The Schwinger Model in Light-Cone Gauge

A. Bassetto (*)

CERN, Theory Division, CH-1211 Geneva 23, Switzerland

INFN, Sezione di Padova, Padua, Italy

G. Nardelli

Dipartimento di Fisica, Università di Trento, 38050 Povo (Trento), Italy

INFN, Gruppo Collegato di Trento, Italy

E. Vianello

Dipartimento di Fisica "G.Galilei", Via Marzolo 8 – 35131 Padua, Italy

Abstract

The Schwinger model, defined in the space interval $-L \leq x \leq L$, with (anti)periodic boundary conditions, is canonically quantized in the light-cone gauge $A_- = 0$ by means of equal-time (anti)commutation relations. The transformation diagonalizing the complete Hamiltonian is explicitly constructed, thereby giving spectrum, chiral anomaly and condensate. The structures of Hilbert spaces related both to free and to interacting Hamiltonians are completely exhibited. Besides the usual massive field, two chiral massless fields are present, which can be consistently expunged from the physical space by means of a subsidiary condition of a Gupta–Bleuler type. The chiral condensate does provide the correct non-vanishing value in the decompactification limit $L \to \infty$.

PACS: 11.10 Kk, 11.15-q. *Keywords*: Schwinger model, light-cone gauge.

(*) On leave of absence from Dipartimento di Fisica "G.Galilei", Via Marzolo 8 – 35131 Padova, Italy.

I. INTRODUCTION

The Schwinger model [1] is a celebrated theoretical laboratory, possessing many non-trivial properties [2]. The literature is so copious that we refrain from providing further references.

The reduction to two dimensions entails tremendous simplifications in quantum electrodynamics and, as a consequence, the Schwinger model turns out to be the only local gauge theory with non-trivial coupling, where exact operatorial solutions can be found. In addition, it is believed that some interesting features that this model exhibits persist in more realistic field theories.

As an example, all the problems related to the spontaneous breakdown of the chiral symmetry, and the corresponding chiral anomaly, are very similar to the ones in four-dimensional QCD (QCD_4), with the obvious advantage that, in this simplified two-dimensional world, these phenomena appear in a much more transparent way. Moreover, the infinite degeneracy of the vacuum state due to chiral symmetry is analogous to that found in QCD_4 so that one is naturally led to introduce θ-vacua.

Nevertheless, it should be emphasized that, although similar, the intimate reasons of the appearance of these phenomena are deeply different in the two theories. It is common belief that the infrared behaviour of QCD_4 is at the root of the confining force between quarks and antiquarks. In the Schwinger model, such a confining force is a "free bonus" of the theory: electrodynamics naturally confines in $1+1$ dimensions, due to the linear rise of the Coulomb potential. Consequently, the Schwinger model is a formidable laboratory to study the effects of confinement, but certainly not the causes.

The aim of this paper is to develop a rigorous operatorial treatment of the Schwinger model, when considered in the space interval $-L \leq x \leq L$, with periodic boundary conditions for the vector field and antiperiodic ones for fermions. The compactification of the spatial dimension will naturally provide an infrared regulator of the theory, without spoiling gauge invariance.

We choose to quantize the model in the “strong” light-cone gauge (LCG) $A_- = 0$, by imposing an *equal-time* canonical algebra [3], [4]. To our knowledge, such a choice has never been considered so far. There are several reasons that make the LCG particularly interesting. First of all, contrary to a common prejudice, this gauge is not manifestly unitary when the theory is quantized on space-like surfaces. In addition, the direction in which the theory is compactified has to be carefully chosen. We recall indeed that axial gauges cannot be defined on compact manifolds (i.e. compactifying both space and time) without introducing singularities in the vector potentials, due to Singer’s theorem; nevertheless, partial compactifications are possible, provided they occur in a direction different from the one specifying the gauge choice: in the present case, to compactify the range of x^- would be inconsistent. On the other hand, equal-*time* quantization suggests to regularize the theory with respect to infrared singularities by compactifying the space direction.

In the Schwinger model, a manifestly unitary formulation can instead be obtained in the Coulomb gauge. Thus, as far as the content of physical degrees of freedom is concerned, LCG can be seen as an intermediate choice between Coulomb and Lorentz gauges, but with the great advange that the Faddeev–Popov sector decouples in the non-Abelian generalization of the model, making the transition to the non-Abelian case smoother.

When the theory in LCG is quantized by equal-time commutation relations, Gauss’ law does not hold strongly; rather, the Gauss operator obeys a free-field equation and entails the presence in the Fock space of unphysical degrees of freedom. The vanishing of Gauss’ operator has to be imposed weakly, by a mechanism that is reminiscent of the Gupta–Bleuler quantization scheme for electrodynamics in Feynman gauge, but with the great advantage that it can be naturally extended to the non-Abelian case.

In the present case, thanks to compactification of the spatial dimension, we have an even richer structure of the Hilbert space, as topological excitations (zero modes) come into the game.

The plan of the paper is as follows.

In Sect. 2 we review the canonical quantization in LCG; we define the space of states for the free theory as the direct product of a Fock space $\mathcal{F}_A$ containing the frequency modes, obtained through repeated action of creation operators on a Fock vacuum, times a "quantum-mechanical" $\mathcal{L}^2$ space of the zero-mode sector. Then we introduce the interaction, namely we consider the Schwinger model in LCG and provide the relevant equal-time commutation relations.

Section 3 is devoted to quantization at $t = 0$; products of quantum operators are defined and regularized by means of gauge-invariant point splitting techniques. In particular, the expressions for the currents and for the kinetic terms occurring in the Hamiltonian are carefully discussed.

Section 4 deals with the diagonalization of the complete Hamiltonian. The unitary transformation, which diagonalizes the Hamiltonian, relating free and interacting fields, is explicitly exhibited. This is made possible by the partially compact nature of the manifold (cylinder). The free-fields sector contains, besides the "physical" massive scalar field with mass $e/\sqrt{\pi}$, two additional "unphysical" chiral massless fields, with the same chirality, but opposite signs in the commutators. Consequently, the Fock space will have an indefinite metric.

In Sect. 5 we discuss the temporal evolution of the fields. All the operatorial solutions are explicitly exhibited and we check that they satisfy the correct Euler–Lagrange equations of motion. In the same section, it is also shown that Gauss' law is implemented by defining a physical Hilbert space in which the total charge vanishes and by selecting a suitable zero-norm combination of ghost-like and "longitudinal" degrees of freedom, in the same way as the Gupta–Bleuler formalism in the Feynman gauge.

The structure of the "physical" Hilbert space is discussed in Sect. 6. We show that the same unitary operator diagonalizing the Hamiltonian performs the mapping between free and interacting Fock spaces. A set of degenerate vacua is obtained, all of them being related by a residual gauge transformation. Then, the requirement of gauge invariance naturally leads to the definition of θ-vacua. Moreover a positive definite inner product is derived, and it is shown that the specific ghost-particle combinations forced by the (weakly) vanishing of Gauss' operator always have vanishing norm. The physical Hilbert space is eventually defined as a quotient space. Finally, chiral anomaly and chiral condensate are discussed. As a final check it is shown that the chiral condensate reproduces the correct non-vanishing result of the continuum in the decompactification limit $L \to \infty$.

II. THE SCHWINGER MODEL

A. The free-fermion model

We start by recalling the treatment of free massless fermions in the space interval $[-L, L]$. We assume antiperiodic boundary conditions $\psi(t,-L) = -\psi(t,L)$. The classical Lagrangian is given by

$$\mathcal{L} = i\bar{\psi}(x)\partial_\mu\gamma^\mu\psi(x).$$

Here ψ is a two-component field:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} ,$$

γ^0 and γ^1 are 2×2 matrices such that the algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

is fulfilled.

The action is invariant under global phase and chiral trasformations leading to the well-known classical expression for the conserved vector and axial vector currents.

When performing the canonical quantization the classical variable $\psi(x)_\alpha$ and its conjugate momentum

$$\pi_\alpha(x) \equiv \frac{\partial\mathcal{L}}{\partial(\,\partial_0\psi_\alpha(x)\,)} = i\psi^+_\alpha(x)$$

become operators, obeying the canonical anticommutation relations

$$\{\psi_\alpha(t,\mathrm{x}),\psi^+_\beta(t,y)\} = \delta_{\alpha\beta} e^{\frac{i\pi}{2L}(\mathrm{x}-y)}\delta(\mathrm{x}-y) \ , \qquad \alpha,\beta = 1,2 \ ,$$
$$\{\psi_\alpha(t,\mathrm{x}),\psi_\beta(t,y)\} = \{\psi^+_\alpha(t,\mathrm{x}),\psi^+_\beta(t,y)\} = 0, \tag{1}$$

where $\delta(\mathrm{x}-y)$ denotes the periodic δ-distribution.

Using the explicit representation for the gamma matrices $\gamma^\mu \ , \ \mu = 0,1$:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ ,$$

we get the equations of motion

$$\partial_-\psi_1(x) = 0 \ , \qquad \partial_+\psi_2(x) = 0 \ ,$$

where

$$\mathrm{x}^- = \frac{t-\mathrm{x}}{\sqrt{2}} \quad , \quad \mathrm{x}^+ = \frac{t+\mathrm{x}}{\sqrt{2}} \ ,$$
$$\partial_- = \frac{\partial}{\partial \mathrm{x}^-} = \frac{\partial_0 - \partial_1}{\sqrt{2}} \quad , \quad \partial_+ = \frac{\partial}{\partial \mathrm{x}^+} = \frac{\partial_0 + \partial_1}{\sqrt{2}} \ .$$

The most general solution fulfilling the condition

$$\psi(t,\mathrm{x}) = -\psi(t,\mathrm{x}+2L) \tag{2}$$

is

$$\psi_1(t,\mathrm{x}) = \frac{1}{\sqrt{2L}} \sum_{n=0}^{+\infty} \left(b_n\, e^{-i(n+1/2)\frac{\pi}{L}(t+\mathrm{x})} + d^+_n\, e^{i(n+1/2)\frac{\pi}{L}(t+\mathrm{x})}\right) \ ,$$
$$\psi_2(t,\mathrm{x}) = \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{-1} \left(b_n\, e^{i(n+1/2)\frac{\pi}{L}(t-\mathrm{x})} + d^+_n\, e^{-i(n+1/2)\frac{\pi}{L}(t-\mathrm{x})}\right) \ , \tag{3}$$

("left" and "right" movers).

The canonical anticommutation relations induce the algebra

$$\{b_n, b^+_m\} = \delta_{m,n} \ , \qquad \{d_n \ , d^+_m\} = \delta_{m,n} \ , \qquad \{b_n \ , d^+_m\} = 0 \ ,$$
$$\{b_n \ , b_m\} = \{d_n \ , d_m\} = \{b_n \ , d_m\} = 0 \ . \tag{4}$$

The classical energy–momentum tensor

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_\alpha)} \partial^\nu \psi_\alpha - \mathcal{L} g^{\mu\nu} \tag{5}$$

leads to the corresponding conserved quantities

$$P^0 \equiv H = \int_{-L}^{L} dx\, \Theta^{00}(x) \qquad ; \qquad P^1 = \int_{-L}^{L} dx\, \Theta^{01}(x). \tag{6}$$

We get a quantum description by defining the products as normal-ordered and introducing creation and annihilation operators:

$$H = -\int_{-L}^{L} dx \ : i\bar{\psi}(x)\gamma^1 \partial_1 \psi(x) : = \sum_{n=-\infty}^{+\infty} k_0[n+1/2]\,(b_n^+ \ b_n \ + d_n^+ \ d_n), \tag{7}$$

$$P_1 = \int_{-L}^{L} dx \ : i\psi^+(x)\partial_1 \psi(x) := \sum_{n=-\infty}^{+\infty} k_1[n+1/2]\,(b_n^+ \ b_n \ + d_n^+ \ d_n) \ , \tag{8}$$

where

$$k_0[n+1/2] = \left| n + \frac{1}{2} \right| \frac{\pi}{L} \quad , \quad k_1[n+1/2] = \left(n + \frac{1}{2} \right) \frac{\pi}{L} \ . \tag{9}$$

Electric charge Q and chiral charge Q_5 are given by the following operators:

$$Q = e \int_{-L}^{L} dx : \psi^+(x)\psi(x) : ,$$

$$Q_5 = -e \int_{-L}^{L} dx : \psi^+(x)\gamma^5 \psi(x) : \ .$$

With

$$\gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

Q and Q_5 become

$$Q = e \sum_{n=-\infty}^{+\infty} (b_n^+ \ b_n - d_n^+ \ d_n \ ) \ , \tag{10}$$

$$Q_5 = e \sum_{n=0}^{+\infty} (b_n^+ \ b_n - d_n^+ \ d_n) - e \sum_{n=-\infty}^{-1} (b_n^+ \ b_n - d_n^+ \ d_n) \ . \tag{11}$$

We are now ready to introduce the bosonization. From the fermionic currents

$$j^- = \frac{j^0 - j^1}{\sqrt{2}} = e\sqrt{2} : \psi_1^+ \psi_1 : ,$$
$$j^+ = \frac{j^0 + j^1}{\sqrt{2}} = e\sqrt{2} : \psi_2^+ \psi_2 : ,$$

we get

$$j^- = \frac{e}{\sqrt{2}L} \sum_{p=1}^{\infty} \Big(C_p e^{-i\frac{\pi}{L}p(t+x)} + C_p^+ e^{i\frac{\pi}{L}p(t+x)} \Big) + \frac{e}{\sqrt{2}L} \sum_{n=0}^{\infty} \Big(b_n^+ b_n - d_n^+ d_n \Big) , \tag{12}$$

$$j^+ = \frac{e}{\sqrt{2}L} \sum_{p=-\infty}^{-1} \Big(C_p e^{+i\frac{\pi}{L}p(t-x)} + C_p^+ e^{-i\frac{\pi}{L}p(t-x)} \Big) + \frac{e}{\sqrt{2}L} \sum_{n=-\infty}^{-1} \Big(b_n^+ b_n - d_n^+ d_n \Big) , \tag{13}$$

where

$$\begin{aligned} C_n &= \sum_{m=0}^{n-1} d_m \; b_{n-m-1} + \sum_{m=0}^{\infty} (b_m^+ \; b_{m+n} - d_m^+ \; d_{m+n}) \quad , \quad \text{for} \;\; n > 0 \; , \\ C_n &= \sum_{m=n}^{-1} d_m \; b_{n-m-1} + \sum_{m=-\infty}^{-1} (b_m^+ \; b_{m+n} - d_m^+ \; d_{m+n}) \quad , \quad \text{for} \; n < 0 \; . \end{aligned} \tag{14}$$

One can easily verify that the fusion operators C_n obey the following *bosonic* commutation rules

$$\begin{aligned} &[C_n \; , C_m^+ \;] = |n| \delta_{m,n} \; , \\ &[C_n \; , C_m] = [C_n^+ \; , C_m^+] = 0 \; . \end{aligned} \tag{15}$$

We notice that the operators Q , Q_5 , C_n, act on the fermionic Fock space $\mathcal{F}$ obtained starting from a vacuum state $|0\rangle$ annihilated by the operators b_n and d_n, and that

$$[C_n, Q] = [C_n, Q_5] = [Q, Q_5] = 0. \tag{16}$$

Let us now consider a peculiar set of states $|M, N\rangle$ [5] . The state $|M, N\rangle$ represents M particles $(M < 0)$ or antiparticles $(M > 0)$ of the *left* type in the lowest M energetic levels and N particles $(N < 0)$ or antiparticles $(N > 0)$ of the *right* type in the lowest N levels; namely

$$|M,N\rangle = b^+_{M-1}\cdots b^+_0 b^+_{-N}\cdots b^+_{-1}|0\rangle \quad , \quad \text{for} \quad M<0\,,\ N<0\,,$$
$$|M,N\rangle = d^+_{M-1}\cdots d^+_0 d^+_{-N}\cdots d^+_{-1}|0\rangle \quad , \quad \text{for} \quad M>0\,,\ N>0\,, \tag{17}$$
$$|M,N\rangle = d^+_{M-1}\cdots d^+_0 b^+_{-N}\cdots b^+_{-1}|0\rangle \quad , \quad \text{for} \quad M>0\,,\ N<0\,,$$
$$|M,N\rangle = b^+_{M-1}\cdots b^+_0 d^+_{-N}\cdots d^+_{-1}|0\rangle \quad , \quad \text{for} \quad M<0\,,\ N>0\,.$$

Since

$$Q|M,N\rangle = -e(M+N)|M,N\rangle, \tag{18}$$
$$Q_5|M,N\rangle = -e(M-N)|M,N\rangle, \tag{19}$$

any given state $|M,N\rangle$ can be identified by means of the eigenvalues of Q and Q_5 . It can be verified that for any state $|M,N\rangle$

$$C_n|M,N\rangle = 0\ .$$

Moreover, as the operators C^+_n commute with the charges, the eigenvalues of Q and Q_5 are not modified by applying C^+_n on the states $|M,N\rangle$.

In ref. [6] it has been shown that the fermionic Fock space $\mathcal{F}$ can be decomposed as an infinite direct sum of irreducible Fock representations of the algebra (15), which coincide with the eigenspaces of Q and Q_5.

The Hamiltonian and the momentum (eqs. (7) and (8)) can be expressed in terms of the charges and of the fusion operators through:

$$H = \frac{\pi}{4Le^2}\left(Q^2+Q_5^2\right) + \frac{\pi}{L}\sum_{n\neq 0} C^+_n C_n\ , \tag{20}$$
$$P_1 = \frac{\pi}{2Le^2}Q\ Q_5 + \frac{\pi}{L}\sum_{n\neq 0} \epsilon(n)\ C^+_n C_n\ . \tag{21}$$

In order to express the fermionic operators in terms of the fusion operators, it is useful to introduce the quantities

$$\varphi_1^{(+)}(t,x) = -\sum_{n=1}^{+\infty}\frac{1}{n}C_n e^{-in\frac{\pi}{L}(x+t)}\ ,$$
$$\varphi_1^{(-)}(t,x) = \sum_{n=1}^{+\infty}\frac{1}{n}C^+_n e^{in\frac{\pi}{L}(x+t)}\ , \tag{22}$$

$$\varphi_2^{(+)}(t,x) = \sum_{n=-\infty}^{-1} \frac{1}{n} C_n e^{in\frac{\pi}{L}(t-x)} ,$$
$$\varphi_2^{(-)}(t,x) = - \sum_{n=-\infty}^{-1} \frac{1}{n} C_n^+ e^{-in\frac{\pi}{L}(t-x)} .$$

Then we can define the operators

$$\sigma_\alpha(t,x) = \sqrt{2L} e^{\varphi_\alpha^{(-)}(t,x)} \psi_\alpha(t,x) e^{\varphi_\alpha^{(+)}(t,x)}, \tag{23}$$

which carry the fermionic charge. One can easily derive the relations [5] :

$$\sigma_\alpha^+(t,x)\sigma_\alpha(t,x) = \sigma_\alpha(t,x)\sigma_\alpha^+(t,x) = 1 , \tag{24}$$

$$\{\sigma_1(x), \sigma_2(y)\} = \{\sigma_1(x), \sigma_2^+(y)\} = 0 , \tag{25}$$

$$[Q, \sigma_\alpha(t,x)] = -e\sigma_\alpha(t,x) \quad , \quad [Q_5, \sigma_\alpha(t,x)] = (-1)^\alpha e \sigma_\alpha(t,x) , \tag{26}$$

$$[C_n, \sigma_\alpha(t,x)] = [C_n^+, \sigma_\alpha(t,x)] = 0 . \tag{27}$$

The quantities $\sigma_\alpha \equiv \sigma_\alpha(0,0)$ are usually called *spurions*.

Using eqs. (20), (21), (26) and (27), it can be shown that

$$\sigma_\alpha(t,x) = e^{\Lambda_\alpha(t,x)} \sigma_\alpha e^{\Lambda_\alpha(t,x)} ,$$

where

$$\Lambda_\alpha(t,x) \equiv -\frac{i\pi}{4Le}[Q - (-1)^\alpha Q_5][t - (-1)^\alpha x] .$$

The following relation can be verified:

$$|M,N\rangle = \sigma_1^M \sigma_2^N |0\rangle , \tag{28}$$

where $\sigma_\alpha^{-1} = \sigma_\alpha^+$. It is useful to rescale the fusion operators

$$C_n = i\sqrt{|n|}\gamma_n \quad , \quad C_n^+ = -i\sqrt{|n|}\gamma_n^+ , \tag{29}$$

so that

$$[\gamma_n, \gamma_m^+] = \delta_{m,n} \quad , \quad [\gamma_n, \gamma_m] = [\gamma_n^+, \gamma_m^+] = 0 \; . \tag{30}$$

Defining the bosonic field

$$\begin{aligned} \varphi(x) &= \frac{i}{2\sqrt{\pi}} \left(\varphi_1^{(+)} + \varphi_1^{(-)} - \varphi_2^{(+)} - \varphi_2^{(-)} \right) = \\ &= \frac{1}{\sqrt{2L}} \sum_{n \neq 0} \frac{1}{\sqrt{2k_0}} \left(\gamma_n e^{-ik\cdot x} + \gamma_n^+ e^{ik\cdot x} \right) \; , \end{aligned} \tag{31}$$

the fermionic currents can be rewritten as

$$j^- = \frac{Q + Q_5}{2\sqrt{2}L} - \frac{e}{\sqrt{\pi}} \partial_+ \varphi \; , \tag{32}$$

$$j^+ = \frac{Q - Q_5}{2\sqrt{2}L} - \frac{e}{\sqrt{\pi}} \partial_- \varphi \; , \tag{33}$$

where φ is a ‘quasi-scalar’ field, as it has no zero mode.

B. The free Maxwell field at $t = 0$

As we are considering the theory on the cylinder, we can assume that the algebra of interacting fields at a given time ($t = 0$) is isomorphic to the free-fields algebra. [1]

The free Maxwell field is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{34}$$

In order to quantize it we have to choose a gauge; we choose the light-cone gauge by adding to the classical Lagrangian the following “gauge-fixing” term

$$\mathcal{L}_{gf} = \lambda^{(0)} n_\mu A^\mu \; ,$$

where $n_\mu = \frac{1}{\sqrt{2}}(1, 1)$ and $\lambda^{(0)}$ is a Lagrange multiplier enforcing the gauge condition $n_\mu A^\mu = 0$.

[1] We recall that this assumption would not be allowed in the continuum, owing to Haag’s theorem.

We define

$$A = A_+ = \frac{A_0 + A_1}{\sqrt{2}} \quad , \quad F = F_{01} \ .$$

The Euler–Lagrange equations are :

$$\partial_\mu F^{\mu\nu} + n^\nu \lambda^{(0)} = 0, \tag{35}$$

which, in turn, imply

$$\partial_- \lambda^{(0)} = 0. \tag{36}$$

Following the standard Dirac procedure, we can derive the following independent *equal-time* commutation relation:

$$[A(t,x), F(t,y)] = i\sqrt{2}\delta(x-y). \tag{37}$$

At variance with the Coulomb gauge choice, equal-time quantization in the light-cone gauge entails the presence of unphysical degrees of freedom [4]. Strictly speaking, in order to recover Maxwell's equations, one should impose the condition $\lambda^{(0)} = 0$. However, as $\lambda^{(0)}$ does not commute with A, to impose $\lambda^{(0)} = 0$ in an operatorial sense would be inconsistent with quantization. In analogy with the Gupta–Bleuler formalism, this condition will be imposed weakly.

We start considering the quantization of the bosonic field at $t = 0$.

The Fourier expansion of $A(0,x)$ and $F(0,x)$ is

$$A(0,x) = \sqrt{2}A_1(0,x) = \frac{1}{\sqrt{L}} \sum_{n \neq 0} a_n e^{in\frac{\pi}{L}x} + \frac{\mathbf{a}_0}{\sqrt{L}} \ , \tag{38}$$

$$F(0,x) = \frac{1}{\sqrt{2L}} \sum_{n \neq 0} b_n e^{in\frac{\pi}{L}x} + \frac{\mathbf{b}_0}{\sqrt{2L}} \ , \tag{39}$$

$$a_n^+ = a_{-n} \quad , \quad b_n^+ = b_{-n} \ .$$

From the canonical commutation relations of A and F it follows that

$$[a_n, b_m] = i\delta_{n,-m} \ , \tag{40}$$

$$[\mathbf{a}_0, \mathbf{b}_0] = i, \tag{41}$$

all other commutators vanishing.

As far as zero modes are concerned, we represent (41) in a $\mathcal{L}^2$-space by means of the relations:

$$\mathbf{a}_0 \psi(a_0) = a_0 \psi(a_0) \qquad , \qquad \mathbf{b}_0 \psi(a_0) = -i \frac{d}{da_0} \psi(a_0) \qquad , \qquad \psi \in \mathcal{L}^2.$$

For the frequency modes we shall consider a Fock space $\mathcal{F}_A$ representation in terms of suitable creation and annihilation operators. The free bosonic Hamiltonian is:

$$H^{(0)} = \int_{-L}^{L} dx \left(\frac{1}{2} F^2(0, \mathrm{x}) - \frac{A(0, \mathrm{x})}{\sqrt{2}} \partial_1 F(0, \mathrm{x}) \right) .$$

The operator ordering in this equation is irrelevant in obtaining the commutators that follow. From (38) and (39) we get

$$H^{(0)} = \frac{1}{2} \sum_{n \neq 0} b_n b_{-n} - i \sum_{n \neq 0} k_1 a_{-n} b_n + \frac{1}{2} b_0^2$$

and thereby

$$[H^{(0)}, b_n] = k_0 b_n \ , \qquad \text{for} \quad n > 0 \ ,$$

$$[H^{(0)}, b_n] = -k_0 b_n \ , \qquad \text{for} \quad n < 0 \ ,$$

which suggest an interpretation of b_n as an annihilation operator for $n < 0$ and as a creation operator for $n > 0$. The commutators for a_n follow from analogous considerations.

$\mathcal{F}_A$ is now constructed by means of the following creation and annihilation operators:

$$A_n = a_n \sqrt{\frac{k_0}{2}} + i \frac{b_n}{\sqrt{2k_0}} \qquad , \qquad A_n^+ = a_{-n} \sqrt{\frac{k_0}{2}} - i \frac{b_{-n}}{\sqrt{2k_0}} \ , \qquad \text{for} \quad n < 0 \ ; \tag{42}$$

$$A_n = a_{-n} \sqrt{\frac{k_0}{2}} - i \frac{b_{-n}}{\sqrt{2k_0}} \qquad , \qquad A_n^+ = a_n \sqrt{\frac{k_0}{2}} + i \frac{b_n}{\sqrt{2k_0}} \ , \qquad \text{for} \quad n > 0 \ . \tag{43}$$

From (40) it follows that

$$[A_n, A_m^+] = \delta_{nm} \ , \qquad \text{for} \quad n, m < 0 \ , \tag{44}$$

$$[A_n, A_m^+] = -\delta_{nm} \ , \qquad \text{for} \quad n, m > 0 \ , \tag{45}$$

all other commutators vanishing.

We notice that, since A_n is an annihilation operator and A_n^+ is a creation one, the negative sign in (45) entails the presence of negative norm states in the Fock space $\mathcal{F}_A$. On the other hand we know that redundant degrees of freedom are present, as we have to impose the vanishing of $\lambda^{(0)}$, the superscript (0) denoting free fields. As a consequence we have to consistently define a physical Hilbert space. Let us consider the time evolution of the Gauss operator

$$\lambda^{(0)}(t,\mathrm{x}) = -\sqrt{2}\partial_1 F^{(0)}(t,\mathrm{x}) = -\sqrt{2}e^{-iH_0 t}\partial_1 F(0,\mathrm{x})e^{iH_0 t} \ .$$

We get

$$\lambda^{(0)}(t,\mathrm{x}) = \frac{1}{\sqrt{2L}}\Bigg\{\sum_{n=-\infty}^{-1} k_0\sqrt{k_0}(A_n - A_{-n})e^{-ik_0(t+\mathrm{x})} - \sum_{n=1}^{+\infty} k_0\sqrt{k_0}(A_n^+ - A_{-n}^+)e^{ik_0(t+\mathrm{x})}\Bigg\} \ .$$

The physical states are defined by

$$\lambda^{(0)}{}^{(+)}|phys\rangle = 0 \ ,$$

namely

$$(A_n - A_{-n})|phys\rangle = 0 \ , \qquad \forall n < 0 \ , \tag{46}$$

which is reminiscent of the Gupta–Bleuler condition in the Feynman gauge.

In the vector space of linear combinations of products of creation operators acting on the vacuum, we introduce a positive-definite inner product $\langle\cdot,\cdot\rangle$ such that $[A_n, A_m^*] = \delta_{nm}$, where A_n^* is the adjoint of A_n with respect to the inner product $\langle\cdot,\cdot\rangle$. A Fock space can be constructed by a Cauchy completion with respect to this inner product, which is related to the product $(\cdot,\cdot)$ compatible with (45), by:

$$(\cdot,\cdot) = \langle\cdot,\eta\ \cdot\rangle \ ,$$

where

$$\eta = (-1)^N \qquad , \qquad N = -\sum_{n<0} A^+_{-n} A_{-n} \ .$$

In turn the inner product $(\cdot,\cdot)$ in the physical subspace defined by (46) turns out to be semi-positive-definite. The physical Hilbert space is eventually obtained as the quotient space.

C. The Schwinger model in the light-cone gauge

The Schwinger model is defined by the following Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2}(\bar{\psi}\partial_\mu \gamma^\mu \psi - \partial_\mu \bar{\psi}\gamma^\mu \psi) - j^\mu A_\mu \ . \tag{47}$$

In order to quantize it we have to choose a gauge; we choose the light-cone gauge by adding to the classical Lagrangian the following "gauge-fixing" term

$$\mathcal{L}_{gf} = \lambda n_\mu A^\mu \ ,$$

where $n_\mu = \frac{1}{\sqrt{2}}(1,1)$ and λ is a Lagrange multiplier enforcing the gauge condition $n_\mu A^\mu = 0$.

The classical gauge-fixed Hamiltonian is

$$H = \int_{-L}^{L} dx \left(\frac{1}{2} F^2(x) - \frac{A}{\sqrt{2}} \partial_1 F(x) - \frac{1}{2} \left[i\bar{\psi}(x)\gamma^1 \partial_1 \psi(x) - i\partial_1 \bar{\psi}(x)\gamma^1 \psi(x) \right] + A(x) j^+(x) \right) , \tag{48}$$

where, again,

$$A = A_+ = \frac{A_0 + A_1}{\sqrt{2}} \quad , \quad F = F_{01} \ .$$

The Euler–Lagrange equations are :

$$\partial_\mu F^{\mu\nu} - j^\nu + n^\nu \lambda = 0, \tag{49}$$

which, in turn, imply

$$\partial_- \lambda = 0. \tag{50}$$

Following the standard Dirac procedure, one can derive the following independent *equal-time* (anti)commutation relations:

$$[A(t,\mathrm{x}), F(t,y)] = i\sqrt{2}\delta(\mathrm{x}-y),$$
$$\{\psi(t,\mathrm{x}), \psi^{+}(t,y)\} = e^{\frac{i\pi}{2L}(x-y)}\delta(\mathrm{x}-y),$$

all other (anti)commutators vanishing.

Again, as λ does not commute with A and ψ, the condition $\lambda = 0$ will be imposed weakly.

III. REGULARIZATION OF COMPOSITE QUANTUM OPERATORS AT EQUAL TIME

A. Point splitting regularization

Products of operators are plagued by ultraviolet singularites that need to be regularized. In the following we shall use the standard point splitting technique. To preserve gauge invariance, the following bosonic string

$$e^{-ie\int_{x}^{x+\varepsilon} dy^{\nu} A_{\nu}(y)}$$

will be inserted in the fermionic bilinears. The exponential involves products of operators at the same point, which in turn have to be suitably regularized. This problem will be carefully discussed when needed. For the time being, we *formally* define [7]

$$j^{\mu}(x) = \frac{e}{2}\lim_{\varepsilon\to 0}\Big[\bar{\psi}(x+\varepsilon)\gamma^{\mu}e^{-ie\int_{x}^{x+\varepsilon} dy^{\nu} A_{\nu}(y)}\psi(x) + \\ + \bar{\psi}(x)\gamma^{\mu}e^{-ie\int_{x+\varepsilon}^{x} dy^{\nu} A_{\nu}(y)}\psi(x+\varepsilon)\Big] , \qquad \varepsilon^{2} < 0 \tag{51}$$

and

$$\Big[i\bar{\psi}(x)\gamma^{\mu}\partial_{\mu}\psi(x) - i\partial_{\mu}\bar{\psi}(x)\gamma^{\mu}\psi(x)\Big]_{R} = \lim_{\varepsilon\to 0}\Big(i\bar{\psi}(x+\varepsilon)\gamma^{\mu}e^{-ie\int_{x}^{x+\varepsilon} dy^{\nu} A_{\nu}(y)}\partial_{\mu}\psi(x) - \\ - \; i\partial_{\mu}\bar{\psi}(x)\gamma^{\mu}e^{-ie\int_{x+\varepsilon}^{x} dy^{\nu} A_{\nu}(y)}\psi(x+\varepsilon) - v.e.v.\Big) , \qquad \varepsilon^{2} < 0 , \tag{52}$$

where the vacuum considered here is the one of the fully interacting theory. The regularized quantum Dirac equation is

$$i\gamma^\mu \partial_\mu \psi(x) - e\gamma^\mu \Big[A_\mu(x)\psi(x)\Big]_R = 0 \ , \tag{53}$$

where [2]

$$\Big[A_\mu(x)\psi(x)\Big]_R = \frac{1}{2}\lim_{\varepsilon\to 0}\Big\{A_\mu(x+\varepsilon)\psi(x) + \psi(x)A_\mu(x-\varepsilon)\Big\} \ , \qquad \varepsilon^2 < 0 \ .$$

B. Regularized current

We are now in the position of giving an explicit expression for the quantum fermionic currents. We recall the definition

$$j^-(0,\mathrm{x}) = \frac{1}{2}\lim_{\epsilon\to 0}\left[\sqrt{2}e\psi_1^+(0,\mathrm{x}+\epsilon)e^{-ie\int_{\mathrm{x}}^{\mathrm{x}+\epsilon} A_1(0,y)dy}\psi_1(0,\mathrm{x}) + h.c.\right] .$$

As A_1 and ψ_1 commute at equal time, we can write

$$j^-(0,\mathrm{x}) = \frac{1}{2}\lim_{\epsilon\to 0}\left[j^-_{(0)}(0,\mathrm{x};\epsilon)e^{-ie\int_{\mathrm{x}}^{\mathrm{x}+\epsilon} A_1(0,y)dy} + h.c.\right] ,$$

where

$$j^-_{(0)}(0,\mathrm{x};\epsilon) \equiv e\sqrt{2}\psi_1^+(0,\mathrm{x}+\epsilon)\psi_1(0,\mathrm{x}) \ . \tag{54}$$

Using

$$\psi_1(0,\mathrm{x}) = e^{-\varphi_1^{(-)}(0,\mathrm{x})}e^{\Lambda_1(0,\mathrm{x})}\sigma_1 e^{\Lambda_1(0,\mathrm{x})}e^{-\varphi_1^{(+)}(0,\mathrm{x})}$$

and

$$\Lambda_1(0,\mathrm{x}) = -\frac{i\pi}{4Le}(Q+Q_5)\mathrm{x} \ ,$$

it follows that

$$\Big[\varphi_1^{(-)}\Big]^+ = -\varphi_1^{(+)} \ , \qquad \Lambda_1^+ = -\Lambda_1 \ ,$$

and hence

$$j^-_{(0)} = \frac{e}{\sqrt{2}L}\, e^{\varphi_1^{(-)}(0,x+\epsilon)} e^{-\Lambda_1(0,x+\epsilon)} \sigma_1^+ e^{-\Lambda_1(0,x+\epsilon)} e^{\varphi_1^{(+)}(0,x+\epsilon)} \cdot$$
$$\cdot e^{-\varphi_1^{(-)}(0,x)} e^{\Lambda_1(0,x)} \sigma_1 e^{\Lambda_1(0,x)} e^{-\varphi_1^{(+)}(0,x)} \ .$$

By recalling eqs. (24) to (27) and the relation

$$e^A B = B e^A e^c \ , \qquad \text{if} \quad [A,B] = cB \ , \quad \text{with} \quad c \quad \text{being} \quad \text{a} \quad c \text{ number} \ , \tag{55}$$

we obtain

$$j^-_{(0)} = \frac{e}{\sqrt{2}L}\, e^{-\frac{i\pi}{2L}\epsilon} e^{-2\Lambda_1(0,\epsilon)} e^{\varphi_1^{(-)}(0,x+\epsilon)} e^{\varphi_1^{(+)}(0,x+\epsilon)} e^{-\varphi_1^{(-)}(0,x)} e^{-\varphi_1^{(+)}(0,x)}.$$

From the identity

$$e^A e^B = e^B e^A e^{[A,B]} \ , \qquad \text{if} \quad [A,B] = c \text{ number} \ , \tag{56}$$

we can write

$$e^{\varphi_1^{(+)}(0,x+\epsilon)} e^{-\varphi_1^{(-)}(0,x)} = e^{-\varphi_1^{(-)}(0,x)} e^{\varphi_1^{(+)}(0,x+\epsilon)} e^{[\varphi_1^+(0,x+\epsilon), -\varphi_1^{(-)}(0,x)]} \ .$$

Using the decomposition for the quasi-scalar field φ, we get

$$[\varphi_1^+(0,x+\epsilon), -\varphi_1^{(-)}(0,x)] = \sum_{n=1}^{+\infty} \frac{1}{n} e^{-in\frac{\pi}{L}\epsilon} = -\ln\left(1 - e^{-i\frac{\pi}{L}\epsilon}\right) \ , \tag{57}$$

and therefore

$$j^-_{(0)} = \frac{e}{\sqrt{2}L} \frac{e^{-\frac{i\pi}{2L}\epsilon}}{1 - e^{-i\frac{\pi}{L}\epsilon}} e^{-2\Lambda_1(0,\epsilon)} e^{\left(\varphi_1^{(-)}(0,x+\epsilon) - \varphi_1^{(-)}(0,x)\right)} e^{\left(\varphi_1^{(+)}(0,x+\epsilon) - \varphi_1^{(+)}(0,x)\right)} \ . \tag{58}$$

Neglecting terms that vanish in the limit $\epsilon \to 0$ we get

$$j^-_{(0)}(0,x;\epsilon) = \frac{e}{i\pi\epsilon\sqrt{2}} + \frac{Q + Q_5}{2\sqrt{2}L} + \frac{e}{i\pi\sqrt{2}}\Big(\partial_1\varphi_1^{(-)}(0,x) + \partial_1\varphi_1^{(+)}(0,x)\Big).$$

Moreover

$$e^{-ie\int_x^{x+\epsilon} A_1(0,y)dy} = 1 - ie\epsilon A_1(0,x) + \ldots$$

and therefore

$$j^-_{(0)}(0,x;\epsilon)e^{-ie\int_x^{x+\epsilon}A_1(0,y)dy} = \frac{e}{i\pi\epsilon\sqrt{2}} - \frac{e^2A_1(0,x)}{\sqrt{2}\pi} + \frac{Q+Q_5}{2\sqrt{2}L} + \\ + \frac{e}{i\pi\sqrt{2}}\Big(\partial_1\varphi_1^{(-)}(0,x) + \partial_1\varphi_1^{(+)}(0,x)\Big) \ .$$

From the definition of $j^-(0,x)$ it follows that

$$j^-(0,x) = -\frac{e^2A_1(0,x)}{\sqrt{2}\pi} + \frac{Q+Q_5}{2\sqrt{2}L} + \frac{e}{\sqrt{2}L}\sum_{n=1}^{+\infty}\Big(i\sqrt{|n|}\gamma_n e^{-in\frac{\pi}{L}x} - i\sqrt{|n|}\gamma_n^+ e^{in\frac{\pi}{L}x}\Big) \ . \qquad (59)$$

An analogous calculation gives, taking

$$[\varphi_2^{(+)}(0,x+\epsilon)\,,\,-\varphi_2^{(-)}(0,x)] = \sum_{n=1}^{+\infty}\frac{1}{n}e^{in\frac{\pi}{L}\epsilon} = -ln\left(1-e^{i\frac{\pi}{L}\epsilon}\right) \qquad (60)$$

into account,

$$j^+(0,x) = \lim_{\epsilon\to 0}\Big[\sqrt{2}e\psi_2^+(0,x+\epsilon)e^{-ie\int_x^{x+\epsilon}A_1(0,y)dy}\psi_2(0,x) + h.c.\Big] = \\ = \frac{e^2A_1(0,x)}{\sqrt{2}\pi} + \frac{Q-Q_5}{2\sqrt{2}L} + \frac{e}{\sqrt{2}L}\sum_{n=-\infty}^{-1}\Big(i\sqrt{|n|}\gamma_n e^{-in\frac{\pi}{L}x} - i\sqrt{|n|}\gamma_n^+ e^{in\frac{\pi}{L}x}\Big) \ . \qquad (61)$$

C. Regularized kinetic term

The fermionic kinetic term entering the quantum Hamiltonian is

$$H_\psi = -\frac{1}{2}\int_{-L}^{L}dx\Big[i\bar\psi(0,x)\gamma^1\partial_1\psi(0,x) - i\partial_1\bar\psi(0,x)\gamma^1\psi(0,x)\Big]_R = \\ = \frac{1}{2}\int_{-L}^{L}dx\Big[i\psi_1^+(0,x)\partial_1\psi_1(0,x) - i\psi_2^+(0,x)\partial_1\psi_2(0,x) + h.c.\Big]_R \ ,$$

where

$$\Big[i\psi_1^+(0,x)\partial_1\psi_1(0,x) - i\psi_2^+(0,x)\partial_1\psi_2(0,x) + h.c.\Big]_R = \\ = \lim_{\epsilon\to 0}\Big\{i\psi_1^+(0,x+\epsilon)e^{-ie\int_x^{x+\epsilon}A_1(0,y)dy}\partial_1\psi_1(0,x) \\ -i\psi_2^+(0,x+\epsilon)e^{-ie\int_x^{x+\epsilon}A_1(0,y)dy}\partial_1\psi_2(0,x) + h.c. - v.e.v.\Big\}.$$

Setting $z = \mathrm{x} + \epsilon$, we can write

$$\Big[i\psi_1^+(0,\mathrm{x})\partial_1\psi_1(0,\mathrm{x}) - i\psi_2^+(0,\mathrm{x})\partial_1\psi_2(0,\mathrm{x}) + h.c.\Big]_R =$$
$$= \lim_{z\to\mathrm{x}}\left\{e^{-ie\int_{\mathrm{x}}^{z}A_1(0,y)dy}\frac{\partial}{\partial\mathrm{x}}\Big[i\psi_1^+(0,z)\psi_1(0,\mathrm{x}) - i\psi_2^+(0,z)\psi_2(0,\mathrm{x})\Big] + h.c. - v.e.v.\right\} .$$

Equation (54) entails

$$\psi_1^+(0,z)\psi_1(0,\mathrm{x}) = \frac{1}{e\sqrt{2}}j^-_{(0)}(\mathrm{x}, z-\mathrm{x})$$

and, using (58), one gets $2L\frac{\partial}{\partial\mathrm{x}}\Big[\psi_1^+(0,z)\psi_1(0,\mathrm{x})\Big] =$

$$= \frac{\partial}{\partial\mathrm{x}}\left[e^{-\frac{i\pi}{2L}(z-\mathrm{x})}\frac{1}{1-e^{-\frac{i\pi}{L}(z-\mathrm{x})}}e^{-2\Lambda_1(0,z-\mathrm{x})} : e^{(\varphi_1(0,z)-\varphi_1(0,\mathrm{x}))} :\right] ,$$

where

$$\varphi_1 = \varphi_1^{(+)} + \varphi_1^{(-)}$$

and

$$: e^{(\varphi_1(0,z)-\varphi_1(0,\mathrm{x}))} := e^{\left(\varphi_1^{(-)}(0,z)-\varphi_1^{(-)}(0,\mathrm{x})\right)}e^{\left(\varphi_1^{(+)}(0,z)-\varphi_1^{(+)}(0,\mathrm{x})\right)} .$$

Neglecting terms that vanish in the limit $z \to \mathrm{x}$, we obtain $2L\frac{\partial}{\partial\mathrm{x}}\Big[\psi_1^+(0,z)\psi_1(0,\mathrm{x})\Big] =$

$$\left(\tfrac{1}{i\frac{\pi}{L}(z-\mathrm{x})^2} + \text{const.}\right)e^{-\frac{i\pi}{2L}(z-\mathrm{x})}e^{-2\Lambda_1(z-\mathrm{x})} : e^{(\varphi_1(0,z)-\varphi_1(0,\mathrm{x}))} : +$$
$$+\tfrac{1}{i\frac{\pi}{L}(z-\mathrm{x})}e^{-2\Lambda_1(z-\mathrm{x})}\left\{\left(-2\tfrac{\partial}{\partial\mathrm{x}}\Lambda_1(z-\mathrm{x}) + \tfrac{i\pi}{2L}\right\} : e^{(\varphi_1(0,z)-\varphi_1(0,\mathrm{x}))} : -\right.$$
$$\left. -\partial_1\varphi_1^{(-)}(0,\mathrm{x}) : e^{(\varphi_1(0,z)-\varphi_1(0,\mathrm{x}))} : - : e^{(\varphi_1(0,z)-\varphi_1(0,\mathrm{x}))} : \partial_1\varphi_1^{(+)}(0,\mathrm{x})\right\}.$$

By expanding in Taylor series one finds

$$\psi_1^+(0,\mathrm{x}+\epsilon)\partial_1\psi_1(0,\mathrm{x}) = \frac{i}{4\pi} : \Big(\partial_1\varphi_1(0,\mathrm{x})\Big)^2 : -\frac{i\pi}{16L^2e^2}(Q+Q_5)^2 -$$
$$-\frac{Q+Q_5}{4Le}\partial_1\varphi_1(0,\mathrm{x}) - \frac{i}{4\pi}\partial_1^2\varphi_1(0,\mathrm{x}) + \frac{1}{2i\pi\epsilon^2} + \text{const.} . \qquad (62)$$

An analogous calculation gives

$$\psi_2^+(0,\mathrm{x}+\epsilon)\partial_1\psi_2(0,\mathrm{x}) = -\frac{i}{4\pi}:\left(\partial_1\varphi_2(0,\mathrm{x})\right)^2: + \frac{i\pi}{16L^2e^2}(Q-Q_5)^2 - \\ -\frac{Q-Q_5}{4Le}\partial_1\varphi_2(0,\mathrm{x}) + \frac{i}{4\pi}\partial_1^2\varphi_2(0,\mathrm{x}) - \frac{1}{2i\pi\epsilon^2} + \text{const.}\ . \qquad (63)$$

In the previous formulae, positive and negative energy components $\varphi_i^{(+)}$ and $\varphi_i^{(-)}$ refer to the free fermion theory.

In the light-cone gauge the *left* component of the fermionic field does not interact: in the full Hamiltonian φ_1 is decoupled. The coupling with the gauge field only affects $\varphi_2 = \varphi_2^{(+)} + \varphi_2^{(-)}$: the two frequencies mix under time evolution. The vacuum involved in the *v.e.v.* subtraction refers to the interacting theory.

If we define

$$H_\psi^{(0)} = \frac{1}{2}\int_{-L}^{L} dx : \left[i\psi_1^+(0,\mathrm{x})\partial_1\psi_1(0,\mathrm{x}) - i\psi_2^+(0,\mathrm{x})\partial_1\psi_2(0,\mathrm{x}) + h.c.\right] : ,$$

where the dots denote normal ordering with respect to the above-mentioned frequencies, we easily get

$$H_\psi^{(0)} = \frac{\pi}{4Le^2}(Q^2+Q_5^2) + \frac{\pi}{L}\sum_{n\neq 0}|n|\gamma_n^+\gamma_n\ .$$

$H_\psi^{(0)}$ takes care of the first two terms in eqs. (62) and (63). The third and fourth contributions vanish, thanks to the periodicity of φ. The singular $1/\epsilon^2$ terms require an expansion of the phase factor up to the second order:

$$e^{-ie\int_{\mathrm{x}}^{\mathrm{x}+\epsilon}A_1(0,y)dy} = 1 - ieA_1(0,\mathrm{x})\epsilon - \frac{ie}{2}\partial_1 A_1(0,\mathrm{x})\epsilon^2 - \frac{e^2}{2}A_1^2(0,\mathrm{x})\epsilon^2 + \ldots\ .$$

The term A_1^2 needs a suitable definition.

Putting everything together, one can realize that the single poles in ϵ cancel under Hermitian conjugation; double pole and finite constants, being c numbers, are subtracted by *v.e.v.*.

We eventually obtain

$$H_\psi = \mathcal{N}\left[\sum_{n\neq 0}k_0\gamma_n^+\gamma_n - \int_{-L}^{L}dx\frac{e^2}{2\pi}A_1^2(0,\mathrm{x})\right] + \frac{\pi}{4Le^2}(Q^2+Q_5^2)\ ,$$

or, using (38), (39):

$$H_\psi = \mathcal{N}\left[\sum_{n\neq 0} k_0\gamma_n^+\gamma_n - \frac{e^2}{2\pi}\sum_{n\neq 0} a_n a_{-n}\right] + \frac{\pi}{4Le^2}(Q^2+Q_5^2) - \frac{e^2}{2\pi}a_0^2 \ . \tag{64}$$

The symbol $\mathcal{N}$ has been introduced because the products appearing there need to be regularized. Since the *v.e.v.* involves the vacuum of the interacting theory, we shall see in the next section that the ordering $\mathcal{N}$ is nothing but the normal ordering with respect to the operators that diagonalize the full Hamiltonian.

IV. DIAGONALIZATION OF THE INTERACTING HAMILTONIAN

We consider the classical Hamiltonian (48). Using eqs. (61), (64), (38) and (39), we get the following quantum expression:

$$\begin{aligned} H = &\frac{1}{2}\mathbf{b_0}^2 + \frac{e^2}{2\pi}\mathbf{a_0}^2 + \frac{\pi}{4Le^2}(Q^2+Q_5^2) + \frac{1}{\sqrt{2L}}\mathbf{a_0}(Q-Q_5) \\ &+\mathcal{N}\left[\frac{1}{2}\sum_{n\neq 0} b_n b_{-n} + i\sum_{n\neq 0} k_1 a_n b_{-n} + \frac{e^2}{2\pi}\sum_{n\neq 0} a_n a_{-n} + \right. \\ &\left. + \frac{ie}{\sqrt{\pi}}\sum_{n=-\infty}^{-1}\sqrt{2k_0}(\gamma_n a_n - \gamma_n^+ a_{-n}) + \sum_{n\neq 0} k_0\gamma_n^+\gamma_n\right]. \end{aligned} \tag{65}$$

Setting $m = \frac{e}{\sqrt{\pi}}$, $\omega = \sqrt{k_0^2+m^2}$ and defining, for $n<0$, the operators

$$\begin{aligned} \chi_n &= \gamma_n^+ - \frac{\sqrt{2\pi k_0}}{e}b_n \ , \\ \Sigma_n &= \frac{1}{\sqrt{\omega}}\left(\frac{\sqrt{\pi}}{e\sqrt{2}}(\omega+k_0)b_n - \frac{ie}{\sqrt{2\pi}}a_n - \sqrt{k_0}\gamma_n^+\right) \ , \\ \Sigma_{-n} &= \frac{1}{\sqrt{\omega}}\left(\frac{\sqrt{\pi}}{e\sqrt{2}}(\omega-k_0)b_{-n} - \frac{ie}{\sqrt{2\pi}}a_{-n} + \sqrt{k_0}\gamma_n\right) \ , \end{aligned} \tag{66}$$

it is easy to check the following commutation relations:

$$[\Sigma_n, \Sigma_m^+] = \delta_{mn} \quad , \quad [\Sigma_{-n}, \Sigma_{-m}^+] = \delta_{mn} \ , \tag{67}$$

$$[\chi_n, \chi_m^+] = -\delta_{mn} \ , \tag{68}$$

all other commutators vanishing. In terms of these operators, the frequency part of the Hamiltonian (65) takes the form

$$H_1 = \mathcal{N}\Bigg[\frac{1}{2}\sum_{n\neq 0} b_n b_{-n} + i\sum_{n\neq 0} k_1 a_n b_{-n} + \frac{e^2}{2\pi}\sum_{n\neq 0} a_n a_{-n} +$$

$$+\frac{ie}{\sqrt{\pi}}\sum_{n=-\infty}^{-1}\sqrt{2k_0}(\gamma_n a_n - \gamma_n^+ a_{-n}) + \sum_{n\neq 0} k_0 \gamma_n^+ \gamma_n\Bigg] =$$

$$= \sum_{n=1}^{+\infty} k_0 \gamma_n^+ \gamma_n - \sum_{n=-\infty}^{-1} k_0 \chi_n^+ \chi_n + \sum_{n\neq 0}\sqrt{k_0^2 + m^2}\ \Sigma_n^+ \Sigma_n\ .$$

H_1 takes the form of the Fock Hamiltonian of a massive boson field (apart from the zero mode) with $m = \frac{e}{\sqrt{\pi}}$ and of two massless chiral bosons, one of which having commutation relations leading to an indefinite metric. We notice that the term with negative sign in H_1 and the commutator of χ and χ^+, entail that γ_n, for $n > 0$, and χ_n both evolve in time with a positive frequency. As we shall see, this property allows us to express the selection of a physical subspace in terms of annihilation operators. Inserting in (66) the expressions of a_n and b_n in terms of the creation and annihilation operators of the gauge fields, we obtain, for $n < 0$:

$$\chi_n = \gamma_n^+ + \frac{ik_0\sqrt{\pi}}{e}(A_n - A_{-n})\ ,$$
$$\Sigma_n = -\sqrt{\frac{k_0}{\omega}}\gamma_n^+ - \frac{ie}{2\sqrt{\pi k_0 \omega}}(A_n + A_{-n}) - \frac{i\sqrt{\pi k_0\omega}}{2e}\left(1 + \frac{k_0}{\omega}\right)(A_n - A_{-n})\ ,$$
$$\Sigma_{-n} = \sqrt{\frac{k_0}{\omega}}\gamma_n - \frac{ie}{2\sqrt{\pi k_0 \omega}}(A_n^+ + A_{-n}^+) - \frac{i\sqrt{\pi k_0\omega}}{2e}\left(1 - \frac{k_0}{\omega}\right)(A_{-n}^+ - A_n^+)\ .$$

One can verify that the following unitary operator:

$$U = e^{i\sum_{n<0}\frac{k_0\sqrt{\pi}}{e}\left[\gamma_n^+(A_n^+ - A_{-n}^+) + \gamma_n(A_n - A_{-n})\right]}$$
$$\cdot e^{\sum_{n<0}\theta(A_n A_{-n}^+ - A_n^+ A_{-n})} e^{\frac{\pi}{2}\sum_{n<0}(\gamma_n A_{-n} - A_{-n}^+\gamma_n^+)}\ ,$$

with

$$\theta = \ln\left(\frac{e}{\sqrt{\pi k_0 \omega}}\right)\ ,$$

maps creation and annihilation operators of the free original fields into new operators that diagonalize the interacting Hamiltonian. As a matter of fact, for $n < 0$, it can be checked that:

$$UA_{-n}U^{-1} = -\chi_n \ ,$$

$$UA_nU^{-1} = i\Sigma_n \ ,$$

$$U\gamma_nU^{-1} = i\Sigma_{-n} \ ,$$

$$U\gamma_{-n}U^{-1} = i\gamma_{-n} \ .$$

It is easy to realize that from

$$A_p|M,N\rangle = \gamma_p|M,N\rangle = 0 \qquad , \qquad p \neq 0 \ ,$$

it follows that

$$\Sigma_{\pm p}U|M,N\rangle = \chi_p U|M,N\rangle = \gamma_{-p}U|M,N\rangle = 0 \qquad , \qquad p < 0 \ . \tag{69}$$

Therefore U maps the base vectors of the free-field Fock space obtained by repeated action of A_p^+ and γ_p^+ on the states $|M,N\rangle$, in eigenstates of H_1 obtained by repeated action of $\Sigma_{\pm p}^+$, χ_p^+ and γ_{-p}^+ $(p<0)$ on the states $U|M,N\rangle$.

One obtains the zero mode of the massive boson by diagonalizing

$$H_0 = \frac{1}{2}\mathbf{b_0}^2 + \frac{e^2}{2\pi}\mathbf{a_0}^2 + \frac{\pi}{4Le^2}(Q^2+Q_5^2) + \frac{1}{\sqrt{2L}}\mathbf{a_0}(Q-Q_5) \ .$$

By defining

$$\Sigma_0 = \alpha + \frac{Q-Q_5}{2i\sqrt{L}m^{\frac{3}{2}}} = \frac{1}{\sqrt{2}}\left(-i\sqrt{m}\mathbf{a_0} + \frac{\mathbf{b_0}}{\sqrt{m}} + \frac{Q-Q_5}{i\sqrt{2L}m^{\frac{3}{2}}}\right) \ ,$$
$$\Sigma_0^+ = \alpha^+ - \frac{Q-Q_5}{2i\sqrt{L}m^{\frac{3}{2}}} = \frac{1}{\sqrt{2}}\left(i\sqrt{m}\mathbf{a_0} + \frac{\mathbf{b_0}}{\sqrt{m}} - \frac{Q-Q_5}{i\sqrt{2L}m^{\frac{3}{2}}}\right) \ , \tag{70}$$

the commutation relations $[\alpha,\alpha^+]=1$, $[\Sigma_0,\Sigma_0^+]=1$ follow and the Hamiltonian H_0 can be written as

$$H_0 = \frac{e}{\sqrt{\pi}}\Sigma_0^+\Sigma_0 + \frac{\pi}{2e^2L}QQ_5 \ .$$

We are thus led to the following diagonal form of the complete Hamiltonian:

$$H = \sum_{n=1}^{+\infty} k_0\gamma_n^+\gamma_n - \sum_{n=-\infty}^{-1} k_0\chi_n^+\chi_n + \sum_{n=-\infty}^{+\infty}\sqrt{k_0^2+m^2}\ \Sigma_n^+\Sigma_n + \frac{\pi}{2e^2L}QQ_5 \ ,$$

α and Σ_0 being related by a unitary operator:

$$\Sigma_0 = U_0 \alpha U_0^{-1} \ ,$$

where

$$U_0 = e^{\frac{i}{2\sqrt{L}m^{3/2}}(Q-Q_5)(\alpha+\alpha^+)} \ .$$

We remark that Q , Q_5 and the spurion σ_1 commute with U_0, whereas

$$\underline{\sigma}_2 = U_0 \sigma_2 U_0^{-1} = e^{-ie\frac{1}{\sqrt{L}m^{3/2}}(\alpha+\alpha^+)} \sigma_2. \tag{71}$$

We have the following commutators between the operators σ_1 , $\underline{\sigma}_2$, Q , Q_5 , Σ_0 Σ_0^+

$$[\Sigma_0, \Sigma_0^+] = 1 \ ,$$

$$\{\sigma_1, \underline{\sigma}_2\} = 0 \ ,$$

$$\begin{aligned} [Q, \sigma_1] &= -e\sigma_1 \quad , \quad [Q_5, \sigma_1] = -e\sigma_1 \ , \\ [Q, \underline{\sigma}_2] &= -e\underline{\sigma}_2 \quad , \quad [Q_5, \underline{\sigma}_2] = e\underline{\sigma}_2 \ , \end{aligned} \tag{72}$$

all other commutators vanishing.

Starting from the classical expression of the momentum

$$P \equiv P^1 = \int_{-L}^{L} dx \Theta^{01} = \int_{-L}^{L} dx \left[\frac{i}{2} \Big(\psi^+(0,\mathrm{x}) \partial^1 \psi(0,\mathrm{x}) - \partial^1 \psi^+(0,\mathrm{x}) \psi(0,\mathrm{x}) \Big) + \right.$$
$$\left. + \frac{1}{\sqrt{2}} F(0,\mathrm{x}) \partial^1 A(0,\mathrm{x}) \right] \ ,$$

the same transformation (66) leads to the following diagonal expression

$$P = -\sum_{n=1}^{+\infty} k_0 \gamma_n^+ \gamma_n + \sum_{n=-\infty}^{-1} k_0 \chi_n^+ \chi_n + \sum_{n \neq 0} k_1 \Sigma_n^+ \Sigma_n - \frac{\pi}{2Le^2} Q Q_5 \ .$$

We notice that the modes γ and χ have a negative momentum (owing to the negative sign in the χ commutator). Therefore they describe bosonic degrees of freedom of the same *left* chirality.

V. THE TEMPORAL EVOLUTION

The temporal evolution can easily be determined, starting from the commutation relations

$$
\begin{aligned}
&[H,\Sigma_{0,\pm n}] = -\omega\Sigma_{0,\pm n} \quad , \quad [H,\Sigma^+_{0,\pm n}] = \omega\Sigma^+_{0,\pm n} \ , \\
&[H,\chi_n] = -k_0\chi_n \quad , \quad [H,\chi^+_n] = k_0\chi^+_n \ , \\
&[H,\gamma_{-n}] = -k_0\gamma_{-n} \quad , \quad [H,\gamma^+_{-n}] = k_0\gamma^+_{-n} \ , \\
&[H,\sigma_1] = -\frac{\pi}{4eL}\Big[(Q+Q_5)\sigma_1 + \sigma_1(Q+Q_5)\Big] \ , \\
&[H,\underline{\sigma_2}] = \frac{\pi}{4eL}\Big[(Q-Q_5)\underline{\sigma_2} + \underline{\sigma_2}(Q-Q_5)\Big] \ , \\
&[H,Q] = [H,Q_5] = 0 \ .
\end{aligned}
\tag{73}
$$

Setting

$$
\Sigma(t,x) = \frac{1}{\sqrt{2L}}\sum_{n=-\infty}^{+\infty}\frac{1}{\sqrt{2\omega}}\left(\Sigma_n e^{-i(\omega t - k_1 x)} + \Sigma^+_n e^{i(\omega t - k_1 x)}\right) \tag{74}
$$

and

$$
\chi(t,x) = \frac{1}{\sqrt{2L}}\sum_{n=-\infty}^{-1}\frac{1}{\sqrt{2k_0}}\left(\chi_n e^{-ik_0(t+x)} + \chi^+_n e^{ik_0(t+x)}\right) , \tag{75}
$$

from (73) and the canonical commutation relations, it is easy to see that $\Sigma(t,x)$ satisfies the Klein–Gordon equation:

$$
(\Box + m^2)\Sigma(t,x) = 0 \tag{76}
$$

and

$$
\Big[\Sigma(t,x),\partial_0\Sigma(t,y)\Big] = i\delta(x-y) \quad , \quad \Big[\Sigma(t,x),\Sigma(t,y)\Big] = \Big[\partial_0\Sigma(t,x),\partial_0\Sigma(t,y)\Big] = 0 \ .
$$

Here $\chi(t,x)$ is a massless chiral (left) field satisfying the algebra

$$
\Big[\chi(t,x),\partial_0\chi(t,y)\Big] = \frac{-i}{2L}\sum_{n=-\infty}^{-1} e^{-\epsilon k_0}\cos k_0(x-y) \ ,
$$

$$\left[\chi(t,x),\chi(t,y)\right] = \frac{i}{2L}\sum_{n=-\infty}^{-1}\frac{\sin k_0(x-y)}{k_0} \ ,$$

$$\left[\partial_0\chi(t,x),\partial_0\chi(t,y)\right] = \frac{i}{2L}\sum_{n=-\infty}^{-1} k_0\sin k_0(x-y) \ .$$

The temporal evolution of the original fields can now be obtained in the Heisenberg picture in the usual way; for instance

$$A(t,x) = e^{iHt}A(0,x)e^{-iHt} \ ,$$

and analogous ones for the other fields. We easily get

$$A(t,x) = -\frac{2}{m}\partial_+\left[\Sigma(t,x)+\chi(t,x)\right] - \frac{Q-Q_5}{\sqrt{2}m^2L}, \tag{77}$$

which in turn implies

$$F(t,x) = m\Sigma(t,x). \tag{78}$$

Therefore the electric field F becomes massive

$$(\Box + m^2)F(t,x) = 0.$$

We now turn our attention to the fermionic field. As far as the ψ_1 component is concerned, one immediately recovers a free temporal evolution:

$$\psi_1(t,x) = \tfrac{1}{\sqrt{2L}}e^{-\varphi_1^{(-)}(t,x)}e^{-\frac{i\pi}{4eL}(Q+Q_5)(t+x)}\sigma_1 e^{-\frac{i\pi}{4eL}(Q+Q_5)(t+x)}e^{-\varphi_1^{(+)}(t,x)} \ , \tag{79}$$

$$\varphi_1^{(-)}(t,x) = -\sum_{n=1}^{+\infty}\frac{i}{\sqrt{n}}\gamma_n^+ e^{ik_0(t+x)} \ ,$$
$$\varphi_1^{(+)}(t,x) = -\sum_{n=1}^{+\infty}\frac{i}{\sqrt{n}}\gamma_n e^{-ik_0(t+x)} \ .$$

For the ψ_2 component we get instead

$$\psi_2(0,x) = \frac{1}{\sqrt{2L}}\exp\left\{ i\sum_{n=-\infty}^{-1}\frac{1}{\sqrt{|n|}}\left[\chi_n + \sqrt{\frac{k_0}{\omega}}\left(\Sigma_n+\Sigma_{-n}^+\right)\right]e^{ik_1x}\right\}.$$

$$\cdot e^{i\frac{\pi}{4Le}(Q-Q_5)x} e^{i\sqrt{\frac{\pi}{mL}}\Sigma_0^+} \underline{\sigma}_2 e^{i\sqrt{\frac{\pi}{mL}}\Sigma_0} e^{i\frac{\pi}{4Le}(Q-Q_5)x} e^{-\frac{\pi}{2mL}} \cdot$$

$$\cdot \exp\left\{ i \sum_{n=-\infty}^{-1} \frac{1}{\sqrt{|n|}} \left[\chi_n^+ + \sqrt{\frac{k_0}{\omega}} (\Sigma_n^+ + \Sigma_{-n}) \right] e^{-ik_1 x} \right\} =$$

$$= \frac{e^Z}{\sqrt{2L}} \exp\left\{ i \sum_{n=-\infty}^{-1} \frac{1}{\sqrt{|n|}} \left[\chi_n^+ e^{-ik_1 x} + \sqrt{\frac{k_0}{\omega}} \left(\Sigma_n^+ e^{-ik_1 x} + \Sigma_{-n}^+ e^{ik_1 x} \right) \right] \right\} \cdot$$

$$\cdot e^{i\frac{\pi}{4Le}(Q-Q_5)x} e^{i\sqrt{\frac{\pi}{mL}}\Sigma_0^+} \underline{\sigma}_2 e^{i\sqrt{\frac{\pi}{mL}}\Sigma_0} e^{i\frac{\pi}{4Le}(Q-Q_5)x} \cdot$$

$$\cdot \exp\left\{ i \sum_{n=-\infty}^{-1} \frac{1}{\sqrt{|n|}} \left[\chi_n e^{ik_1 x} + \sqrt{\frac{k_0}{\omega}} \left(\Sigma_n e^{ik_1 x} + \Sigma_{-n} e^{-ik_1 x} \right) \right] \right\} ,$$

where

$$\begin{aligned} Z =& -\frac{\pi}{2mL} - \left[\sum_{n=-\infty}^{-1} \frac{1}{\sqrt{|n|}} \chi_n e^{in\frac{\pi}{L}x} \ , \ \sum_{p=-\infty}^{-1} \frac{1}{\sqrt{|p|}} \chi_p^+ e^{-ip\frac{\pi}{L}x} \right] + \\ & + \frac{\pi}{L} \left[\sum_{n=-\infty}^{-1} \frac{1}{\sqrt{\omega}} \Sigma_n e^{in\frac{\pi}{L}x} \ , \ \sum_{p=-\infty}^{-1} \frac{1}{\sqrt{\omega}} \Sigma_p^+ e^{-ip\frac{\pi}{L}x} \right] = \\ =& -\frac{\pi}{2mL} + \sum_{n=-\infty}^{-1} \left(\frac{1}{|n|} - \frac{\pi}{L}\frac{1}{\omega} \right) = -\frac{\pi}{2mL} + \sum_{n=1}^{+\infty} \frac{m^2}{n\sqrt{\left(\frac{n\pi}{L}\right)^2 + m^2} \left(\sqrt{\left(\frac{n\pi}{L}\right)^2 + m^2} + n\frac{\pi}{L} \right)} \end{aligned} \quad (80)$$

is a finite constant.[2]

Using the relation

$$e^{iHt} \underline{\sigma}_2 e^{-iHt} = e^{\frac{i\pi}{4Le}(Q-Q_5)t} \underline{\sigma}_2 e^{\frac{i\pi}{4Le}(Q-Q_5)t} ,$$

one can eventually find the temporal evolution of $\psi_2 : \psi_2(t,x) = e^{iHt}\psi_2(0,x)e^{-iHt} =$

$$\begin{aligned} =& \frac{e^Z}{\sqrt{2L}} e^{2i\sqrt{\pi}\left[\chi^{(-)}(t,x)+\Sigma^{(-)}(t,x)\right]} e^{\frac{i\pi}{4Le}(Q-Q_5)(t+x)} \ \underline{\sigma}_2 \ \cdot \\ & \cdot e^{\frac{i\pi}{4Le}(Q-Q_5)(t+x)} e^{2i\sqrt{\pi}\left[\chi^{(+)}(t,x)+\Sigma^{(+)}(t,x)\right]} , \end{aligned} \quad (81)$$

---

[2] We remark that, in the decompactification limit $L \to \infty$, the factor $\frac{e^Z}{\sqrt{2L}}$ would diverge, producing an infinite renormalization constant for the field ψ_2.

where

$$\chi^{(-)}(t,x) = \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{-1} \frac{1}{\sqrt{2k_0}} \chi_n^+ e^{ik_0(t+x)} ,$$

$$\chi^{(+)}(t,x) = \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{-1} \frac{1}{\sqrt{2k_0}} \chi_n e^{-ik_0(t+x)} ,$$

$$\Sigma^{(-)}(t,x) = \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{+\infty} \frac{1}{\sqrt{2\omega}} \Sigma_n^+ e^{i(\omega t - k_1 x)} ,$$

$$\Sigma^{(+)}(t,x) = \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{+\infty} \frac{1}{\sqrt{2\omega}} \Sigma_n e^{-i(\omega t - k_1 x)} .$$

It can be checked that the field ψ_2 satisfies the regularized Dirac equation (53).

Finally, for λ we get

$$\lambda(t,x) = e^{iHt}\lambda(0,x)e^{-iHt} =$$
$$= \frac{Q}{\sqrt{2L}} + \frac{m}{\sqrt{2L}} \sum_{n=-\infty}^{-1} \sqrt{k_0} \left[i\left(\gamma_{-n} - \chi_n\right) e^{-ik_0(t+x)} - i\left(\gamma_{-n}^+ - \chi_n^+\right) e^{ik_0(t+x)} \right] ,$$

in agreement with both Euler–Lagrange equations for the electric field; λ cannot vanish strongly as it has non-vanishing commutators with A and ψ. In order to recover the correct Maxwell equations in the weak sense we have to define the physical states by imposing the following conditions

$$Q|phys\rangle = 0 , \tag{82}$$

$$\lambda^{(+)}|phys\rangle = 0 , \tag{83}$$

$\lambda^{(+)}$ being the positive frequency component of $\lambda(x)$.

Equation (83) is equivalent to

$$(\gamma_{-n} - \chi_n)|phys\rangle = 0 \quad , \quad \forall n < 0. \tag{84}$$

VI. THE STRUCTURE OF THE HILBERT SPACE

A. The vacuum sector

The operators H, P, Q and Q_5 commute pair-wise. The common eigenstates can be obtained by the repeated action of the creation operators $\Sigma_{0,\pm p}^+$, γ_{-p}^+ , χ_p^+ $(p < 0)$ to the states

$$|\Omega_{mn}\rangle = S\sigma_1^m\sigma_2^n|0\rangle = \sigma_1^m\underline{\sigma}_2^n S|0\rangle \ , \qquad m,n = 0,\pm1,\pm2,\dots \quad ,$$

where $|0\rangle$ is defined by the conditions

$$\alpha|0\rangle = A_n|0\rangle = \gamma_n|0\rangle = Q|0\rangle = Q_5|0\rangle = 0$$

and

$$S = UU_0 \ .$$

$|\Omega_{mn}\rangle$ obeys the equations

$$\Sigma_{0,\pm p}|\Omega_{mn}\rangle = \gamma_{-p}|\Omega_{mn}\rangle = \chi_p|\Omega_{mn}\rangle = 0 \ ,$$

$$\langle\Omega_{mn}|\Omega_{pq}\rangle = \langle 0|\sigma_2^{-n}\sigma_1^{-m}S^+S\sigma_1^p\sigma_2^q|0\rangle = (-1)^{(p-m)n}\langle 0|\sigma_1^{(p-m)}\sigma_2^{(q-n)}|0\rangle =$$
$$= (-1)^{(p-m)n}\langle 0|p-m,q-n\rangle = \delta_{pm}\delta_{qn}\langle 0|0\rangle \ ,$$

so that

$$\langle\Omega_{mn}|\Omega_{pq}\rangle = \delta_{pm}\delta_{qn} \ , \quad \text{if} \quad \langle 0|0\rangle = 1 \ ,$$

and

$$Q|\Omega_{mn}\rangle = -e(m+n)|\Omega_{mn}\rangle \ ,$$
$$Q_5|\Omega_{mn}\rangle = -e(m-n)|\Omega_{mn}\rangle \ ,$$
$$H|\Omega_{mn}\rangle = \frac{\pi}{2L}(m^2-n^2)|\Omega_{mn}\rangle \ ,$$
$$P|\Omega_{mn}\rangle = -\frac{\pi}{2L}(m^2-n^2)|\Omega_{mn}\rangle \ .$$

Among them, only the states

$$|\Omega_{-n\,n}\rangle = S\sigma_1^{-n}\sigma_2^n|0\rangle$$

satisfy the condition

$$Q|\Omega_{-n\,n}\rangle = 0 \ ,$$

and therefore are physical.

For such states

$$Q_5|\Omega_{-n\,n}\rangle = 2n|\Omega_{-n\,n}\rangle \ ,$$

$$H|\Omega_{-n\,n}\rangle = P|\Omega_{-n\,n}\rangle = 0 \ .$$

An infinite set of vacua, labelled by the integer n, are translation-invariant. Such states are related by a residual gauge transformation. As a matter of fact, if we define

$$|\Omega_n\rangle = e^{i\pi\frac{n(n-1)}{2}}|\Omega_{-n\,n}\rangle = e^{i\pi\frac{n(n-1)}{2}}\sigma_1^{-n}\underline{\sigma}_2^n S|0\rangle = (\sigma_1^{-1}\underline{\sigma}_2)^n S|0\rangle \ ,$$

we get

$$T_1|\Omega_n\rangle = |\Omega_{n+1}\rangle \ ,$$

with

$$T_1 = \sigma_1^{-1}\underline{\sigma}_2 \ .$$

The unitary operators $T_m = (T_1)^m$ generate residual gauge transformations with winding number m :

$$T_m A(x) T_m^+ = A(x) - \frac{m\pi\sqrt{2}}{eL} \ ,$$

$$T_m \psi_\alpha(x) T_m^+ = (-1)^m e^{im\frac{\pi}{L}(t+x)}\psi_\alpha(x) \ .$$

In order to construct gauge-invariant vacua (up to a phase factor), $|\theta\rangle$ vacua are introduced:

$$|\theta\rangle = \sum_{n=-\infty}^{+\infty} e^{in\theta}|\Omega_n\rangle \ ,$$

which diagonalize the T_m operators

$$T_m|\theta\rangle = e^{-im\theta}|\theta\rangle \ .$$

$|\theta\rangle$ vacua are necessary to recover the validity of cluster decomposition. Different values of θ give rise to inequivalent representations.

B. The physical subspace

We notice that the indefinite inner product $(\cdot,\cdot)$, which is preserved by the U operator, entails the presence of negative norm states. On the other hand, the scalar product $\langle\cdot,\cdot\rangle$, is not preserved by U and the relations (67), (68) no longer hold for the scalar product $\langle\cdot,\cdot\rangle$, if the adjoint operation is defined according to the indefinite inner product.

A Fock space $\mathcal{V}$ can be constructed by repeated action of creation operators $\Sigma^+_{\pm,0}$, χ^+_n γ^+_{-n} $(n<0)$ on a $|\theta\rangle$ vacuum, endowed with a positive-definite inner product $(\cdot,\cdot)_+$, which preserves eqs. (67) and such that the commutator (68) becomes canonical if χ^+ is replaced by the adjoint of χ with respect to $(\cdot,\cdot)_+$. In this case

$$(\cdot,\cdot) = (\cdot,\eta'\cdot)_+ \qquad , \quad \text{with} \quad \eta' = (-1)^{N_\chi} \ ,$$

where

$$N_\chi = -\sum_{n=-\infty}^{-1} \chi^+_n \chi_n \ .$$

If we select the physical space $\mathcal{V}_p$ with the condition (84), we can easily see that the eigenstates $|\phi_n\rangle$ of the operator

$$N_{\chi\gamma} = \sum_{n=-\infty}^{-1} \left(\gamma^+_{-n}\gamma_{-n} - \chi^+_n\chi_n\right)$$

$$N_{\chi\gamma}|\phi_n\rangle = n|\phi_n\rangle \ ,$$

have vanishing norm for $n \neq 0$ and do not contribute to physical quantities. The physical Hilbert space can be defined as usual as a quotient space.

C. Chiral anomaly and fermionic condensate

From the temporal evolution of the fermion fields, it is easy to derive the expressions for the currents

$$j^0(x) = j^1_5(x) = \frac{Q}{2L} + \frac{ie}{2\sqrt{\pi L}} \sum_{n\neq 0} \frac{k_1}{\sqrt{\omega}} \left(\Sigma_n e^{-i(\omega t - k_1 x)} - \Sigma^+_n e^{i(\omega t - k_1 x)} \right) +$$
$$+ \frac{ie}{2\sqrt{\pi L}} \sum_{n=-\infty}^{-1} \sqrt{k_0} \left[(\gamma_{-n} - \chi_n) e^{-ik_0(t+x)} - (\gamma^+_{-n} - \chi^+_n) e^{ik_0(t+x)} \right] ,$$

$$j^1(x) = j^0_5(x) = -\frac{Q_5}{2L} + \frac{ie}{2\sqrt{\pi L}} \sum_{n=-\infty}^{+\infty} \sqrt{\omega} \left(\Sigma_n e^{-i(\omega t - k_1 x)} - \Sigma^+_n e^{i(\omega t - k_1 x)} \right) -$$
$$- \frac{ie}{2\sqrt{\pi L}} \sum_{n=-\infty}^{-1} \sqrt{k_0} \left[(\gamma_{-n} - \chi_n) e^{-ik_0(t+x)} - (\gamma^+_{-n} - \chi^+_n) e^{ik_0(t+x)} \right] .$$

Using (74), (76) and (78), we get

$$\partial_\mu j^\mu_5(x) = \frac{e^2}{\pi} F(x) .$$

As a consequence the chiral charge

$$q_5 = -\int_{-L}^{L} dx j^0_5(x) = Q_5 - im^{\frac{3}{2}} \sqrt{L} \left(\Sigma_0 e^{-imt} - \Sigma^+_0 e^{imt} \right)$$

is not conserved and therefore cannot be the generator of chiral transformations. Instead, the generator is the "free" chiral charge Q_5, which commutes with the Hamiltonian. On the other hand q_5 is gauge-invariant, while Q_5 is not. The $|\theta\rangle$ vacua are not invariant under the action of Q_5, leading to a spontaneous chiral symmetry breaking

$$e^{\frac{i}{e}\alpha Q_5} |\theta\rangle = |\theta + 2\alpha\rangle .$$

Let us now introduce the fermionic condensate

$$\frac{\langle\theta|\bar{\psi}(x)\psi(x)|\theta\rangle}{\langle\theta|\theta\rangle} = \frac{1}{\langle\theta|\theta\rangle} \left[\langle\theta|\psi^+_1(x)\psi_2(x)|\theta\rangle + \langle\theta|\psi^+_2(x)\psi_1(x)|\theta\rangle \right] .$$

Using (79) and (81) it is easy to see that

$$\langle\theta|\psi^+_1(x)\psi_2(x)|\theta\rangle = \frac{e^Z}{2L} \langle\theta| e^{\frac{i\pi}{2Le}Q(t+x)} \sigma^+_1 \underline{\sigma}_2 e^{\frac{i\pi}{2Le}Q(t+x)} |\theta\rangle =$$
$$= \frac{e^Z}{2L} \langle\theta|\sigma^+_1 \underline{\sigma}_2|\theta\rangle = \frac{e^Z}{2L} e^{-i\theta} \langle\theta|\theta\rangle ,$$

$$\langle\theta|\psi^+_2(x)\psi_1(x)|\theta\rangle = \frac{e^Z}{2L} \langle\theta| e^{-\frac{i\pi}{2Le}Q(t+x)} \underline{\sigma}^+_2 \sigma_1 e^{-\frac{i\pi}{2Le}Q(t+x)} |\theta\rangle =$$
$$= \frac{e^Z}{2L} \langle\theta|\underline{\sigma}^+_2 \sigma_1|\theta\rangle = \frac{e^Z}{2L} e^{i\theta} \langle\theta|\theta\rangle$$

and, therefore,

$$\frac{\langle\theta|\bar{\psi}(x)\psi(x)|\theta\rangle}{\langle\theta|\theta\rangle} = \frac{e^Z}{L}\cos\theta \ .$$

In the decompactification limit $L \to \infty$ one recovers the well-known result [8]:

$$\lim_{L\to\infty} \frac{e^Z}{L}\cos\theta = \frac{m}{2\pi}e^\gamma \cos\theta \ ,$$

γ being the Euler–Mascheroni constant.

ACKNOWLEDGMENTS

We thank Gary McCartor for many useful discussions.

REFERENCES

[1] J. Schwinger, *Phys. Rev.* **128**, 2425 (1962).

[2] J.H. Lowenstein and J.A. Swieca, *Ann. Phys. (N.Y.)* **68**, 172 (1971).

[3] A. Bassetto, M. Dalbosco, I. Lazzizzera and R. Soldati, *Phys. Rev.* **D31**, 2012 (1985).

[4] A. Bassetto, G. Nardelli and R. Soldati, *Yang-Mills theories in algebraic non-covariant gauges* (World Scientific, Singapore, 1991).

[5] Y. Nakawaki, *Prog. Theor. Phys.* **64**, 1828 (1980); Y. Nakawaki, *Prog. Theor. Phys.* **70**, 1105 (1983).

[6] D.A. Uhlenbrock, *Commun. Math. Phys.* **4**, 64 (1967).

[7] K. Johnson, *Nucl. Phys.* **25**, 431 (1961).

[8] G. McCartor, *Int. J. Mod. Phys.* **A12**, 1091 (1997) and references therein.