\beta}}^i)^* = \lambda \delta_{\alpha\beta} , \quad (6)$$

where λ is a real number.

Equation (6) explicitly exhibits an $SU(N_f, N_f) \times U(1)_B$ flavour invariance [5], acting on the $2N_f$ component vectors (Q, \tilde{Q}^*) , including a Cartan generator corresponding to the $U(1)_R$. Moreover there is an obvious invariance under dilation. Finally the symmetry of the moduli space is $G_e \times SU(N_c)_G$, where

$$G_e = U(N_f, N_f) \times D . \quad (7)$$

Notice that holomorphy is not preserved by the action of G_e .

Due to the constraint (6), possibly with the interchange of the role of Q and \tilde{Q} , and by making an $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R \times SU(N_c)$ transformation, any solution of (5) can be put in the form

$$(Q_\alpha^i, \tilde{Q}_{\tilde{\alpha}}^{i*}) = \begin{cases} \left(\sqrt{a_i^2 + \lambda^2} \delta_\alpha^i, a_i \delta_\alpha^i \right) & \text{for } i = 1 \cdots N_c ; \\ (0, 0) & \text{for } i = N_c + 1 \cdots N_f . \end{cases} \quad (8)$$

with a_i real and positive. Actually these solutions belong to two kinds of orbits of $U(N_f, N_f) \times SU(N_c) \times D$, corresponding to the two cases where λ is zero and not.

Consider first the case $\lambda \neq 0$ ²; then using the appropriate element in the subgroup $U(1, 1)^{N_c}$ of $U(N_f, N_f)$, one gets

$$Q_\alpha^i = \begin{cases} \delta_\alpha^i & \text{for } i = 1 \cdots N_c ; \\ 0 & \text{for } i = N_c + 1 \cdots N_f ; \end{cases} \quad \tilde{Q}_{\tilde{\alpha}}^i = 0 . \quad (9)$$

We shall refer to the orbit associated with this point as the time-like orbit or *baryonic orbit*. When $\lambda = 0$, the situation is different since one cannot use non-compact symmetries to cancel some components, but only to fix all the non-vanishing values as follows

$$Q_\alpha^i = \tilde{Q}_{\tilde{\alpha}}^i = \begin{cases} \delta_\alpha^i & \text{for } \alpha = 1 \cdots r \leq N_c ; \\ 0 & \text{elsewhere .} \end{cases} \quad (10)$$

²There are two conjugated baryonic branches corresponding to $\lambda > 0$ and $\lambda < 0$. Each of them forms a single orbit under the symmetry group. In the following we deal with the case $\lambda > 0$. The case $\lambda < 0$ is obtained by exchanging the roles of Q and \tilde{Q}^* .

These orbits will be called *mesonic*. Therefore, the vacua manifold has been identified as a finite set of $SU(N_f, N_f) \times U(1)_B \times SU(N_c) \times D$ orbits. Each of these can be represented by the quotient of the symmetry group (acting transitively on the orbit) by the stabilizer (or little group) of one point. Thus we have to identify the stabilizer of each representative point considered earlier to characterize each orbit.

For the baryonic orbit, the little group associated to the vacuum (9) takes a simple structure of direct product

$$H_e = SU(N_f - N_c, N_f) \times U(1) \times SU(N_c)_D , \quad (11)$$

where $SU(N_c)_D$ is the diagonal combination of the gauge $SU(N_c)_G$ and an $SU(N_c)$ subgroup of $U(N_f, N_f)$. The $U(1)$ is a combination of the $U(1)_B$ and an element of the Cartan subalgebra of $SU(N_f, N_f)$. Here the gauge group is completely broken; then after eliminating the spurious massless scalars associated to the Higgs mechanism, the real dimension of the coset G_e/H_e is $4N_f N_c - 2N_c^2 + 2$.

In order to further characterize the geometry of the baryonic branch of moduli space, we first extract a flat subspace associated to the diagonal $Gl(1, \mathbb{C})$ factor in the coset and then introduce as coordinates the $N_c \times (2N_f - N_c)$ complex matrix z and define its transformation under $U(N_f, N_f)$ as follows. First parametrize the Lie algebra of $U(N_f, N_f)$ in the representation of dimension $2N_f$ by

$$\begin{pmatrix} a & m \\ -\eta m^\dagger & -d \end{pmatrix} , \quad (12)$$

where η is the $(N_f - N_c, N_f)$ signature, a is an $N_c \times N_c$ anti-Hermitian matrix, d is a $(2N_f - N_c) \times (2N_f - N_c)$ matrix verifying $d\eta = -\eta d^\dagger$ and m is an $N_c \times (2N_f - N_c)$ matrix. The action on the matrix z takes the non-linear infinitesimal form

$$\delta z = m + az + zd + z\eta m^\dagger z . \quad (13)$$

A finite transformation takes the homographic form

$$z \rightarrow (Az + B)(Cz + D)^{-1} . \quad (14)$$

The elements of the coset are then parametrized by the exponentials $e^{t(z)}$, where

$$t(z) = \begin{pmatrix} 0 & z \\ -\eta z^\dagger & 0 \end{pmatrix} . \quad (15)$$

The action (13) of $U(N_f, N_f)$ on z yields a non-linear action on the coset

$$g : z \rightarrow z_g, \quad e^{t(z)} \rightarrow e^{t(z_g)}. \quad (16)$$

This action is transitive. There is only one $U(N_f, N_f)$ -invariant Kähler potential up to a Kähler transformation, namely: $\mathcal{K} = \text{tr} \ln(1 + z\eta z^\dagger)$.

Let us now check the correspondence between the massless scalar fields and the moduli fields. As already discussed previously, since there is no superpotential and the whole gauge group is Higgsed around the vacuum (9), all the scalars are moduli with the exception of the $N_c^2 - 1$ complex scalars given by $T_{i\beta}^A Q_\beta^i$, associated to the $SU(N_c)$ massive vector multiplets. All the $4N_f N_c - 2N_c^2 + 2$ remaining scalar fields are massless, and their number coincides with the real dimension of G_e/H_e .

For the *mesonic orbit*, the pattern is more complicated since the little group associated to the vacuum (10) now has a structure of semi-direct product, $H_r \times SU(N_c - r)_G$, with:

$$H_r = SU(N_f - r, N_f - r) \times U(1)^2 \times SU(r)_D \otimes \tilde{H}, \quad (17)$$

where $SU(r)_D$ is the diagonal combination of the gauge and flavour $SU(r)$ subgroups, the two $U(1)$'s are combinations of the $U(1)_B$, an element of the Cartan subalgebra of $SU(N_f, N_f)$ and an element of the Cartan subalgebra of $SU(N_c)_G$. The semi-direct factor \tilde{H} is a flavour nilpotent subgroup with generators transforming as $(2(N_f - r), \bar{r}) \oplus (\overline{2(N_f - r)}, r)$ under $U(N_f - r, N_f - r) \times SU(r)_D$ and the Abelian subalgebra defined by their commutators, which transform as $(1, \text{Adj} \oplus 1)$.

Notice that in this case, at least if $r < N_c - 1$, an $SU(N_c - r)$ gauge subgroup remains unbroken. After eliminating the degrees of freedom moving to the massive vector multiplet, the real dimension of the coset G_e/H_r is $4N_f r - 2r^2$, which is the dimension of the moduli space. On the mesonic orbit, the action of $SU(N_c)$ can be complexified; nevertheless, the action of $U(N_f, N_f) \times SU(N_c)_G$ is already transitive. Let us check this dimension from the massless spectrum around the vacuum (10). As discussed in the previous section, starting from $4N_f N_c$ scalars, one has to exclude $2(2N_c r - r^2)$ scalars, corresponding to the Higgsed part of $SU(N_c)/SU(N_c - r)$, and $4(N_f - r)(N_c - r)$ massless scalars (corresponding to $N_f - r$ flavours of quarks and antiquarks), which are charged under the unbroken gauge group $SU(N_c - r)$. Hence the remaining $4N_f r - 2r^2$, *i.e.* $\dim(G_e/H_r)$, ones are gauge singlets.

The mesonic orbits are stratified by the index r . The stratum corresponding to $r = N_c$ is such that the gauge group is completely broken. In fact the mesonic orbits correspond to the “infinitely boosted” baryonic orbit. Indeed let us choose a point (8) on the baryonic orbit, *i.e.* with $\lambda \neq 0$, and apply r transformations in $U(1,1)$ of rapidities θ_i such that

$$e^{\theta_i} \left(a_i^2 + \sqrt{\tilde{a}_i^2 + \lambda^2} \right) = e^\theta , \quad (18)$$

where θ goes to infinity. Then using the dilation invariance of the baryonic orbit, one can rescale the vacuum solution by a common factor $e^{-\theta}$. This entails that the r chosen directions are taken to unity and the remaining ones to zero. The vacuum solution eventually tends to the mesonic solution (10). Hence the moduli space is the closure of the baryonic orbit, *i.e.* by applying appropriate boosts and a global dilation to the baryonic orbit one can converge to all the strata of the mesonic orbits. In the same way, by applying infinite boosts, we can go from a mesonic orbit with a stratification index r to a more singular one with lower index. Geometrically, these boosts correspond to the shrinking of some circles in the moduli space. From a physical point of view, the stratification index is related to the number of massless singlets of the theory. As the stratification index goes from r to $r - 1$, the corresponding orbits differ by a dimension $4(r - N_f) - 2$.

We now turn to the analysis of the dual theory, or magnetic theory. It possesses a gauge group $SU(N_f - N_c)$ and contains N_f chiral superfields, denoted q_α^i and $\tilde{q}_{\tilde{\alpha}}^{\tilde{j}}$, in the fundamental and antifundamental representations, together with a gauge-singlet matrix field $M^{i\tilde{j}}$. The superpotential of the magnetic theory is [1]

$$W = \text{tr} (q^i M^{i\tilde{j}} \tilde{q}^{\tilde{j}}) , \quad (19)$$

and the Kähler potential is

$$\mathcal{K} = \text{tr} (q^\dagger q) + \text{tr} (\tilde{q}^\dagger \tilde{q}) + K(M^\dagger M) , \quad (20)$$

corresponding to the assumption of a flat metric for the quarks. The Kähler manifold for the meson fields M should be deduced from the duality with the electric theory, as discussed later.

The moduli space of the magnetic theory is defined as the set of possible

vacua; as such, it is identified with the solutions of the equations:

$$\begin{aligned}
D^A &\equiv (q_\alpha^i)^* T_{\alpha\beta}^A q_\beta^i - \tilde{q}_{\tilde{\alpha}}^{\tilde{j}} T_{\alpha\beta}^A (\tilde{q}_{\tilde{\beta}}^{\tilde{j}})^* = 0 ; \\
F_{q_\alpha^i} &\equiv M^{i\tilde{j}} \tilde{q}_{\tilde{\alpha}}^{\tilde{j}} = 0 ; \\
F_{\tilde{q}_{\tilde{\alpha}}^{\tilde{j}}} &\equiv q_\alpha^i M^{i\tilde{j}} = 0 ; \\
F_{M^{i\tilde{j}}} &\equiv q_\alpha^i \tilde{q}_{\tilde{\alpha}}^{\tilde{j}} = 0 .
\end{aligned} \tag{21}$$

First of all, as one of the requirements of duality, the magnetic theory possesses the same anomaly-free global symmetries as the electric one $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R$, which transform the dual quark superfields as $(\tilde{N}_f, 1, N_c/(N_f - N_c), N_c/N_f)$, the antiquark ones as $(1, N_f, -N_c/(N_f - N_c), N_c/N_f)$ and the gauge-singlet superfields M as $(N_f, \tilde{N}_f, 0, 2(N_f - N_c)/N_f)$. The equations of the moduli space are invariant under those symmetries; in particular the invariance under $U(N_f) \times U(N_f)$ allows using the general form of solutions (8), where N_c is replaced by $N_f - N_c$. Then, the last equation in (21) implies that $a_i = 0$, *i.e.* the moduli space of the magnetic theory possesses a baryonic solution in the non-singular case, $\lambda \neq 0$ ³. Using the dilation invariance of the equations one can get the canonical solution on this orbit:

$$q_{0,\alpha}^i = \begin{cases} \delta_\alpha^i & \text{for } i = 1 \cdots N_c ; \\ 0 & \text{for } i = N_f - N_c \cdots N_f ; \end{cases} \quad \tilde{q}_{0,\tilde{\alpha}}^{\tilde{j}} = 0 . \tag{22}$$

We can now use this information to solve for M . It is easy to see that

$$M = \begin{pmatrix} 0 & M_0 \end{pmatrix} , \tag{23}$$

where M_0 is an $N_f \times N_c$ complex matrix. We have identified the regular part of the magnetic moduli space as a set of $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R \times SU(N_f - N_c)_G \times D$ orbits. Indeed the magnetic vacua are obtained by the transitive action of this symmetry group on the canonical solutions. The action of $U \in U(N_f)$ on the solutions is given by $q \rightarrow Uq$ and $M \rightarrow MU^\dagger$. The stabilizer of the canonical value of q is

$$H_m = SU(N_c) \times SU(N_f - N_c)_D \times SU(N_f) \times U(1)^2 , \tag{24}$$

where $SU(N_c)$ is a subgroup of $SU(N_f)$; $SU(N_f - N_c)_D$ is the diagonal combination of the gauge-group and the flavour transformations; and the

³As for the electric case, the baryonic branch consists of two baryonic orbits. We explicitly describe one of them.

two $U(1)$'s are combinations of the $U(1)_R$, the $U(1)_B$ and an element of the Cartan subalgebra of $SU(N_f)$. It is easy to see that the action of an element $h \in H_m$ of the stabilizer on the canonical field $M = (0 M_0)$ gives an element $(0 M'_0)$ in the canonical form. The magnetic moduli space has a structure of a fibred space. After the extraction of the gauge degrees of freedom of the completely broken gauge group, the regular part has a base manifold

$$G_m/H_m = \frac{SU(N_f) \times U(1)_R \times U(1)_B \times D}{SU(N_c) \times SU(N_f - N_c)_D \times U(1)^2}, \quad (25)$$

whose real dimension is $2N_f N_c - 2N_c^2 + 2$.

The fibres are subspaces of $\mathbb{C}^{N_f \times N_c}$. More precisely the moduli space is parametrized by pairs (U, M) , where U is a representative of the G_m/H_m and $M = (0 M_0)U^\dagger$. The corresponding vacuum is given by $q = Uq_0$ and M . The action of the stabilizer H_m leaves the base G_m/H_m invariant but acts as an automorphism on the fibres.

Let us describe the base manifold. As a homogeneous space it is characterized by its tangent space at the origin. The tangent space is parametrized by $N_f \times N_f$ matrices

$$t(u) = \begin{pmatrix} 0 & u \\ -u^\dagger & 0 \end{pmatrix}, \quad (26)$$

where u is a $(N_f - N_c) \times N_c$ complex matrix such that $U = e^{t(u)}$ are the elements of the coset G_m/H_m . This base manifold is a complex manifold. As the fibres are complex, this proves that the regular orbit of the magnetic moduli space is a complex manifold.

Finally, the real dimension of this fibred space is $4N_c N_f - 2N_c^2 + 2$, in accordance with the expected dimension of the electric moduli space and also with the number of massless states deduced from the scalar mass matrix. Indeed, the magnetic theory has $6N_f^2 - 4N_f N_c$ scalars. Since the gauge group $SU(N_f - N_c)$ is completely broken, one has to exclude $2(N_f - N_c)^2 - 2$ states in the vector multiplets. Finally, the rank of the Hessian F_{ij} matrix is $2N_f(N_f - N_c)$, independently of the M_0 matrix elements. This leave precisely $4N_f N_c - 2N_c^2 + 2$ massless states.

The singularities of the magnetic moduli space are obtained when the values of q and \tilde{q} are $q = \tilde{q} = 0$. Hence any matrix M is a solution of (21). The set of matrices $\{M\}$ is stratified by their rank r . In this letter, we concentrate on the baryonic orbits, and we shall discuss the singularities corresponding to mesonic orbits in a forthcoming paper [10].

It is appealing to try to understand the relation between the electric and magnetic moduli spaces. We have seen that the magnetic moduli space is parametrized by the coordinates u of the base manifold and the coordinates along the fibres M_0 . The following $N_c \times (2N_f - N_c)$ complex matrix can be constructed:

$$z_m = (u^T \ M_0^T) . \quad (27)$$

Consider the homographic action of the group $U(N_f, N_f)$ on z_m , analogous to (14). The magnetic moduli space is invariant under this non-linear action of $U(N_f, N_f)$. Indeed all the equations in (21) are invariant since $U(N_f, N_f)$ is an automorphism of the parameter space z_m of their solutions. Consider the image of the origin $z_m = (0, 0)$. As the action on z_m is the same as (14) on the electric baryonic orbit we know that the image of the origin is the whole baryonic orbit. Therefore the baryonic branches of the electric and magnetic moduli spaces are isomorphic as non-compact complex manifolds if and only if the matrices (27) are restricted to the $U(N_f, N_f)$ orbits. The dualising map reads simply $z \leftrightarrow z_m$. This identification is consistent with the fact that the subspace of the u submatrices is given in both the electric and magnetic theories by the coset (25). The corresponding restriction of the matrices M_0 must result from the Kähler manifold for the meson fields M associated to the Kähler potential (20). The Kähler manifold satisfying these requirements is the homogeneous space $U(N_f, N_f)/U(N_f) \times U(N_f)$. The mesonic branches of the moduli space close the baryonic one [10].

We therefore conclude that the differential geometric description of the electric and magnetic moduli spaces shows an isomorphism provided that the mesonic fields M manifold is suitably defined. This isomorphism can be extended to the induced Kählerian structure that we now turn to discuss.

Let us adopt the prescription given in [8]. The induced Kähler potential is deduced from the induced metric that one can calculate by restricting the Kähler potential to the moduli space. Replacing the parametrization of the baryonic moduli space

$$\begin{pmatrix} Q \\ \tilde{Q}^* \end{pmatrix} \Big|_{D^A=0} = e^{t(z)} \begin{pmatrix} \mathbf{1}_{N_c} \\ 0 \end{pmatrix} , \quad (28)$$

into (4), one obtains the induced Kähler potential

$$\mathcal{K}_{eff} = \text{tr} (Q^\dagger Q + \tilde{Q}^\dagger \tilde{Q}) \Big|_{D^A=0} = \text{tr} (e^{t(z)^\dagger} e^{t(z)} P_{N_c}) , \quad (29)$$

where P_{N_c} projects onto the first N_c components. This expression gives an explicit description of the low-energy theory.

Let us now deal with the dual theory, where the parametrization of the baryonic solutions of (21) is

$$\begin{aligned} q &= e^{t(u)} q_0 ; \\ \tilde{q} &= 0 ; \\ M &= M_0 e^{t(u)\dagger} . \end{aligned} \tag{30}$$

The low-energy physics is entirely defined by the restriction of the Kähler potential (20) to these flat directions. This yields

$$\mathcal{K}_{eff} = \text{tr} (e^{t(u)\dagger} e^{t(u)} P_{N_f - N_c}) + K(e^{t(u)} M_0^\dagger M_0 e^{t(u)\dagger}) , \tag{31}$$

where $P_{N_f - N_c}$ projects onto the first $N_f - N_c$ components. It is easily checked from $z_m = (u^T M_0^T)$ that (29) and (31), near the origin, exactly coincide with a Euclidean metrics. It should be noticed that in spite of the Minkowskian signature of the moduli manifold, the metrics defined by (29) and (31) have Euclidean signature.

We have only obtained these results by looking at the scalar sector of the theories. This approach to the low-energy effective theory is puzzling because the parametrization of the solutions in terms of the moduli is not holomorphic. The flatness condition $D^A = 0$ are not holomorphic (*cf* the presence of the Kähler function in (2)). The $U(N_f, N_f)$ action is not analytic either. Therefore we cannot directly extend our formalism onto a superfield one, which would manifestly preserve supersymmetry. This unsatisfactory aspect of our approach is now under investigation.

The extension of this analysis to $SO(N_C)$ gauge supersymmetric theories [11] with N_f flavours of quarks transforming as fundamental representations is straightforward [10]. Basically, the non-compact symmetries $U(N_f, N_f)$ are replaced by their real subgroup $Sp(2N_f, \mathbb{R})$ acting on $(\Re Q_\alpha^i \Im Q_\alpha^i)$.

In conclusion, we have shown the equivalence between the low-energy description of the electric and magnetic $SU(N_c)$ supersymmetric gauge theories from a more geometrical point of view. It would be of great interest to analyse the actions in terms of conformal field theories.

Acknowledgements

We thank J.P. Derendinger, E. Dudas and R. Stora for helpful and stimulating discussions.

References

- [1] N. Seiberg, *Phys. Rev.* **D49** (1994) 6857, *Nucl. Phys.* **B435** (1995) 129.
- [2] V. Novikov, M. Shifman, A. Vainshtein and V. Zakharov, *Nucl. Phys.* **B229** (1983) 381 and *Phys. Lett.* **B166** (1986) 329.
- [3] N. Seiberg, *Phys. Lett.* **B318** (1993) 469 and *hep-th/9408013*, Proc. of PASCOS 94.
- [4] N. Seiberg and E. Witten, *Nucl. Phys.* **B426** (1994) 19.
- [5] F. Buccella, J.P. Derendinger, S. Ferrara and C. Savoy, *Phys. Lett.* **115B** (1982) 375.
- [6] C. Procesi and G.W. Schwarz, *Phys. Lett.* **B161** (1985) 117; R. Gatto and G. Sartori, *Phys. Lett.* **157B** (1985) 389 and *Commun. Math. Phys.* **109** (1987) 327.
- [7] G. Kempf and L. Ness, *Lecture Notes in Mathematics* , vol. 732 (Springer, Heidelberg, 1978).
- [8] I. Affleck, M. Dine and N. Seiberg, *Nucl. Phys.* **B241** (1984) 493 and **B256** (1985) 557.
- [9] E. Poppitz and L. Randall, *Phys. Lett.* **B336** (1994) 402.
- [10] Ph. Brax, C. Grojean and C.A. Savoy, *in preparation*.
- [11] K. Intriligator and N. Seiberg, *Nucl. Phys.* **B444** (1995) 125.