

# HETEROTIC / TYPE I DUALITY AND D-BRANE INSTANTONS

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## ABSTRACT

We study heterotic/type I duality in  $d = 8, 9$  uncompactified dimensions. We consider the special (“BPS saturated”)  $\mathcal{F}^4$  and  $\mathcal{R}^4$  terms in the effective one-loop heterotic action, which are expected to be non-perturbatively exact. Under the standard duality map these translate to tree-level, perturbative and non-perturbative contributions on the type I side. We check agreement with the one-loop open string calculation, and discuss the higher-order perturbative contributions, which arise because of the mild non-holomorphicities of the heterotic elliptic genus. We put the heterotic world-sheet instanton corrections in a form that can be motivated as arising from a D-brane instanton calculation on the type I side.

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# 1 Introduction

The conjectured duality between the type I and heterotic  $SO(32)$  string theories [1, 2] occupies a special position in the web of dualities. Together with the  $SL(2, \mathbf{Z})$  symmetry of type IIB, it is the only duality that relates two string theories in their critical dimension. It can thus be analyzed in flat space-time, without the complications of curved-compactification geometry. Furthermore, it is a duality between two drastically different perturbative expansions. In heterotic theory there is a single diagram of given genus, and ultraviolet divergences are cutoff by restricting the world-sheet moduli to a fundamental domain. In type I theory, on the other hand, there are several unoriented surfaces with boundaries at any given order, and ultraviolet finiteness results from subtle cancellations of their contributions [3]. Finally, it can be argued [4] that the heterotic/type I duality is the central piece of the duality web, from which all other dualities can be derived modulo mild geometrical assumptions.

One of the aims of the present paper will be to strengthen the existing evidence [1, 2, 5, 6, 7, 8, 9] for the equivalence of the type I and heterotic  $SO(32)$  theories. We will in particular extend and sharpen the recent analysis by two of us [7] of special  $\mathcal{F}^4$  and  $\mathcal{R}^4$  terms of the effective action in  $d = 8, 9$  uncompactified dimensions<sup>1</sup>. As we will argue in the following section, there are good reasons to believe that the one-loop heterotic calculation of these couplings is *exact*. The only identifiable source of non-perturbative corrections are heterotic five-brane instantons [24]. These need a six-dimensional compact space to wrap around, and hence cannot contribute to the effective action in  $d > 4$ . The situation on the type I side is on the other hand different: first, space-time supersymmetry does not *commute* with the genus expansion, so that different terms of a superinvariant can be generated at different orders [6]. Secondly, in  $d < 9$  there are non-perturbative corrections from D1 instantons, which are the duals of wrapped heterotic world-sheets [2, 5]. Not surprisingly, the one-loop heterotic calculation translates therefore under the standard duality map [1], into a sum of tree-level, perturbative and non-perturbative contributions.

The pure-gauge one-loop corrections on the type I side have been computed previously in refs. [25, 7]. They are given by a ten-dimensional super Yang-Mills expression with a particular regularization of the (naively quadratically-divergent) decompactification limit. The same contribution on the heterotic side comes as we will explain below from a sum of infinite towers of BPS states, whose net effect is to unfold the fundamental domain of the heterotic integral into the strip. This trick is well-known from the study of

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<sup>1</sup>The structure of these higher-derivative operators in theories with sixteen supercharges has been of interest recently [10, 11, 12, 13] for a different reason: they are closely related to the velocity-dependent interactions of branes [14, 15, 16, 17, 18, 19, 20, 21, 22], which are being analyzed vigorously in testing the Matrix Theory conjecture [23].

finite-temperature partition functions [26, 27]. The decompactification limit also agrees [6], even though it is regularized differently in the two expansions: the strip is replaced by a fundamental domain on the heterotic side, and by the disk and projective-plane diagrams on the type I side. It is a very interesting question, whether an analogous geometric regularization exists for the divergent loop of eleven-dimensional supergravity [28].

In what concerns the heterotic world-sheet instanton corrections, we will put them in this paper in a form that can be plausibly motivated on the type I side. It is however an open (and we feel instructive) problem, to learn how to calculate these corrections directly from first principles. The logic can in fact be turned upside down: *assuming* heterotic/type I duality, we can use the heterotic expression as a guide to elucidate the rules of D-instanton calculus. These rules have been the subject of many interesting papers recently [29, 30, 31, 32, 33, 34, 35, 36, 37, 38]. All of them involve inevitably some guesswork, since in contrast to conventional field theory there is no functional-integral formulation at one's disposal. One particular tricky point concerns the correct counting of multiply-wrapped Euclidean branes [30, 33, 35, 37]. Not surprisingly, what we find here is that one must include all supersymmetric maps of the D-string world-sheet onto the compactification torus, modulo (local and global) reparametrizations of the former. This is of course the heterotic world-sheet prescription, which ensures in particular invariance under the  $O(d,d)$  symmetry of space-time. It is tempting to conjecture that this is the correct prescription in all instances, provided one extends reparametrization invariance to include gauge transformations, when gauge fields live on the world volume of the brane.

This paper is organized as follows: Section 2 describes the one-loop heterotic calculation of special  $\mathcal{F}^4$  and  $\mathcal{R}^4$  terms, for vacua with sixteen unbroken (real) supercharges, and its relation to the (almost holomorphic) elliptic genus [39, 40, 42, 43]. Section 3 reviews rather rapidly how the one-loop type I calculation of the pure-gauge  $\mathcal{F}^4$  terms reduces to a (regularized) ten-dimensional super Yang-Mills expression [25]. In section 4 we employ the unfolding trick to compare the heterotic and type I results in  $d = 9$  non-compact dimensions. We also explain how the mild non-holomorphicities of the elliptic genus translate to higher-order perturbative corrections on the type I side. In section 5 we move on to  $d = 8$ , where world-sheet instantons start to contribute. We express their contribution in terms of the elliptic genus of the complex structure that is induced from target space onto the string world-sheet. This form can be motivated as arising from an instantonic D-brane calculation on the type I side, as we explain in section 6. Section 7 contains some concluding remarks. A few useful formulae are collected in the appendix.

## 2 One-Loop Heterotic Thresholds

The terms that will be of interest to us are those obtained by dimensional reduction from the ten-dimensional superinvariants, whose bosonic parts read [44, 6]

$$\begin{aligned}
 I_1 &= t_8 \text{tr} \mathcal{F}^4 - \frac{1}{4} \epsilon_{10} B \text{tr} \mathcal{F}^4, & I_2 &= t_8 (\text{tr} \mathcal{F}^2)^2 - \frac{1}{4} \epsilon_{10} B (\text{tr} \mathcal{F}^2)^2 \\
 I_3 &= t_8 \text{tr} \mathcal{R}^4 - \frac{1}{4} \epsilon_{10} B \text{tr} \mathcal{R}^4, & I_4 &= t_8 (\text{tr} \mathcal{R}^2)^2 - \frac{1}{4} \epsilon_{10} B (\text{tr} \mathcal{R}^2)^2 \\
 I_5 &= t_8 (\text{tr} \mathcal{R}^2) (\text{tr} \mathcal{F}^2) - \frac{1}{4} \epsilon_{10} B (\text{tr} \mathcal{R}^2) (\text{tr} \mathcal{F}^2) \quad .
 \end{aligned} \tag{2.1}$$

These are special because they contain anomaly-cancelling CP-odd pieces. As a result anomaly cancellation fixes entirely their coefficients in both the heterotic and the type I effective actions in ten dimensions. Comparing these coefficients is not therefore a test of duality, but rather of the fact that both these theories are consistent [6]. In lower dimensions things are different: the coefficients of the various terms, obtained from a single ten-dimensional superinvariant through dimensional reduction, depend on the compactification moduli. Supersymmetry is expected to relate these coefficients to each other, but is not powerful enough so as to fix them completely. This is analogous to the case of N=1 super Yang-Mills in six dimensions: the two-derivative gauge-field action is uniquely fixed, but after toroidal compactification to four dimensions, it depends on a holomorphic prepotential which supersymmetry alone cannot determine.

On the heterotic side there are good reasons to believe that these dimensionally-reduced terms receive only one-loop corrections. To start with, this is true for their CP-odd anomaly-cancelling pieces [45]. Furthermore it has been argued in the past [46] that there exists a prescription for treating supermoduli, which ensures that space-time supersymmetry commutes with the heterotic genus expansion, at least for vacua with more than four conserved supercharges<sup>2</sup>. Thus we may plausibly assume that there are no higher-loop corrections to the terms of interest. Furthermore, the only identifiable supersymmetric instantons are the heterotic five-branes. These do not contribute in  $d > 4$  uncompactified dimensions, since they have no finite-volume 6-cycle to wrap around. Non-supersymmetric instantons, if they exist, have on the other hand too many fermionic zero modes to make a non-zero contribution. It should be noted that these arguments do not apply to the sixth superinvariant [44, 6]

$$J_0 = t_8 t_8 \mathcal{R}^4 - \frac{1}{8} \epsilon_{10} \epsilon_{10} \mathcal{R}^4, \tag{2.2}$$

which is not related to the anomaly. This receives as we will mention below both perturbative and non-perturbative corrections.

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<sup>2</sup>A notable exception are compactifications with a naively-anomalous U(1) factor [47].

The general form of the heterotic one-loop corrections to these couplings is [39, 40]

$$\mathcal{I}^{het} = -\mathcal{N} \int_F \frac{d^2\tau}{\tau_2^2} (2\pi^2\tau_2)^{d/2} \Gamma_{d,d} \mathcal{A}(\mathcal{F}, \mathcal{R}, \tau) \quad (2.3)$$

where  $\mathcal{A}$  is an (almost) holomorphic modular form of weight zero related to the elliptic genus,  $\mathcal{F}$  and  $\mathcal{R}$  stand for the gauge-field strength and curvature two-forms,  $\Gamma_{d,d}$  is the lattice sum over momentum and winding modes for  $d$  toroidally-compactified dimensions,  $F$  is the usual fundamental domain, and

$$\mathcal{N} = \frac{V^{(10-d)}}{2^{10}\pi^6} \quad (2.4)$$

is a normalization that includes the volume of the uncompactified dimensions [7]. To keep things simple we have taken vanishing Wilson lines on the  $d$ -hypertorus, so that the sum over momenta ( $p$ ) and windings ( $w$ ),

$$\Gamma_{d,d} = \sum_{p,w} e^{-\frac{\pi\tau_2}{2}(p^2+w^2/\pi^2)+i\tau_1 p \cdot w} \quad , \quad (2.5)$$

factorizes inside the integrand. Our conventions are

$$\alpha' = \frac{1}{2} \quad , \quad q = e^{2\pi i\tau} \quad , \quad d^2\tau = d\tau_1 d\tau_2 \quad (2.6)$$

while winding and momentum are normalized so that  $p \in \frac{1}{L}\mathbb{Z}$  and  $w \in 2\pi L \mathbb{Z}$  for a circle of radius  $L$ . The Lagrangian form of the above lattice sum, obtained by a Poisson resummation, reads

$$\Gamma_{d,d} = \left(\frac{2}{\tau_2}\right)^{d/2} \sqrt{\det G} \sum_{n_i, m_i \in \mathbb{Z}} e^{-\frac{2\pi}{\tau_2} \sum_{i,j} (G+B)_{ij} (m_i\tau - n_i)(m_j\bar{\tau} - n_j)} \quad (2.7)$$

with  $G_{ij}$  the metric and  $B_{ij}$  the (constant) antisymmetric-tensor background on the compactification torus. For a circle of radius  $L$  the metric is  $G = L^2$ .

The modular function  $\mathcal{A}$  inside the integrand depends on the vacuum. It is *quartic*, *quadratic* or *linear* in  $\mathcal{F}$  and  $\mathcal{R}$ , for vacua with *maximal*, *half* or a *quarter* of unbroken supersymmetries. The corresponding amplitudes have the property of saturating exactly the fermionic zero modes in a Green-Schwarz light-cone formalism, so that the contribution from left-moving oscillators cancels out [40]<sup>3</sup>. In the covariant NSR formulation this same fact follows from  $\vartheta$ -function identities. As a result  $\mathcal{A}$  should have been holomorphic in  $q$ , but the use of a modular-invariant regulator introduces some extra  $\tau_2$ -dependence [40]. As a result  $\mathcal{A}$  takes the generic form of a finite polynomial in  $1/\tau_2$ , with coefficients that have Laurent expansions with at most simple poles in  $q$ ,

$$\mathcal{A}(\mathcal{F}, \mathcal{R}, \tau) = \sum_{r=0}^{r_{max}} \sum_{n=-1}^{\infty} \frac{1}{\tau_2^r} q^n \mathcal{A}_n^{(r)}(\mathcal{F}, \mathcal{R}). \quad (2.8)$$

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<sup>3</sup>Modulo the regularization,  $\mathcal{A}$  is in fact the appropriate term in the weak-field expansion of the elliptic genus [41, 42, 43]

The poles in  $q$  come from the would-be tachyon. Since this is not charged under the gauge group, the poles are only present in the purely gravitational terms of the effective action. This can be verified explicitly in eq. (2.9) below. The  $1/\tau_2^r$  terms play an important role in what follows. They come from corners of the moduli space where vertex operators, whose fusion can produce a massless state, collide. Each pair of colliding operators contributes one factor of  $1/\tau_2$ . For maximally-supersymmetric vacua the effective action of interest starts with terms having four external legs, so that  $r_{max} = 2$ . For vacua respecting half the supersymmetries (N=1 in six dimensions or N=2 in four) the one-loop effective action starts with terms having two external legs and thus  $r_{max} = 1$ .

Much of what we will say in the sequel depends only on the above generic properties of  $\mathcal{A}$ . It will apply in particular in the most-often-studied case of four-dimensional vacua with N=2. For definiteness we will, however, focus our attention to the toroidally-compactified SO(32) theory, for which [39, 40]

$$\begin{aligned} \mathcal{A}(\mathcal{F}, \mathcal{R}, \tau) = & t_8 \operatorname{tr} \mathcal{F}^4 + \frac{1}{2^7 \cdot 3^2 \cdot 5} \frac{E_4^3}{\eta^{24}} t_8 \operatorname{tr} \mathcal{R}^4 + \frac{1}{2^9 \cdot 3^2} \frac{\hat{E}_2^2 E_4^2}{\eta^{24}} t_8 (\operatorname{tr} \mathcal{R}^2)^2 \\ & + \frac{1}{2^9 \cdot 3^2} \left[ \frac{E_4^3}{\eta^{24}} + \frac{\hat{E}_2^2 E_4^2}{\eta^{24}} - 2 \frac{\hat{E}_2 E_4 E_6}{\eta^{24}} - 2^7 \cdot 3^2 \right] t_8 (\operatorname{tr} \mathcal{F}^2)^2 \\ & + \frac{1}{2^8 \cdot 3^2} \left[ \frac{\hat{E}_2 E_4 E_6}{\eta^{24}} - \frac{\hat{E}_2^2 E_4^2}{\eta^{24}} \right] t_8 \operatorname{tr} \mathcal{F}^2 \operatorname{tr} \mathcal{R}^2 . \end{aligned} \quad (2.9)$$

Here  $t_8$  is the well-known tensor appearing in four-point amplitudes of the heterotic string [48], and  $E_{2k}$  are the Eisenstein series which are (holomorphic for  $k > 1$ ) modular forms of weight  $2k$ . Their explicit expressions are collected for convenience in the appendix. The second Eisenstein series  $\hat{E}_2$  is special, in that it requires non-holomorphic regularization. The entire non-holomorphicity of  $\mathcal{A}$  in eq. (2.9), arises through this modified Eisenstein series.

In the toroidally-compactified heterotic string all one-loop amplitudes with fewer than four external legs vanish identically [49]. Consequently eq. (2.3) gives directly the effective action, without the need to subtract one-particle-reducible diagrams, as is the case at tree level [50]. Notice also that this four-derivative effective action has infrared divergences when more than one dimensions are compactified. Such IR divergences can be regularized in a modular-invariant way with a curved background [51, 52]. This should be kept in mind, even though for the sake of simplicity we will be working in this paper with unregularized expressions.

### 3 One-loop Type-I Thresholds

The one-loop type I effective action has the form

$$\mathcal{I}^I = -\frac{i}{2} (\mathcal{T} + \mathcal{K} + \mathcal{A} + \mathcal{M}) \quad (3.1)$$

corresponding to the contributions of the torus, Klein bottle, annulus and Möbius strip diagrams. Only the last two surfaces (with boundaries) contribute to the  $\mathcal{F}^4$ ,  $(\mathcal{F}^2)^2$  and  $\mathcal{F}^2\mathcal{R}^2$  terms of the action. The remaining two pure gravitational terms may also receive contributions from the torus and from the Klein bottle. Contrary to what happens on the heterotic side, this one-loop calculation is corrected by both higher-order perturbative and non-perturbative contributions.

For the sake of completeness we review here the calculation of pure gauge terms following refs. [25, 7]. To the order of interest only the short BPS multiplets of the open string spectrum contribute. This follows from the fact that the wave operator in the presence of a background magnetic field  $\mathcal{F}_{12} = \mathcal{B}$  reads

$$\mathcal{O} = M^2 + (p_\perp)^2 + (2n + 1)\epsilon + 2\lambda\epsilon \quad (3.2)$$

where  $\epsilon \simeq \mathcal{B} + o(\mathcal{B}^3)$  is a non-linear function of the field,  $\lambda$  is the spin operator projected onto the plane (12),  $p_\perp$  denotes the momenta in the directions 034...9,  $M$  is a string mass and  $n$  labels the Landau levels. The one-loop free energy thus formally reads

$$\mathcal{I}^I = -\frac{1}{2} \int_0^\infty \frac{dt}{t} \text{Str} e^{-\frac{\pi t}{2} \mathcal{O}} \quad (3.3)$$

where the supertrace stands for a sum over all bosonic minus fermionic states of the open string, including a sum over the Chan-Paton charges, the center of mass positions and momenta, as well as over the Landau levels.

Let us concentrate on the spin-dependent term inside the integrand, which can be expanded for weak field

$$e^{-\pi t \lambda \epsilon} = \sum_{n=0}^{\infty} \frac{(-\pi t)^n}{n!} (\lambda \epsilon)^n \quad (3.4)$$

The  $n < 4$  terms vanish for every supermultiplet because of the properties of the helicity supertrace [7], while to the  $n = 4$  term only short BPS multiplets can contribute. The only short multiplets in the perturbative spectrum of the toroidally-compactified open string are the SO(32) gauge bosons and their Kaluza-Klein dependents. It follows after some straightforward algebra that the special  $\mathcal{F}^4$  terms of interest are given by the following (formal) one-loop super Yang-Mills expression

$$\mathcal{I}^I = -\frac{V^{(10-d)}}{3 \cdot 2^{12} \pi^4} \int_0^\infty \frac{dt}{t} (2\pi^2 t)^{\frac{d}{2}-1} \sum_{p \in {}^*\Gamma} e^{-\pi t p^2/2} \times t_8 \text{Tr}_{adj} \mathcal{F}^4 \quad (3.5)$$

where  ${}^*\Gamma$  is the lattice of Kaluza-Klein momenta on a  $d$ -dimensional torus, and the trace is in the adjoint representation of SO(32).

This expression is quadratically UV divergent, but in the full string theory one must remember to (a) regularize contributions from the annulus and Möbius uniformly in the *transverse* closed-string channel, and (b) to subtract the one-particle-reducible diagram corresponding to the exchange of a massless (super)graviton between two  $tr\mathcal{F}^2$  tadpoles, with the trace being here in the fundamental representation of the group.

The net result can be summarized easily, after a Poisson resummation from the open-channel Kaluza-Klein momenta to the closed-channel windings, and amounts to simply subtracting the contribution of the zero-winding sector [25, 7]. Using also the fact that  $\text{Tr}_{adj} \mathcal{F}^4 = 24 \text{tr} \mathcal{F}^4 + 3(\text{tr} \mathcal{F}^2)^2$  we thus derive the final one-loop expression on the type I side

$$\mathcal{I}^I = -\frac{V^{(10)}}{2^{10}\pi^6} \int_0^\infty \frac{dt}{t^2} \sum_{w \in \Gamma \setminus \{0\}} e^{-w^2/2\pi t} \times t_8 \left( \text{tr} \mathcal{F}^4 + \frac{1}{8} (\text{tr} \mathcal{F}^2)^2 \right). \quad (3.6)$$

The conventions for momentum and winding are the same as in the heterotic calculation of the previous section.

The calculation of the gravitational terms is more involved because we have no simple background-field method at our disposal. It can be done in principle following the method described in ref. [8]. There is one particular point we want to stress here: if the one-loop heterotic calculation is exact, and assuming that duality is valid, there should be no world-sheet instanton corrections on the type I side. Such corrections would indeed translate to non-perturbative contributions in the heterotic string [53], and we have just argued above that there should not be any. The dangerous diagram is the torus which can wrap non-trivially around the compactification manifold. The type I torus diagram is on the other hand identical to the type IIB one, assuming there are only graviton insertions. This latter diagram was explicitly calculated in eight uncompactified dimensions in ref. [54], confirming our expectations: the CP-odd invariants only depend on the complex structure of the compactification torus, but not on its Kähler structure. This is not true for the CP-even invariant  $J_0$ .

## 4 Circle Compactification

Let us begin now our comparison of the effective actions with the simplest situation, namely compactification on a circle. There are no world-sheet or D-string instanton contributions in this case, since Euclidean world-sheets have no finite-area manifold in target space to wrap around. Thus the one-loop heterotic amplitude should be expected to match with a perturbative calculation on the type I side. This sounds at first puzzling, since the heterotic theory contains infinitely more charged BPS multiplets than the type I theory in its perturbative spectrum. Indeed, one can combine any state of the  $SO(32)$  current algebra with appropriate  $S^1$ -winding and momentum, so as to satisfy the level-matching condition of physical states. The heterotic theory thus contains short multiplets in arbitrary representations of the gauge group.

The puzzle is resolved by a well-known trick, used previously in the study of string thermodynamics [26, 27], and which trades the winding sum for an unfolding of the fundamental domain into the half-strip,  $-\frac{1}{2} < \tau_1 < \frac{1}{2}$  and  $\tau_2 > 0$ . The trick works as



follows: starting with the Lagrangian form of the heterotic lattice sum,

$$(2\pi^2\tau_2)^{1/2} \Gamma_{1,1} = 2\pi L \sum_{(m,n) \in \mathbb{Z}^2} e^{-2\pi L^2 |m\tau - n|^2 / \tau_2} . \quad (4.1)$$

one decomposes any non-zero pair of integers as  $(m, n) = (jc, -jd)$ , where  $j$  is their greatest common divisor (up to a sign). We will denote the set of all relative primes  $(c, d)$ , modulo an overall sign, by  $\mathcal{S}$ . The lattice sum can thus be written as

$$(2\pi^2\tau_2)^{1/2} \Gamma_{1,1} = 2\pi L \left[ 1 + \sum_{j \in \mathbb{Z} \setminus \{0\}} \sum_{(c,d) \in \mathcal{S}} e^{-2\pi L^2 j^2 |c\tau + d|^2 / \tau_2} \right] . \quad (4.2)$$

Now the set  $\mathcal{S}$  is in one-to-one correspondence with all modular transformations,

$$\tilde{\tau} = \frac{a\tau + b}{c\tau + d} \implies \tilde{\tau}_2 = \frac{\tau_2}{|c\tau + d|^2} \quad (4.3)$$

such that  $-\frac{1}{2} < \tilde{\tau}_1 \leq \frac{1}{2}$ . Indeed the condition  $ad - bc = 1$  has a solution only if  $(c, d)$  belongs to  $\mathcal{S}$ , and the solution is unique modulo a shift and an irrelevant sign

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \pm \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \quad (4.4)$$

By choosing  $l$  appropriately we may always bring  $\tilde{\tau}$  inside the strip, which establishes the above claim.

Using the modular invariance of  $\mathcal{A}$ , we can thus suppress the sum over  $(c, d) \in \mathcal{S}$  and unfold the integration regime for the  $j \neq 0$  part of the expression. This gives

$$\mathcal{I}^{het} = -\frac{V^{(9)}L}{2^9\pi^5} \left[ \int_F \frac{d^2\tau}{\tau_2^2} \mathcal{A} + \int_{\text{strip}} \frac{d^2\tau}{\tau_2^2} \sum_{j \neq 0} e^{-2\pi L^2 j^2 / \tau_2} \mathcal{A} \right] . \quad (4.5)$$

There is one subtle point in this derivation [27]: convergence of the original threshold integral, when  $\mathcal{A}$  has a  $\frac{1}{q}$  pole<sup>4</sup>, requires that we integrate  $\tau_1$  first in the  $\tau_2 \rightarrow \infty$  region. Since constant  $\tau_2$  lines transform however non-trivially under  $\text{SL}(2, \mathbf{Z})$ , the integration over the entire strip would have to be supplemented by a highly singular prescription. The problem could be avoided if integration of the  $m \neq 0$  terms in the Lagrangian sum (i.e. those terms that required a change of integration variable) were absolutely convergent. This is the case for  $L > 1$ , so expression (4.5) should only be trusted in this region.

Let us now proceed to evaluate this expression. The fundamental domain integrals can be performed explicitly by using the formula [40]

$$\int_F \frac{d^2\tau}{\tau_2^2} (\hat{E}_2)^r \Phi_r = \frac{\pi}{3(r+1)} [c_0 - 24(r+1)c_{-1}] \quad (4.6)$$

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<sup>4</sup>(Physical) massless states do not lead to IR divergences in four-derivative operators in nine dimensions

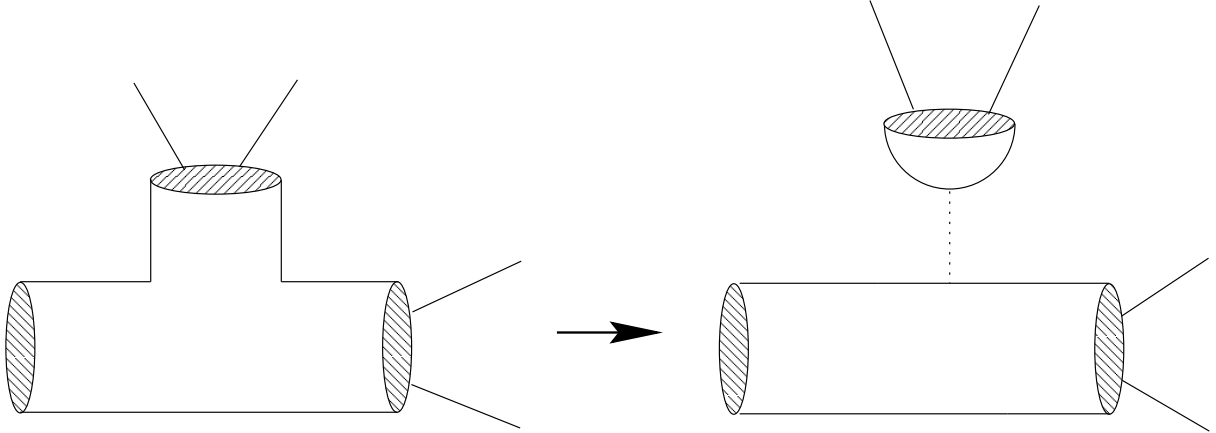


Figure 1: A type I diagram with Euler characteristic  $\chi = -1$ . This contributes to the  $(tr \mathcal{F}^2)^2$  piece of the effective action, only in degeneration limits such as the one depicted above.

where

$$\Phi_r(q) = \sum_{n=-1}^{\infty} c_n q^n \quad (4.7)$$

is any modular form of weight  $-2r$  which is holomorphic everywhere except possibly for a simple pole at zero. As for the strip integration, it picks up only the  $\mathcal{O}(q^0)$  term in the expansion of  $\mathcal{A}$ . Modulo the non-holomorphic regularization, only the SO(32) gauge bosons contribute to the elliptic genus at this order, in agreement precisely with the result of the type I side! For  $k \geq 1$  let us define more generally

$$\int_0^{\infty} \frac{d\tau_2}{\tau_2^{1+k}} \sum_{j \neq 0} e^{-2\pi L^2 j^2 / \tau_2} = \frac{2\Gamma(k)\zeta(2k)}{(2\pi L^2)^k} \equiv \frac{N_k}{L^{2k}}, \quad (4.8)$$

where  $L$  is the radius of the compactification circle. The one-loop SO(32) heterotic action takes finally the form

$$\begin{aligned} \mathcal{I}^{het} = & -\frac{V^{(10)}}{2^{10}\pi^6} \left\{ \frac{\pi}{3} \left[ \mathcal{F}^4 - \frac{1}{8} \mathcal{F}^2 \mathcal{R}^2 + \frac{1}{8} \mathcal{R}^4 + \frac{1}{32} (\mathcal{R}^2)^2 \right] + \right. \\ & + \frac{N_1}{L^2} \left[ \mathcal{F}^4 + \frac{1}{8} (\mathcal{F}^2)^2 - \frac{5}{16} \mathcal{F}^2 \mathcal{R}^2 + \frac{31}{240} \mathcal{R}^4 + \frac{19}{192} (\mathcal{R}^2)^2 \right] - \\ & \left. - \frac{5}{16\pi} \times \frac{N_2}{L^4} \left[ 3(\mathcal{F}^2)^2 - 5 \mathcal{F}^2 \mathcal{R}^2 + 2 (\mathcal{R}^2)^2 \right] + \frac{21}{64\pi^2} \times \frac{N_3}{L^6} (\mathcal{F}^2 - \mathcal{R}^2)^2 \right\}. \end{aligned} \quad (4.9)$$

To simplify notation we have written here  $\mathcal{F}^4$  instead of  $t_8 tr \mathcal{F}^4$ ,  $(\mathcal{F}^2)^2$  instead of  $t_8 tr \mathcal{F}^2 tr \mathcal{F}^2$  etc.

We have expressed the result as an expansion in inverse powers of the compactification volume. Since the heterotic/type I duality map transforms ( $\sigma$ -model) length scales as

$$L_h^2 = L_I^2 / \lambda_I \quad (4.10)$$

with  $\lambda_I$  the open-string loop counting parameter, this expansion can be translated to a genus expansion on the type I side. The Euler number of a non-orientable surface is given by  $\chi = 2 - 2g - B - C$  where  $g$  is the number of holes,  $B$  the number of boundaries and  $C$  the number of cross-caps. The leading term corresponds to the disk and projective plane diagrams and is completely fixed by ten-dimensional supersymmetry and anomaly cancellation [6]. To check this one must remember to transform the metric in both  $V^{(10)}$  and the tensor  $t_8$  appropriately. Notice that the type I sphere diagram, which is the same as in type IIB, only contributes to the  $J_0$  invariant which we are not considering here. The subleading  $\mathcal{O}(L^{-2})$  terms correspond to the annulus, Möbius strip, Klein bottle and torus diagrams, all with  $\chi = 0$ . For zero background curvature these agree with the type I calculation [7] as described in section 3.

The last two terms in the expansion (4.9) correspond to diagrams with  $\chi = -1, -2$ . These contributions must be there if the duality map of ref. [1] does not receive higher-order corrections. Such corrections could anyway always be absorbed by redefining fields on the type I side, so that if duality holds, there must exist some regularization scheme in which these higher-genus contributions do arise. These terms do on the other hand come from the boundary of moduli space. For instance the  $\chi = -1$  contribution to the  $(\mathcal{F}^2)^2$  term comes from the boundary of moduli space shown in figure 1. It could thus be conceivably eliminated in favour of some lower-dimension operators in the effective action.

It is in any case striking that a single heterotic diagram contains contributions from different topologies on the type I side. Notice in particular that the divergent  $w = 0$  term in the one-loop field theoretic calculation, regularized on the heterotic side by replacing the strip by a fundamental domain, is regularized on the type I side by replacing the annulus by the disk.

## 5 Two-torus Compactification

The next simplest situation corresponds to compactification on a two-dimensional torus. There are in this case world-sheet instanton contributions on the heterotic side, and our aim in this and the following sections will be to understand them as (Euclidean) D-string trajectory contributions on the type I side. The discussion can be extended with little effort to toroidal compactifications in lower than eight dimensions. New effects are only expected to arise in four or fewer uncompactified dimensions, where the solitonic heterotic instantons or, equivalently, the type I D5-branes can contribute.

The target-space torus is characterized by two complex moduli, the Kähler-class

$$T = T_1 + iT_2 = \frac{1}{\alpha'}(B_{89} + i\sqrt{G}) \quad (5.1)$$

and the complex structure

$$U = U_1 + iU_2 = (G_{89} + i\sqrt{G})/G_{88} , \quad (5.2)$$

where  $G_{\mu\nu}$  and  $B_{\mu\nu}$  are the  $\sigma$ -model metric and antisymmetric tensor on the heterotic side. The one-loop thresholds now read

$$\mathcal{I}^{het} = \frac{V^{(8)}}{2^9 \pi^4} \int_F \frac{d^2 \tau}{\tau_2} \Gamma_{2,2} \mathcal{A}(\mathcal{F}, \mathcal{R}, \tau) , \quad (5.3)$$

where the lattice sum takes the form [55]

$$\Gamma_{2,2} = \frac{T_2}{\tau_2} \times \sum_{M \in \text{Mat}(2 \times 2, \mathbb{Z})} e^{2\pi i T \det M} e^{-\frac{\pi T_2}{\tau_2 U_2} |(1 \ U) M \begin{pmatrix} \tau \\ -1 \end{pmatrix}|^2} . \quad (5.4)$$

The exponent in the above sum is (minus) the Polyakov action,

$$S_{\text{Polyakov}} = \frac{1}{4\pi\alpha'} \int d^2\sigma (\sqrt{g} G_{\mu\nu} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu + i B_{\mu\nu} \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu) , \quad (5.5)$$

evaluated for the topologically non-trivial mapping of the string world-sheet onto the target-space torus,

$$\begin{pmatrix} X^8 \\ X^9 \end{pmatrix} = M \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \equiv \begin{pmatrix} m_1 & n_1 \\ m_2 & n_2 \end{pmatrix} \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} . \quad (5.6)$$

The entries of the matrix  $M$  are integers, and both target-space and world-sheet coordinates take values in the (periodic) interval  $(0, 2\pi]$ . To verify the above assertion one needs to use the metrics

$$G_{\mu\nu} = \frac{\alpha' T_2}{U_2} \begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix} , \quad g^{\alpha\beta} = \frac{1}{\tau_2^2} \begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix} . \quad (5.7)$$

The Polyakov action is invariant under global reparametrizations of the world-sheet,

$$\begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} \rightarrow \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix} , \quad (5.8)$$

which transform

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} , \quad \text{and } M \rightarrow M \begin{pmatrix} d & b \\ c & a \end{pmatrix} . \quad (5.9)$$

Following Dixon, Kaplunovsky and Louis [55], we decompose the set of all matrices  $M$  into orbits of  $\text{PSL}(2, \mathbb{Z})$ , which is the group of the above transformations up to an overall sign. There are three types of orbits,

invariant :  $M = 0$

degenerate :  $\det M = 0, M \neq 0$

non – degenerate :  $\det M \neq 0$

A canonical choice of representatives for the degenerate orbits is

$$M = \begin{pmatrix} 0 & j_1 \\ 0 & j_2 \end{pmatrix} \quad (5.10)$$

where the integers  $j_1, j_2$  should not both vanish, but are otherwise arbitrary. Distinct elements of a degenerate orbit are in one-to-one correspondence with the set  $\mathcal{S}$ , i.e. with modular transformations that map the fundamental domain inside the strip, as in section 4. In what concerns the non-degenerate orbits, a canonical choice of representatives is

$$M = \pm \begin{pmatrix} k & j \\ 0 & p \end{pmatrix} \quad \text{with } 0 \leq j < k \quad , \quad p \neq 0 \quad . \quad (5.11)$$

Distinct elements of a non-degenerate orbit are in one-to-one correspondence with the fundamental domains of  $\tau$  in the upper-half complex plane.

Trading the sum over orbit elements for an extension of the integration region of  $\tau$ , we can thus express eqs. (5.3,5.4) as follows

$$\begin{aligned} \mathcal{I}^{het} = & -\frac{V^{(8)}T_2}{2^9\pi^4} \times \left\{ \int_F \frac{d^2\tau}{\tau_2^2} \mathcal{A} + \int_{\text{strip}} \frac{d^2\tau}{\tau_2^2} \sum_{(j_1, j_2) \neq (0,0)} e^{-\frac{\pi T_2}{\tau_2 U_2} |j_1 + j_2 U|^2} \mathcal{A} \right. \\ & \left. + 2 \int_{\mathbb{C}^+} \frac{d^2\tau}{\tau_2^2} \sum_{\substack{0 \leq j < k \\ p \neq 0}} e^{2\pi i T p k} e^{-\frac{\pi T_2}{\tau_2 U_2} |k\tau - j - pU|^2} \mathcal{A} \right\} \equiv \mathcal{I}_{pert} + \mathcal{I}_{inst}. \end{aligned} \quad (5.12)$$

The three terms inside the curly brackets are constant, power-suppressed and exponentially-suppressed in the large compactification-volume limit. They correspond to tree-level, higher perturbative and non-perturbative, respectively, contributions on the type I side. The discussion of the perturbative contributions follows exactly the analogous discussion in section 4. The only difference is the replacement of eq. (4.8) by

$$\begin{aligned} \int_0^\infty \frac{d\tau_2}{\tau_2^{1+k}} \sum_{(j_1, j_2) \neq (0,0)} e^{-\frac{\pi T_2}{\tau_2 U_2} |j_1 + j_2 U|^2} &= \Gamma(k) \left( \frac{U_2}{\pi T_2} \right)^k \sum_{(j_1, j_2) \neq (0,0)} |j_1 + j_2 U|^{-2k} \\ &= \frac{2\Gamma(k)\zeta(2k)}{(\pi T_2)^k} E(U, k). \end{aligned} \quad (5.13)$$

where  $E(U, k)$  are generalized Eisenstein series [56]. In the open-string channel of the type I side this takes into account properly the (double) sum over Kaluza-Klein momenta [7]. Notice that the holomorphic anomalies in  $\mathcal{A}$  lead again to higher powers of the inverse volume, which translate to higher-genus contributions on the type I side. Notice also that the  $k = 1$  term has a logarithmic infrared divergence, which must be regularized appropriately, as discussed in the introduction.

We turn now to the novel feature of eight dimensions, namely the contributions of world-sheet instantons. Plugging in the expansion (2.8) of the elliptic genus, we are lead

to consider the integrals

$$I_{n,r} = \int_{\mathbb{C}^+} \frac{d^2\tau}{\tau_2^2} e^{-\frac{\pi T_2}{\tau_2 U_2} |k\tau - j - pU|^2} \frac{1}{\tau_2^r} e^{2i\pi\tau n} \quad (5.14)$$

Doing first the (Gaussian)  $\tau_1$  integral, one finds after some rearrangements

$$I_{n,r} = \frac{1}{k} \sqrt{\frac{U_2}{T_2}} e^{2i\pi n \left(\frac{j+pU_1}{k}\right)} e^{2\pi k p T_2} \int_0^\infty \frac{d\tau_2}{\tau_2^{3/2+r}} e^{-\frac{\pi T_2}{U_2} \left(k + \frac{nU_2}{kT_2}\right)^2 \tau_2} e^{-\pi p^2 T_2 U_2 / \tau_2} \quad (5.15)$$

The  $\tau_2$  integration can now be done using the formula

$$\int_0^\infty \frac{dx}{x^{3/2+r}} e^{-ax - b/x} = \left(-\frac{\partial}{\partial b}\right)^r \sqrt{\frac{\pi}{b}} e^{-2\sqrt{ab}} \quad (5.16)$$

where  $a = \frac{\pi T_2}{U_2} \left(k + \frac{nU_2}{kT_2}\right)^2$  and  $b = \pi p^2 T_2 U_2$  are both proportional to the volume of the compactification torus. The leading term in the large-volume limit is obtained when all derivatives hit the exponential in the above expression. Using (5.16) we find

$$I_{n,r} = \frac{1}{k|p|T_2} \left(\frac{k}{|p|U_2}\right)^r e^{2\pi k(p-|p|)T_2} e^{2i\pi n \left[\frac{j+pU_1}{k} + i|p|\frac{U_2}{k}\right]} \left(1 + o\left(\frac{1}{T_2}\right)\right) \quad (5.17)$$

and plugging back into eq. (5.12) we get

$$\mathcal{I}_{inst}^{het} \simeq -\frac{2V^{(10)}}{2^{10}\pi^6} \sum_{\substack{0 \leq j < k \\ p > 0}} \frac{1}{k p T_2} e^{2\pi i T p k} \mathcal{A}\left(\frac{j+pU}{k}\right) + \text{c.c.} \quad (5.18)$$

This equality is exact for the holomorphic parts of the elliptic genus. Correction terms have the form of an order- $r_{max}$  polynomial in inverse powers of the volume, as we will discuss in a minute.

Expression (5.18) has an elegant rewriting in terms of Hecke operators  $H_N$  [57]. On any modular form  $\Phi_r(z)$  of weight  $-2r$ , the action of a Hecke operator, defined by [58]

$$H_N[\Phi_r](z) = \frac{1}{N^{2r+1}} \sum_{\substack{k,p>0 \\ kp=N}} \sum_{0 \leq j < k} k^{2r} \Phi_r\left(\frac{pz+j}{k}\right), \quad (5.19)$$

gives another modular form of the same weight. The Hecke operator is self-adjoint with respect to the inner product defined by integration of modular forms on a fundamental domain. Using the above definition one finds

$$\mathcal{I}_{inst}^{het} \simeq -\frac{2V^{(10)}}{2^{10}\pi^6} \sum_{N=1}^\infty \frac{1}{T_2} e^{2\pi i N T} H_N[\mathcal{A}](U) + \text{c.c.} \quad (5.20)$$

In the above form the result might be easier to compare with a calculation based on the heterotic matrix string theory [59].

Let us complete now the calculation, by taking into account the sub-leading terms in the large-volume limit. Using eq. (5.16) we can in fact evaluate explicitly the integrals

(5.14). After some long but straightforward algebra the correction terms can all be expressed in terms of the induced moduli

$$\mathcal{U} = \frac{j+pU}{k} \quad \text{and} \quad \mathcal{T} = kpT . \quad (5.21)$$

$$I_{n,1} \rightarrow I_{n,1} \times \left( 1 + \frac{1}{\mathcal{T}_2} (n\mathcal{U}_2 + \frac{1}{2\pi}) \right) , \quad (5.22)$$

$$I_{n,2} \rightarrow I_{n,2} \times \left( 1 + \frac{1}{\mathcal{T}_2} (2n\mathcal{U}_2 + \frac{3}{2\pi}) + \frac{1}{\mathcal{T}_2^2} (n^2\mathcal{U}_2^2 + \frac{3n\mathcal{U}_2}{2\pi} + \frac{3}{4\pi^2}) \right) . \quad (5.23)$$

These terms can be rewritten elegantly by using the operator

$$\square \equiv \mathcal{U}_2^2 \partial_{\mathcal{U}} \bar{\partial}_{\mathcal{U}} \quad (5.24)$$

This is a modular invariant operator, which annihilates holomorphic forms. The correction terms for all  $r = 0, 1, 2$  are summarized by the expression

$$\mathcal{U}_2^r e^{-2i\pi\mathcal{U}n} \left( 1 + \frac{1}{\pi\mathcal{T}_2} \square + \frac{1}{2} \frac{1}{\pi^2\mathcal{T}_2^2} (\square^2 - \square/2) \right) \mathcal{U}_2^{-r} e^{2i\pi\mathcal{U}n} . \quad (5.25)$$

The instanton sum is modified accordingly to

$$\mathcal{I}_{inst}^{het} = -\frac{2V^{(10)}}{2^{10}\pi^6} \sum_{instantons} \frac{1}{\mathcal{T}_2} e^{2\pi i\mathcal{T}} \left( 1 + \frac{1}{\pi\mathcal{T}_2} \square + \frac{1}{2} \frac{1}{\pi^2\mathcal{T}_2^2} (\square^2 - \square/2) \right) \mathcal{A}(\mathcal{U}) + \text{c.c.} . \quad (5.26)$$

One final rearrangement puts this to the form

$$\mathcal{I}_{inst}^{het} = -\frac{2V^{(10)}}{2^{10}\pi^6} \sum_{instantons} \frac{1}{\mathcal{T}_2} e^{2\pi i\mathcal{T}} \left( \sum_{s=0}^{\infty} \frac{1}{s!} \frac{1}{\mathcal{T}_2^s} (-iD)^s (\mathcal{U}_2^2 \bar{\partial}_{\mathcal{U}})^s \right) \mathcal{A}(\mathcal{U}) + \text{c.c.} . \quad (5.27)$$

where here  $D$  is the covariant derivative, which acting on a modular form  $\Phi_r$  of weight  $-2r$  gives a form of weight  $-2r + 2$ ,

$$D\Phi_r = \left( \frac{i}{\pi} \partial_{\mathcal{U}} - \frac{r}{\pi\mathcal{U}_2} \right) \Phi_r . \quad (5.28)$$

Some properties of covariant derivatives are summarized in the appendix.

The virtue of this last rewriting is that the  $s$ th operator in the sum annihilates explicitly the first  $s$  terms in the expansion of the elliptic genus in powers of  $\frac{1}{\mathcal{U}_2}$ . From the general form of  $\mathcal{A}$ , eq. (2.8) we conclude that only the terms with  $s \leq 2$  ( $s \leq 1$ ) contribute in the case of sixteen (eight) unbroken real supercharges. The modular-invariant descendants of the genus, obtained by applying the  $s$ th operator on  $\mathcal{A}$ , determine in fact the corrections to other dimension-eight operators in the effective action. The full effective action can be expressed in terms of generalized holomorphic prepotentials, a result that we will not develop further here.

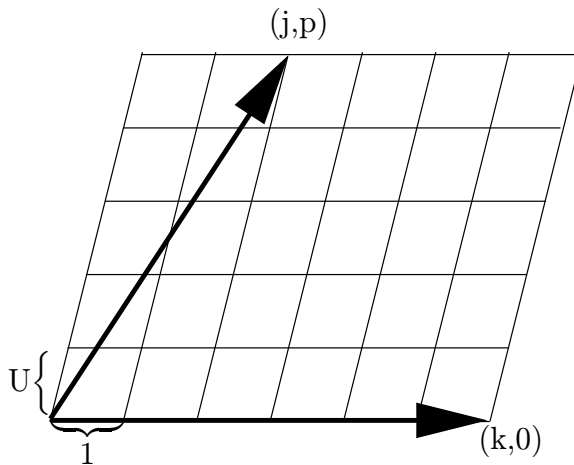


Figure 2: Embedding of the lattice  $\Gamma'$  (D1-brane) in the lattice  $\Gamma$  (compactification torus).

## 6 D-instanton Interpretation

We would now like to understand the above result from the perspective of type I string theory. The world-sheet instantons on the heterotic side map to D-brane instantons, that is Euclidean trajectories of D-strings wrapping non-trivially around the compactification torus. A Euclidean trajectory described by eq. (5.6) defines a sublattice ( $\Gamma'$ ) of the compactification lattice ( $\Gamma$ ). If  $\mathbf{e}_{i=1,2}$  are the two vectors spanning  $\Gamma$ , then  $\Gamma'$  is spanned by the vectors  $\mathbf{e}'_i = M_{ji}\mathbf{e}_j$  (figure 2). Under a change of basis for  $\Gamma$  ( $\Gamma'$ ) the matrix  $M$  transforms by left (right) multiplication with the appropriate elements of  $SL(2, \mathbf{Z})$ . Using reparametrizations of the world-sheet we can thus bring the basis  $\mathbf{e}'_i$  into the canonical form, eq. (5.11), as described in the previous section (see also figure 2).

Now the key remark is that on the heterotic world-sheet we have an induced complex structure and Kähler modulus, which for positive  $p$  are given by

$$\mathcal{U} = \frac{j + pU}{k} \quad \text{and} \quad \mathcal{T} = kpT. \quad (6.1)$$

For negative  $p$ 's, describing anti-instantons, we must take the absolute value of  $p$  and complex conjugate these expressions. One can check these facts by inspection of figure 2, or by computing explicitly the pull-backs of the metric and antisymmetric tensor field,

$$\hat{G}_{\alpha\beta} = G_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu, \quad \hat{B}_{\alpha\beta} = B_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu. \quad (6.2)$$

Notice that  $N = kp$  is the total number of times the world-sheet wraps around the compactification torus. In terms of induced moduli the instanton sum (5.18) takes the form

$$\mathcal{I}_{inst} \simeq -\frac{2V^{(10)}}{2^{10}\pi^6} \sum_{instantons} \frac{1}{\mathcal{T}_2} e^{2\pi i\mathcal{T}} \mathcal{A}(\mathcal{U}) + \text{c.c.} \quad (6.3)$$

The various terms of this expression have a simple interpretation on the type I side.



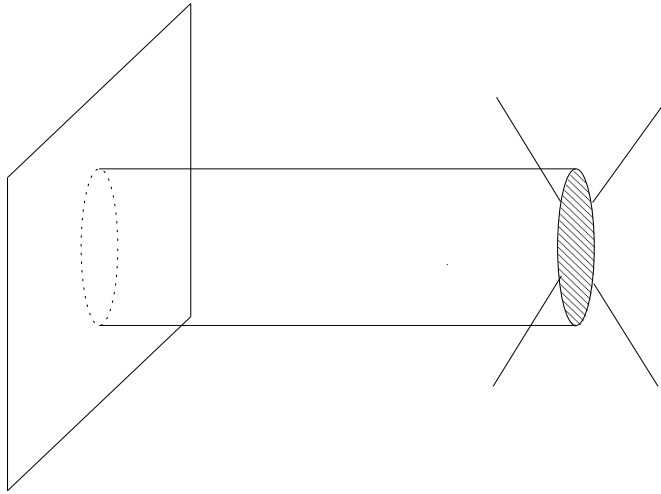


Figure 3: A D1-brane instanton correction to  $trF^4$ .

The action of a wrapped D-string is [61]

$$S_{\text{D-string}} = \frac{1}{2\pi\alpha'\lambda_I} \int d^2\sigma \sqrt{|\det\hat{G}_I|} - \frac{i}{2\pi\alpha'} \int \hat{B}_I \quad (6.4)$$

where  $B_I$  is the type I 2-form coming from the RR sector. Using the heterotic/type I map

$$\mathcal{T}_2^{\text{het}} = \mathcal{T}_2^I / \lambda_I, \quad B^{\text{het}} = B^I \quad (6.5)$$

and the fact that the world-sheet area of the D-string is  $4\pi^2\mathcal{T}_2^I$ , we see that the exponential of this Nambu-Goto action reproduces exactly the exponential in the instanton sum, eq. (6.3). The inverse factor of the volume comes from the integration of the longitudinal translation zero modes. Finally the elliptic genus of the D-brane complex structure, should come from the functional integration over the (second quantized) string fields in the instanton background. A typical diagram contributing to the  $\mathcal{F}^4$  coupling is shown in figure 3. For the purely holomorphic pieces of the elliptic genus the result is topological, so it should be expected to coincide with the heterotic  $\sigma$ -model calculation of refs. [39, 40, 42, 43]. Put differently, massive string modes and higher-order terms in the effective D-string action are expected to play no role in the calculation. It is an interesting and open problem to obtain this result directly on the type I side.

The other interesting lesson from expression (5.18) concerns the counting of distinct instanton solutions. The prescription in this case is to include all supersymmetric (holomorphic) wrappings modulo world-sheet reparametrizations of the D-brane. One may conjecture that this prescription stays valid for higher-dimensional branes, provided one also mods out world-volume gauge symmetries when present. This statement sounds obvious for world-sheet instantons on the heterotic side, but is non-trivial when considering for example the solitonic five-brane.

## 7 Concluding Remarks

Perhaps the most interesting question raised in this paper, is the calculation of the D-brane instanton contribution to the effective action. Although the topological nature of this calculation makes it plausible that the (leading) answer should be proportional to the elliptic genus, as suggested by the heterotic/type I duality, it would be very interesting to see how this will come about from explicit string diagrams. This is important, since it would open the way for doing other semiclassical D-brane instanton calculations, particularly in the background of the type I D5-brane. This latter is a heterotic zero-size instanton [62], for which the field-theoretic calculation rules remain to be found.

Another interesting check would be the explicit evaluation of the higher-order perturbative contributions. Depending on the world-sheet regularization, these could appear through corrections to lower-dimension operators, as in the case of vacua with eight unbroken supercharges [8]. We believe that the presence of these terms is enforced by supersymmetric Ward identities, and it would be interesting to derive these in detail. Similar issues actually arose in the study of D4-D0 brane scattering [19], where the background geometry seems to require a subleading two-loop open-string contribution. Finally, we find particularly intriguing the way in which string theory regularizes what seems otherwise as a field-theoretic super Yang-Mills expression. It could be very interesting to contemplate similarly the eleven-dimensional supergravity loop [28], whose regularization may admit an analogous geometric interpretation.

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## A Modular functions

Holomorphic modular forms  $\Phi_r(\tau)$  of weight  $-2r$  are invariant under  $\tau \rightarrow \tau + 1$  and transform as

$$\Phi_r \rightarrow \tau^{2r} \Phi_r \quad \text{under} \quad \tau \rightarrow -\frac{1}{\tau} \quad . \quad (\text{A.1})$$

The set of modular forms, relevant for our purposes, are the Eisenstein series

$$E_{2k} = -\frac{(2k)!}{(2\pi i)^{2k} B_{2k}} G_{2k} \quad , \quad (\text{A.2})$$

with  $B_{2k}$  the Bernoulli numbers and

$$G_{2k}(\tau) = \sum_{(m,n) \neq 0} (m\tau + n)^{-2k} \quad . \quad (\text{A.3})$$

for  $k > 1$ . For  $k = 1$  the Eisenstein series diverges. Its modular invariant regularization, denoted by a hat and used in this paper, is

$$\hat{G}_2(\tau) = \lim_{s \rightarrow 0} \sum_{(m,n) \neq 0} (m\tau + n)^{-2} |m\tau + n|^{-s} \quad . \quad (\text{A.4})$$

The (hatted) Eisenstein series are modular forms of weight  $2k$ . The ring of holomorphic modular forms is generated by  $E_4$  and  $E_6$ . If we include (non-holomorphic) covariant derivatives (to be discussed below) then the generators of this ring are  $\hat{E}_2, E_4, E_6$ .

Expressed as power series in  $q = \exp(2i\pi\tau)$ , the first few of the Eisenstein series are

$$E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n} \quad (\text{A.5})$$

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} \quad (\text{A.6})$$

$$E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} \quad . \quad (\text{A.7})$$

The modified first Eisenstein series is

$$\hat{E}_2 = E_2 - \frac{3}{\pi\tau_2} \quad . \quad (\text{A.8})$$

The Dedekind function is

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad (\text{A.9})$$

We can write the (weight 12) cusp form  $\eta^{24}$  and the modular invariant  $j$ -function in terms of  $E_4$  and  $E_6$

$$\eta^{24} = \frac{1}{2^6 \cdot 3^3} [E_4^3 - E_6^2] \quad , \quad j = \frac{E_4^3}{\eta^{24}} = \frac{1}{q} + 744 + \dots \quad (\text{A.10})$$

There is a (non-holomorphic) covariant derivative that maps modular forms of weight  $-2r$  to forms of weight  $-2r + 2$ :

$$\Phi_{r-1} = \left( \frac{i}{\pi} \partial_\tau - \frac{r}{\pi\tau_2} \right) \Phi_r = -2 \left( q\partial_q + \frac{r}{2\pi\tau_2} \right) \Phi_r \equiv D \Phi_r \quad . \quad (\text{A.11})$$

The covariant derivative satisfies the Leibnitz rule:

$$D (\Phi_{r_1} \Phi_{r_2}) = \Phi_{r_1} D\Phi_{r_2} + (D\Phi_{r_1}) \Phi_{r_2} \quad . \quad (\text{A.12})$$

Note that a double derivative on a weight  $-2r$  form is

$$D^2 \Phi_r \equiv \left( \frac{i}{\pi} \partial_\tau - \frac{r-1}{\pi\tau_2} \right) \left( \frac{i}{\pi} \partial_\tau - \frac{r}{\pi\tau_2} \right) \Phi_r \quad . \quad (\text{A.13})$$

The following formulae allow the computation of the covariant derivative of any form:

$$D \hat{E}_2 = \frac{1}{6} E_4 - \frac{1}{6} \hat{E}_2^2 \quad , \quad D E_4 = \frac{2}{3} E_6 - \frac{2}{3} \hat{E}_2 E_4 \quad (\text{A.14})$$

$$D E_6 = E_4^2 - \hat{E}_2 E_6 \quad . \quad (\text{A.15})$$

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