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A REPRESENTATION THEORETIC APPROACH TO THE WZW VERLINDE FORMULA

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Abstract
By exploring the description of chiral blocks in terms of co-invariants, a derivation of the Verlinde formula for WZW models is obtained which is entirely based on the representation theory of affine Lie algebras. In contrast to existing proofs of the Verlinde formula, this approach works universally for all untwisted affine Lie algebras. As a by-product we obtain a homological interpretation of the Verlinde multiplicities as Euler characteristics of complexes built from invariant tensors of finite-dimensional simple Lie algebras.
Our results can also be used to compute certain traces of automorphisms on the spaces of chiral blocks.

1 Introduction and summary

Spaces of chiral blocks are finite-dimensional vector spaces that arise naturally in the study of moduli spaces of flat connections over complex curves; therefore they emerge in various contexts in physics and mathematics. These vector spaces form a vector bundle over the moduli space of smooth projective complex curves with marked points. The aim of this paper is to further elucidate the structure of such spaces by using the representation theory of affine Lie algebras.

In physics, spaces of chiral blocks appear in the following guises. In three-dimensional topological Chern–Simons gauge theories with space-time equal to the product of $\mathbb{R}$ (describing time) and a complex curve, they arise as the spaces of physical states that are obtained when quantizing the theory in the temporal gauge. In two-dimensional conformal field theory the chiral blocks are the basic constituents of correlation functions, which are the quantities of prime interest in any quantum field theory. More precisely, correlation functions on closed orientable Riemann surfaces are obtained as bilinear combinations of chiral blocks, while correlators on surfaces that have boundaries and / or are unorientable can be expressed in terms of linear combinations of chiral blocks. Correlators satisfying boundary conditions that correspond to the presence of so-called D-branes are expressible in terms of chiral blocks, or close relatives thereof, as well.

In this paper we consider the spaces of chiral blocks which are associated to WZW (Wess–Zumino–Witten) conformal field theories. A WZW theory is specified by the choice of a finite-dimensional simple Lie algebra $\bar{\mathfrak{g}}$ and a positive integer k. In this case of our interest each of the marked points of the curve is labelled with an integrable weight at level k of the untwisted affine Lie algebra $\mathfrak{g}$ that is associated to $\bar{\mathfrak{g}}$ via the loop construction. The chiral blocks of WZW theories are also of interest in algebraic geometry. Namely, the space of WZW chiral blocks can be interpreted as the space of holomorphic sections in the k-th tensor power of a line bundle over the moduli space of flat $\bar{\mathfrak{g}}$-connections over the curve. Chiral WZW blocks can therefore be regarded as non-abelian generalizations of theta functions. In this paper we use the fact that the space of chiral blocks can be described in terms of co-invariants of certain integrable modules over $\mathfrak{g}$ (this characterization of chiral blocks will be reviewed in subsection 2.5). Based on this description we can apply tools from the representation theory of the affine Lie algebra $\mathfrak{g}$ to study the structure of chiral blocks. As a crucial ingredient we will introduce a suitable central extension of the so-called block algebra $\bar{\mathfrak{g}}(\mathcal{C})$, which by definition (for details see subsection 2.1) consists of the $\bar{\mathfrak{g}}$-valued algebraic functions on the (punctured) curve $\mathcal{C}$.

The most fundamental information about a sheaf of chiral blocks is its rank; there exist closed expressions for this quantity, which are commonly referred to as *Verlinde formulæ* [1,2]. Owing to factorization theorems [3–6], the problem of computing these numbers can be reduced to the case of a curve of genus zero. Accordingly it will be assumed throughout this paper that the genus is zero. The main result of this paper is a purely representation-theoretic argument for deriving the Verlinde formula for chiral blocks at genus zero. In contrast to existing algebraic proofs of the Verlinde formula (which we will briefly list in section 6) we work as long as possible in the framework of infinite-dimensional Lie algebras. This enables us to obtain the Verlinde formula in a uniform manner for *all* choices of the underlying finite-dimensional simple Lie algebra $\bar{\mathfrak{g}}$, including the cases of $\bar{\mathfrak{g}} = F_4$, E_6, E_7, E_8 for which no rigorous algebraic proof had been known so far. Another advantage of our approach is that we can derive the formula for an arbitrary number of marked points without invoking (genus-preserving) factorization rules; this can be interpreted as an independent check of these factorization rules at fixed genus zero. As a further by-product we obtain a description of the spaces of chiral blocks in terms of a complex of invariant tensors of $\bar{\mathfrak{g}}$ with vanishing Euler characteristic.

The structure of our approach is as follows. We first determine the spaces of (genus zero) two-point blocks by making use of the explicit form of the two-point block algebra; this is done in section 3. In section 4 the problem for an arbitrary number of marked points is reduced to the case of two marked points and to the calculation of certain branching rules. We then derive an integral formula for the latter and use it to find an integral formula for the dimension of the space of chiral blocks. When doing so, we have to make an assumption about the existence of a suitable completion of the modules. Finally, in section 5, we employ a generalized Poisson resummation rule to cast the integral formula into the usual form of the Verlinde formula, i.e. as a finite sum over elements of the matrix S that describes the modular transformation properties of the characters (which are the one-point blocks on the torus).

We conclude the paper in section 6 with some remarks which set our approach into the context of related work. In particular we present a complex of co-invariants that characterizes the chiral blocks, which has the property that the vanishing of its Euler characteristic is equivalent to the Verlinde formula. We also comment briefly on a possible extension to non-unitary theories. In section 2 the necessary representation-theoretic background is summarized. Some technical aspects have been relegated to appendices.

2 Chiral blocks as co-invariants

2.1 The algebras $\mathfrak{g}$ and $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$

A basic ingredient in the definition of chiral blocks is an untwisted affine Lie algebra $\mathfrak{g}$. (In conformal field theory, the semidirect sum of $\mathfrak{g}$ with the Virasoro algebra plays the rôle of the chiral symmetry algebra of WZW theories.) For the purposes of this paper we regard $\mathfrak{g}$ as the centrally extended loop algebra

$$\mathfrak{g} := \bar{\mathfrak{g}} \otimes \mathbb{C}(\!(t)\!) \oplus \mathbb{C}K\,, \tag{2.1}$$

where $\mathbb{C}(\!(t)\!)$ denotes the ring of Laurent series in some indeterminate t and $\bar{\mathfrak{g}}$ is a finite-dimensional simple Lie algebra, which is isomorphic to, and will be identified with, the horizontal (i.e. zero mode) subalgebra of $\mathfrak{g}$. The Lie bracket relations of $\mathfrak{g}$ read $[\bar{x}{\otimes}f, \bar{y}{\otimes}g] = [\bar{x},\bar{y}]{\otimes}fg + \kappa(\bar{x},\bar{y})\,\mathrm{Res}_0(\mathrm{d}f\,g)\,K$ for $\bar{x},\bar{y}\in\bar{\mathfrak{g}}$ and $f,g\in\mathbb{C}(\!(t)\!)$ (here κ denotes the Killing form of $\bar{\mathfrak{g}}$) and $[K,\bar{x}{\otimes}f]=0$, i.e. $K\in\mathfrak{g}$ is a central element. We also introduce the subalgebras

$$\mathfrak{g}^+ := \bar{\mathfrak{g}} \otimes t\,\mathbb{C}[[t]]\,, \qquad \mathfrak{g}^- := \bar{\mathfrak{g}} \otimes t^{-1}\mathbb{C}[t^{-1}] \tag{2.2}$$

of $\mathfrak{g}$ ($\mathbb{C}[t]$ and $\mathbb{C}[[t]]$ denote polynomials and arbitrary power series in t, respectively); then as a vector space $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g} \,=\, \mathfrak{g}^- \oplus \bar{\mathfrak{g}} \oplus \mathbb{C}K \oplus \mathfrak{g}^+\,. \tag{2.3}$$

The subalgebras $\mathfrak{g}^\pm$ must not be confused with the maximal nilpotent subalgebras $\mathfrak{g}_\pm$ that appear in the triangular decomposition

$$\mathfrak{g} \;=\; \mathfrak{g}_- \oplus \mathfrak{g}_\circ \oplus \mathfrak{g}_+ \tag{2.4}$$

of $\mathfrak{g}$ into the Cartan subalgebra $\mathfrak{g}_\circ$ and the nilpotent subalgebras $\mathfrak{g}_\pm$ that correspond to the positive and negative $\mathfrak{g}$-roots, respectively; one has $\mathfrak{g}_\pm \cong \mathfrak{g}^\pm \oplus \bar{\mathfrak{g}}_\pm$.

Concerning our characterization (2.1) of $\mathfrak{g}$, two remarks are in order. First, apart from defining the grading that corresponds to the power of t, in the present context the outer derivation $D = -L_0$ of the affine algebra will not play any particular rôle. Accordingly we did not include D in the definition (2.1), even though it is e.g. needed in order for $\mathfrak{g}$ to possess a non-degenerate invariant bilinear form and roots of finite multiplicity [7]. Second, strictly speaking it is the subalgebra $\mathring{\mathfrak{g}} = \bar{\mathfrak{g}} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K$ of $\mathfrak{g}$ that is generated when one allows only for Laurent polynomials rather than arbitrary Laurent series which should be referred to as the affine Lie algebra. However, as will be explained in the following subsection, we will only deal with $\mathfrak{g}$-representations for which every vector of the associated module (representation space) is annihilated by all but finitely many generators of the subalgebra $\mathring{\mathfrak{g}}^+$ of $\mathring{\mathfrak{g}}$. As a consequence, any such representation of $\mathring{\mathfrak{g}}$ can be naturally promoted to a representation of the larger algebra $\mathfrak{g}$. For the present purposes the distinction between $\mathfrak{g}$ and $\mathring{\mathfrak{g}}$ is therefore immaterial. Note that unlike in the case of $\mathring{\mathfrak{g}}$, the subalgebras $\mathfrak{g}^+$ and $\mathfrak{g}^-$ of $\mathfrak{g}$ are not isomorphic; in particular, $\mathfrak{g}^-$ is a subalgebra of $\mathring{\mathfrak{g}}$, while $\mathfrak{g}^+$ is not.

The physical states of a WZW conformal field theory can be completely described in terms of the representation theory of the affine algebra $\mathfrak{g}$; e.g. the WZW primary fields $\Phi \equiv \Phi_\Lambda$ correspond to the highest weight vectors of integrable irreducible highest weight modules $\mathcal{H}_\Lambda$ of $\mathfrak{g}$. However, the description of chiral blocks also involves another infinite-dimensional Lie algebra, which we will call the *block algebra* and denote by $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$. The block algebra is the Lie algebra of $\bar{\mathfrak{g}}$-valued algebraic functions on the (punctured) curve. More precisely, it is defined as follows. To any open subset U of the Riemann sphere $\mathbb{P}^1$ one associates the ring $\mathcal{F}(U)$ of algebraic functions on U and the vector space $\bar{\mathfrak{g}}(U) := \bar{\mathfrak{g}} \otimes \mathcal{F}(U)$; here and below all tensor products are taken over the complex numbers, unless stated otherwise. The open subset of interest to us

$$\mathbb{P}^1_{(m)} := \mathbb{P}^1 \setminus \{p_1, p_2, \ldots, p_m\}\,, \tag{2.5}$$

where $\{p_i \,|\, i=1,2,\ldots,m\}$ is a finite set of pairwise distinct non-singular points on $\mathbb{P}^1$. The points p_i correspond to the positions of the WZW primary fields Φ_{Λ_i} whose correlation function is obtained from the chiral blocks we are interested in; they are called the *insertion points* or the *parabolic points*, and $\mathbb{P}^1_{(m)}$ is referred to as a punctured Riemann sphere. The corresponding vector space

$$\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)}) := \bar{\mathfrak{g}} \otimes \mathcal{F}(\mathbb{P}^1_{(m)}) \tag{2.6}$$

becomes a Lie algebra when endowed with the natural bracket

$$[\bar{x}{\otimes}f, \bar{y}{\otimes}g] := [\bar{x},\bar{y}] \otimes fg \qquad \text{for } \bar{x},\bar{y} \in \bar{\mathfrak{g}} \text{ and } f,g \in \mathcal{F}(\mathbb{P}^1_{(m)})\,. \tag{2.7}$$

A basis $\mathcal{B}$ of the block algebra $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$ is given by

$$\mathcal{B} = \bar{\mathcal{B}} \times \big(\{z^0\} \cup \bigcup_{i=1}^{m} \{(z-z_i)^n \,|\, n \in \mathbb{Z}_{<0}\} \big) \,, \tag{2.8}$$

where $\bar{\mathcal{B}} = \{\bar{x}^a \,|\, a = 1, 2, ..., \dim \bar{\mathfrak{g}}\}$ is a basis of the Lie algebra $\bar{\mathfrak{g}}$, z is the global coordinate of $\mathbb{C} \subset \mathbb{P}^1$, and z_i are the values of z at the insertion points p_i.

As a side remark we mention that the block algebra (2.6) admits a natural central extension, which we will employ in section 3. A similar remark applies to the block algebras that arise when $\mathbb{P}^1$ is replaced by some Riemann surface of higher genus. The corresponding centrally extended Lie algebras are known as higher genus affine Lie algebras or as generalized Krichever–Novikov algebras of affine type [8, 9].

2.2 $\mathfrak{g}$-modules

For the applications we have in mind, we will need to consider modules (representation spaces) over the block algebra which come from modules over the affine Lie algebra $\mathfrak{g}$. Most of the $\mathfrak{g}$-modules that we are interested in here share the specific properties that the action of the Cartan subalgebra $\mathfrak{g}_\circ$ can be diagonalized in such a way that the resulting weight spaces are finite-dimensional when the full Cartan subalgebra (i.e. including the derivation D) is considered, and such that each weight of the module can be obtained from a finite set $\{\mu_\ell\}$ of weights by subtraction of a finite number of positive $\mathfrak{g}$-roots. The collection of all such $\mathfrak{g}$-modules forms the objects of a category, called the *category* $\mathcal{O}$ (see e.g. Chap. 9.1 of [7]); this category is closed under forming finite direct sums or tensor products, submodules and quotients. Every module V in $\mathcal{O}$ is in particular *restricted*, i.e. each element $v \in V$ is annihilated by the step operators for all but a finite number of positive $\mathfrak{g}$-roots; moreover, the subalgebra $\mathfrak{g}^+$ of $\mathfrak{g}$ acts locally nilpotently.

Among the modules in $\mathcal{O}$ there are in particular the highest weight modules V for which by definition there exists a highest weight vector v_Λ which is annihilated by $\mathfrak{g}_+$, i.e. $\mathfrak{g}_\alpha v_\Lambda = 0$ for all positive $\mathfrak{g}$-roots α, which is an eigenvector of $\mathfrak{g}_\circ$, i.e. $hv_\Lambda = \Lambda(h) v_\Lambda$ for all $h \in \mathfrak{g}_\circ$, and for which the action of $\mathfrak{g}_-$ yields the whole module, $\mathsf{U}(\mathfrak{g}_-)v_\Lambda = V$. In particular, for highest weight modules the set $\{\mu_\ell\}$ of distinguished weights contains only a single element, the highest weight Λ. Every $\mathfrak{g}$-module with highest weight Λ can be obtained as a suitable quotient from the *Verma module* $\mathcal{V}_\Lambda := \mathsf{U}(\mathfrak{g}) \otimes_{\mathsf{U}(\mathfrak{g}_+ \oplus \mathfrak{g}_\circ)} v_\Lambda$, which as a $\mathfrak{g}_-$-module is isomorphic to the free module $\mathsf{U}(\mathfrak{g}_-) \otimes_{\mathbb{C}} v_\Lambda$.

The space of physical states of a (chiral) WZW theory is the direct sum of integrable irreducible highest weight modules $\mathcal{H}_\Lambda$ of $\mathfrak{g}$ which all have one and the same eigenvalue of the central generator $K \in \mathfrak{g}$, i.e. the same level. Throughout the paper we therefore keep some fixed value k of the level. For integrability of the module $\mathcal{H}_\Lambda$, the highest weight Λ must be dominant integral, which implies that k must be a non-negative integer, and the horizontal projection $\bar{\Lambda}$ of Λ must lie in the set

$$\mathrm{P_k} := \{ \bar{\Lambda} \in L_\mathrm{w} \,|\, (\bar{\Lambda}, \bar{\alpha}^{(i)}) \geq 0 \text{ for all } i = 1, 2, ..., \mathrm{rank}\, \bar{\mathfrak{g}},\ (\bar{\Lambda}, \bar{\theta}) \leq \mathrm{k} \} \,, \tag{2.9}$$

i.e. belong to the integral weights in the closure of the dominant Weyl alcove at level k (here L_{w} denotes the weight lattice of the horizontal subalgebra $\bar{\mathfrak{g}}$, i.e. the lattice in $\bar{\mathfrak{g}}_\circ^\star$ that is spanned by the fundamental $\bar{\mathfrak{g}}$-weights, and $\bar\alpha^{(i)}$ are the simple roots and $\bar\theta$ the highest root of $\bar{\mathfrak{g}}$). Note that $\mathrm{P_k}$ is a finite set (e.g. at level 0 there is only a single integrable module, the trivial one-dimensional module $\mathcal{H}_0$ with highest weight $\Lambda = 0$).

In conformal field theory terms, the WZW chiral blocks are the chiral constituents of the m-point correlation functions

$$\mathsf{B}_{\{\Lambda_i\},\{p_i\}} = \langle\, \Phi_{\Lambda_1}(p_1)\, \Phi_{\Lambda_2}(p_2) \cdots \Phi_{\Lambda_m}(p_m)\, \rangle \tag{2.10}$$

for primary WZW fields. These fields are associated with integrable irreducible highest weight modules of $\mathfrak{g}$. As a consequence, when studying chiral blocks, one has to deal with a collection of irreducible highest weight modules $\mathcal{H}_{\Lambda_i}$ $(i = 1, 2, ... , m)$ of $\mathfrak{g}$ which are at level k and satisfy $\bar\Lambda_i \in \mathrm{P_k}$, and analyze the tensor product space (over $\mathbb{C}$)

$$\mathcal{H}_{\vec\Lambda}^{(m)} := \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2} \otimes \cdots \otimes \mathcal{H}_{\Lambda_m} . \tag{2.11}$$

This tensor product is in a natural way a module over the m-fold direct sum $\mathfrak{g}^m \equiv \mathfrak{g}\oplus\mathfrak{g}\oplus\cdots\oplus\mathfrak{g}$.

In addition to irreducible highest weight modules, we will occasionally also have to deal with other objects in the category $\mathcal{O}$, namely with the so-called *generalized* or *parabolic Verma modules*

$$\mathcal{P}_\Lambda := \mathsf{U}(\mathfrak{g}) \otimes_{\mathsf{U}(\mathfrak{p}^+)} \bar{\mathcal{H}}_{\bar\Lambda} . \tag{2.12}$$

Here $\bar{\mathcal{H}}_{\bar\Lambda}$ denotes the irreducible $\bar{\mathfrak{g}}$-module with highest weight $\bar\Lambda$, while $\mathfrak{p}^+$ is an arbitrary parabolic subalgebra of $\mathfrak{g}$. More specifically, we consider the case where

$$\mathfrak{p}^+ = \bar{\mathfrak{g}} \oplus \mathbb{C}K \oplus \mathfrak{g}^+ . \tag{2.13}$$

Then $\mathfrak{g} = \mathfrak{p}^+ \oplus \mathfrak{g}^-$ with $\mathfrak{g}^- = \bar{\mathfrak{g}} \otimes t^{-1}\mathbb{C}[t^{-1}]$ as defined in (2.2), and for any integrable $\mathfrak{g}$-weight Λ the Poincaré–Birkhoff–Witt theorem implies a natural isomorphism

$$\mathcal{P}_\Lambda \equiv \mathsf{U}(\mathfrak{g}) \otimes_{\mathsf{U}(\mathfrak{p}^+)} \bar{\mathcal{H}}_{\bar\Lambda} \;\cong\; \mathsf{U}(\mathfrak{g}^-) \otimes \bar{\mathcal{H}}_{\bar\Lambda} \tag{2.14}$$

of $\mathfrak{g}^-$-modules. In particular, $\mathcal{P}_\Lambda$ is free as a $\mathfrak{g}^-$-module.

Finally another class of modules will play a rôle, which are not restricted and hence in particular not in the category $\mathcal{O}$, but still integrable. These $\mathfrak{g}$-modules, called *evaluation modules*, are finite-dimensional and of level zero; we will encounter them in subsection 4.2.

For later reference we also present the characters of some of the $\mathfrak{g}$-modules of our interest. The character of an integrable irreducible highest weight module $\mathcal{H}_\Lambda$ is given by the Weyl–Kac character formula

$$\chi_\Lambda = \mathcal{X} \cdot \sum_{w\in W} \epsilon(w)\, \mathrm{e}^{w(\Lambda+\rho)} , \tag{2.15}$$

where the summation is over the Weyl group W of $\mathfrak{g}$, ρ is the Weyl vector of $\mathfrak{g}$ and

$$\mathcal{X} := \big(\sum_{w\in W} \epsilon(w)\, \mathrm{e}^{w(\rho)} \big)^{-1} = \mathrm{e}^{-\rho} \prod_{\alpha>0} (1-\mathrm{e}^{-\alpha})^{-\mathrm{mult}\,\alpha} . \tag{2.16}$$

Both $\mathcal{X}$ and the second factor in (2.15) are totally antisymmetric under the Weyl group W, so that χ_Λ is W-invariant. The character of the Verma module with highest weight Λ is $\mathrm{e}^{\Lambda+\rho}\mathcal{X}$. Accordingly we will refer to the quantity (2.16) as the 'universal Verma character' of $\mathfrak{g}$.

2.3 The tensor product $\mathcal{H}^{(m)}_{\vec\Lambda}$ as a $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$-module

We would now like to endow the vector space (2.11) with the structure of a $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$-module. To this end we have to choose a local holomorphic coordinate ζ_i around each insertion point p_i such that $\zeta_i(p_i)=0$; for instance, in terms of the global coordinate z of $\mathbb{C}\subset\mathbb{P}^1$, we can take $\zeta_i=z-z_i$ when $p_i\neq\infty$, while for $p_i=\infty$ we can take $\zeta_i=z^{-1}$.

For any $\bar{x}\otimes f$ with $\bar{x}\in\bar{\mathfrak{g}}$ and $f\in\mathcal{F}(\mathbb{P}^1_{(m)})$ we expand f in these local coordinates so as to obtain Laurent series $f_{p_i}=f_{p_i}(\zeta_i)$. By linear extension this defines, for each $i\in\{1,2,\dots,m\}$, a ring homomorphism from $\mathcal{F}(\mathbb{P}^1_{(m)})$ to $\mathbb{C}(\!(\zeta_i)\!)$, and hence by identifying the indeterminate t of the loop construction with the local coordinate ζ_i, the local realizations

$$x_i := \bar{x}\otimes f_{p_i} \tag{2.17}$$

can be regarded as elements of the loop algebra

$$\bar{\mathfrak{g}}_{\rm loop} = \bar{\mathfrak{g}}\otimes\mathbb{C}(\!(t)\!) = \mathfrak{g}^-\oplus\bar{\mathfrak{g}}\oplus\mathfrak{g}^+\,, \tag{2.18}$$

and, as such, as elements of the affine Lie algebra $\mathfrak{g}$ (2.3). Doing so, along with any vector $v_i\in\mathcal{H}_{\Lambda_i}$ also $R_{\Lambda_i}(x_i)v_i$, where R_{Λ_i} is the $\mathfrak{g}$-representation associated to the irreducible $\mathfrak{g}$-module $\mathcal{H}_{\Lambda_i}$, is a vector in $\mathcal{H}_{\Lambda_i}$. Moreover, even though the block algebra $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$ is not centrally extended, we can obtain a representation $R^{(m)}_{\vec\Lambda}$ of $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$ on $\mathcal{H}^{(m)}_{\vec\Lambda}$, namely by defining the action of $\bar{x}\otimes f\in\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$ on the element $v_1\otimes v_2\otimes\cdots\otimes v_m\in\mathcal{H}^{(m)}_{\vec\Lambda}$ by

$$\big(R^{(m)}_{\vec\Lambda}(\bar{x}\otimes f)\big)(v_1\otimes v_2\otimes\cdots\otimes v_m) := \sum_{i=1}^m v_1\otimes v_2\otimes\cdots\otimes R_{\Lambda_i}(x_i)v_i\otimes\cdots\otimes v_m\,. \tag{2.19}$$

To verify that this yields a $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$-representation we compute

$$\begin{aligned}\big(R^{(m)}_{\vec\Lambda}(\bar{x}\otimes f)R^{(m)}_{\vec\Lambda}(\bar{y}\otimes g)\big)&(v_1\otimes v_2\otimes\cdots\otimes v_m) - \big(R^{(m)}_{\vec\Lambda}(\bar{y}\otimes g)R^{(m)}_{\vec\Lambda}(\bar{x}\otimes f)\big)(v_1\otimes v_2\otimes\cdots\otimes v_m)\\ &= \sum_{i=1}^m v_1\otimes v_2\otimes\cdots\otimes[R_{\Lambda_i}(x_i),R_{\Lambda_i}(y_i)]v_i\otimes\cdots\otimes v_m\\ &= \sum_{i=1}^m v_1\otimes v_2\otimes\cdots\otimes R_{\Lambda_i}([\bar{x},\bar{y}]\otimes f_{p_i}g_{p_i})v_i\otimes\cdots\otimes v_m\\ &\qquad + \mathrm{k}\,\kappa(\bar{x},\bar{y})\,\Big(\sum_{i=1}^m \mathrm{Res}_{p_i}(\mathrm{d}f\,g)\Big)\,v_1\otimes v_2\otimes\cdots\otimes v_m\,,\end{aligned} \tag{2.20}$$

where in the first equality we observed that terms acting on different tensor factors of $\mathcal{H}^{(m)}_{\vec\Lambda}$ cancel and in the second equality we inserted the bracket relations of the affine Lie algebra $\mathfrak{g}$. Now the terms in (2.20) that involve the level k cancel as a consequence of the residue formula, while the other terms add up to $R^{(m)}_{\vec\Lambda}([\bar{x}\otimes f,\bar{y}\otimes g])(v_1\otimes v_2\otimes\cdots\otimes v_m)$, where the Lie bracket is the one of $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$ as defined in (2.7). Hence as promised, for any choice $\vec\zeta\equiv(\zeta_1,\zeta_2,\dots,\zeta_m)$ of local coordinates at the parabolic points we have a representation $R^{(m)}_{\vec\Lambda}=R^{(m)}_{\vec\Lambda;\vec\zeta}$ of the block algebra $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$.

Note that the cancellation of the terms coming from the central extension only works if all modules $\mathcal{H}_{\Lambda_i}$ have the same level. Under the same condition the result does not only hold for irreducible highest weight modules, but analogously also for any module on which the central element K acts as a multiple of the identity, and hence in particular for Verma modules and their quotients and for direct sums of such modules.

2.4 Co-invariants

Before we can introduce chiral blocks, we need one more ingredient, the notion of a co-invariant. For any Lie algebra $\mathfrak{h}$ we denote by $\mathsf{U}(\mathfrak{h})$ its universal enveloping algebra and by

$$\mathsf{U}^+(\mathfrak{h}) := \mathfrak{h}\,\mathsf{U}(\mathfrak{h}) \tag{2.21}$$

the augmentation ideal of $\mathsf{U}(\mathfrak{h})$. Then for any $\mathfrak{h}$-module V, the quotient module

$$\lfloor V \rfloor_{\mathfrak{h}} := V \,/\, \mathsf{U}^+(\mathfrak{h})\, V \tag{2.22}$$

is known as the *space of co-invariants* of V with respect to $\mathfrak{h}$. (Strictly speaking, in place of (2.22) one should write

$$\lfloor V \rfloor_{\mathfrak{h}} := V \,/\, R(\mathsf{U}^+(\mathfrak{h}))\, V\,, \tag{2.23}$$

where R denotes the representation by which $\mathfrak{h}$ acts on the module V. Different actions of $\mathfrak{h}$ on one and the same underlying vector space will of course give rise to different spaces of co-invariants.)

Let us list briefly a few basic facts about co-invariants (for further properties of co-invariants see Appendix A).

• The vector space $\lfloor V \rfloor_{\mathfrak{h}}$ can be characterized as the largest quotient module of V on which $\mathfrak{h}$ acts trivially.

• The space of co-invariants of the tensor product $V \otimes W$ of two $\mathfrak{h}$-modules V and W equals their tensor product over $\mathsf{U}(\mathfrak{h})$:

$$\lfloor V \otimes W \rfloor_{\mathfrak{h}} \equiv \lfloor V \otimes_{\mathbb{C}} W \rfloor_{\mathfrak{h}} = V \otimes_{\mathsf{U}(\mathfrak{h})} W\,. \tag{2.24}$$

Indeed, both spaces are by definition equal to the quotient of $V \otimes W$ by the subspace that is spanned by the vectors $(xv){\otimes}w + v{\otimes}(xw)$ with $v \in V$, $w \in W$ and $x \in \mathfrak{h}$.[1]

• The concept of co-invariants generalizes the notion of invariant tensors to the case of non-fully reducible modules. More precisely, when $\mathfrak{h}$ is a finite-dimensional semi-simple Lie algebra and V is fully reducible, then $\lfloor V \rfloor_{\mathfrak{h}}$ is isomorphic to the space of invariant tensors of V. Thus the dimension of $\lfloor V \rfloor_{\mathfrak{h}}$ is given by the number of singlets contained in V; in particular, one has the formula

$$\dim\big(\lfloor \bar{\mathcal{H}}_{\bar\Lambda} \otimes \bar{\mathcal{H}}_{\bar\Lambda'} \rfloor_{\mathfrak{h}}\big) = \delta_{\bar\Lambda',\bar\Lambda^+} \tag{2.25}$$

for the $\mathfrak{h}$-co-invariants of the tensor product of two finite-dimensional highest weight modules over $\mathfrak{h}$; a distinguished representative for the corresponding non-trivial co-invariant is given by $v_{\bar\Lambda}{\otimes}v_{-(\bar\Lambda')^+}$, where v_Λ is the highest weight vector of $\bar{\mathcal{H}}_{\bar\Lambda}$ and $v_{-(\bar\Lambda')^+}$ the lowest weight vector of $\bar{\mathcal{H}}_{\bar\Lambda'}$.

[1] In the definition of the tensor product $\otimes_{\mathsf{U}(\mathfrak{h})}$ we have to include the canonical anti-involution of $\mathsf{U}(\mathfrak{h})$ that is defined by $\mathbf{1} \mapsto \mathbf{1}$ and $x \mapsto -x$ for all $x \in \mathfrak{h}$, so as to obtain the structure of a right $\mathsf{U}(\mathfrak{h})$-module on V.

2.5 The definition of chiral blocks

We are now in a position to give a precise definition of the chiral blocks [3–5, 10, 11]. We have seen that any choice $\vec{\zeta} \equiv (\zeta_1, \zeta_2, \dots, \zeta_m)$ of local coordinates at the parabolic points leads to a representation $R^{(m)}_{\vec{\Lambda};\vec{\zeta}}$ of the block algebra $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$ on $\mathcal{H}^{(m)}_{\vec{\Lambda}}$. We start by introducing the spaces

$$B_{\vec{\zeta}} = \lfloor \mathcal{H}^{(m)}_{\vec{\Lambda}} \rfloor_{R^{(m)}_{\vec{\Lambda};\vec{\zeta}}(\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)}))} \equiv \mathcal{H}^{(m)}_{\vec{\Lambda}} \,/\, R^{(m)}_{\vec{\Lambda};\vec{\zeta}}(\mathsf{U}^+(\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})))\, \mathcal{H}^{(m)}_{\vec{\Lambda}} \tag{2.26}$$

of co-invariants. Now these spaces of course depend on the choice of local coordinates $\vec{\zeta}$. On the other hand, as already mentioned the chiral blocks play the rôle of the building blocks for correlation functions in conformal field theory; it is a fundamental physical requirement that those correlation functions should depend covariantly on the choice of coordinates. Here we will impose the natural stronger requirement that even the chiral blocks transform covariantly under a change of the local coordinates at the parabolic points.

To get rid of the coordinate dependence we make use of the group

$$U := \{u \in \mathbb{C}[[z]] \,|\, u(0) = 0,\ \tfrac{\mathrm{d}u}{\mathrm{d}z}(0) \neq 0\} \tag{2.27}$$

of local coordinate changes. The group $U^m \equiv U \times U \times \cdots \times U$ acts transitively on the set of all collections $\vec{\zeta}$ of local coordinates by sending each local coordinate ζ_i to the local coordinate $\zeta_i \circ u_i$ with $u_i \in U$. Moreover, for any $\vec{u} \in U^m$ one can find a map $\Gamma_{\vec{u}}$: $\mathcal{H}^{(m)}_{\vec{\Lambda}} \to \mathcal{H}^{(m)}_{\vec{\Lambda}}$ with the property that

$$\Gamma_{\vec{u}}\, R^{(m)}_{\vec{\Lambda};\vec{\zeta}}(x) = R^{(m)}_{\vec{\Lambda};\vec{\zeta}\circ\vec{u}}(x)\, \Gamma_{\vec{u}} \tag{2.28}$$

for all $x \in \bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$, and this map is unique up to a scalar. Now via the local realizations (2.17) (obtained by identifying the local coordinates at the parabolic points with the indeterminate of the loop construction for the relevant summand of $\mathfrak{g}^m \equiv \mathfrak{g} \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}$) one associates to any choice $\vec{\zeta}$ of local coordinates a subalgebra $\mathfrak{g}^{(m)}_{\vec{\zeta}}$ of $\mathfrak{g}^m$ which is isomorphic to the block algebra $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$. The map $\Gamma_{\vec{u}}$ has the property that it restricts to an isomorphism

$$\Gamma_{\vec{u}}: \quad \mathfrak{g}^{(m)}_{\vec{\zeta}} \mathcal{H}^{(m)}_{\vec{\Lambda}} \stackrel{\cong}{\to} \mathfrak{g}^{(m)}_{\vec{\zeta}\circ\vec{u}}\, \mathcal{H}^{(m)}_{\vec{\Lambda}} \tag{2.29}$$

and therefore induces a map on spaces of co-invariants taken with respect to the different actions of the block algebra on the space $\mathcal{H}^{(m)}_{\vec{\Lambda}}$ that are associated to the different choices of local coordinates.

We now interpret the chiral blocks as the *equivalence classes* of co-invariants under the action of the group U^m, and denote the blocks by

$$B = \lfloor \mathcal{H}^{(m)}_{\vec{\Lambda}} \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})} \equiv \mathcal{H}^{(m)}_{\vec{\Lambda}} \,/\, \mathsf{U}^+(\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)}))\, \mathcal{H}^{(m)}_{\vec{\Lambda}}\,, \tag{2.30}$$

where we think of $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$ as the abstract block algebra $\bar{\mathfrak{g}} \otimes \mathcal{F}(\mathbb{P}^1_{(m)})$ without specifying its embedding into $\mathfrak{g}^m$. For the purposes of this paper, we regard this prescription as the definition of the spaces $B \equiv B_{\{\Lambda_i\},\{p_i\}}$ of chiral blocks.

In conformal field theory terminology, taking $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$-co-invariants corresponds to the procedure of imposing the Ward identities of the current algebra on the correlation functions.

Let us also remark that in certain contexts it is actually more natural to consider the dual of the space $\lfloor\mathcal{H}_\Lambda^{(m)}\rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})}$, i.e. the space of $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$-invariants in the (algebraic) dual of $\mathcal{H}_\Lambda^{(m)}$. As already mentioned, in algebraic geometry and in quantum Chern–Simons theory, chiral blocks describe holomorphic sections in line bundles over moduli spaces; the relation to invariants is via infinite-dimensional Borel–Weil–Bott theory in which the full algebraic dual of a highest weight module appears. Here we are only interested in the dimensions, which are finite and hence are the same for invariants and co-invariants.

Manifestly, the definition (2.30) does not depend on the choice of local coordinates. Moreover, any two spaces of chiral blocks with the same number of insertion points and the same highest weights are isomorphic. (The relevant isomorphism is canonical when there is a conformal mapping of $\mathbb{P}^1$ that maps the two sets of insertion points bijectively on each other.) Later on we will, however, often work with specific representatives, i.e. prescribe a specific choice of coordinates. In this paper, we will focus on a genus zero curve for which a (quasi-) global holomorphic coordinate exists. Any choice of such a global coordinate gives a natural set of local coordinates at the parabolic points.

3 Two-point blocks

The Verlinde formula provides a closed expression for the dimensions $\dim B$ of spaces of chiral blocks. To enter the calculation of such dimensions, let us first investigate the special situation where the number of insertion points is $m=2$. Recall that by definition the chiral blocks are independent of the choice of the global holomorphic coordinate z. For definiteness, we will work with specific representatives of the blocks, namely by choosing the global coordinate in such a way that the two insertion points are at $z=0$ and at $z=\infty$. Then the block algebra is nothing but the (polynomial) loop algebra

$$\bar{\mathfrak{g}}(\mathbb{P}^1\backslash\{0,\infty\}) = \bar{\mathfrak{g}} \otimes \mathbb{C}[z,z^{-1}] =: \mathfrak{z}\,, \tag{3.1}$$

where the indeterminate of the loop construction is given by the global coordinate z on $\mathbb{C}=\mathbb{P}^1\backslash\{\infty\}$. Now as we have seen in Subsection 2.3, the tensor product $\mathcal{H}_\Lambda^{(2)} \equiv \mathcal{H}_{\Lambda_1}\otimes\mathcal{H}_{\Lambda_2}$ is a $\mathfrak{z}$-module. In order to describe the $\mathfrak{z}$-co-invariants of this module it is, however, most desirable that not only the tensor product, but both factors $\mathcal{H}_{\Lambda_1}$ and $\mathcal{H}_{\Lambda_2}$ can be regarded as modules over the block algebra individually. As is clear from the calculation (2.20), this can definitely not be achieved with the block algebra as defined so far. Therefore in addition to $\mathfrak{z}$ we introduce a central extension of $\mathfrak{z}$ by a one-dimensional center $\mathbb{C}\hat{K}$, so as to obtain a centrally extended loop algebra $\hat{\mathfrak{z}} = \mathfrak{z} \oplus \mathbb{C}\hat{K}$. The bracket relations of $\hat{\mathfrak{z}}$ read

$$[\bar{x}{\otimes}f, \bar{y}{\otimes}g] = [\bar{x},\bar{y}]\otimes fg + \kappa(\bar{x},\bar{y})\,\mathrm{Res}_0(\mathrm{d}f\, g)\,\hat{K}\,. \tag{3.2}$$

By identifying the local coordinates at 0 and ∞, respectively, with the indeterminate of the loop construction, we obtain two different embeddings $\imath_0$ and $\imath_\infty$ of $\mathfrak{z}$ as a vector space into $\mathfrak{g}$. We can extend these maps to two isomorphisms between $\hat{\mathfrak{z}}$ and $\mathfrak{g}$ as Lie algebras, which read explicitly

$$\imath_0(\bar{x}{\otimes}z^n) := \bar{x}{\otimes}t^n\,, \qquad \imath_0(\hat{K}) := K \tag{3.3}$$

($\bar{x} \in \bar{\mathfrak{g}}$, $n \in \mathbb{Z}$) and

$$\imath_\infty(\bar{x} \otimes z^n) := \bar{x} \otimes t^{-n} \,, \qquad \imath_\infty(\hat{K}) := -K \,, \tag{3.4}$$

respectively. (Note that one should carefully distinguish between the block algebra as an abstract Lie algebra and the embedding of the block algebra into $\mathfrak{g}^m$ via local coordinates. For instance, in the two-point situation considered here, the block algebra $\mathfrak{z}$ is isomorphic to the algebra $\bar{\mathfrak{g}} \otimes \mathbb{C}(t)$ of $\bar{\mathfrak{g}}$-valued Laurent *polynomials* that is contained in $\mathring{\mathfrak{g}}$. But for a generic choice of the coordinate z, $\mathfrak{z}$ is embedded into $\mathfrak{g}$ as a subalgebra that is isomorphic but not identical to $\bar{\mathfrak{g}} \otimes \mathbb{C}(t)$ and in particular involves arbitrary Laurent series, as is generically needed in order to be able to define inverses and hence to have a *group* of local coordinate changes. This is the main motivation why in this paper we regard $\mathfrak{g}$ rather than its subalgebra $\mathring{\mathfrak{g}}$ as the affine Lie algebra.) [2]

Using the embeddings (3.3) and (3.4), each of the two irreducible highest weight modules $\mathcal{H}_{\Lambda_1}$ and $\mathcal{H}_{\Lambda_2}$ of $\mathfrak{g}$ can be separately endowed with the structure of a $\hat{\mathfrak{z}}$-module. With the help of the isomorphism $\imath_0$ we can transport the triangular decomposition (2.4) of $\mathfrak{g}$ to a triangular decomposition of $\hat{\mathfrak{z}}$. (Alternatively we could choose to use $\imath_\infty$ for this purpose; this would result in a different triangular decomposition.) It then makes sense to talk about highest and lowest weight modules over the centrally extended block algebra $\hat{\mathfrak{z}}$. Let us analyse the structure of the vector spaces $\mathcal{H}_{\Lambda_1}$ and $\mathcal{H}_{\Lambda_2}$ in this spirit. One easily sees that also as a $\hat{\mathfrak{z}}$-module, $\mathcal{H}_{\Lambda_1}$ is an irreducible highest weight module of level k with highest weight Λ_1, while as a $\hat{\mathfrak{z}}$-module $\mathcal{H}_{\Lambda_2}$ is a *lowest* weight module, with lowest weight $-\Lambda_2^+$ and at level $-\mathrm{k}$. [3] We are interested in the tensor product of these two $\hat{\mathfrak{z}}$-modules, and in the co-invariants of the tensor product with respect to the action of $\mathfrak{z}$. Now we make the simple but crucial observation that the tensor product $\mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2}$ has level $\mathrm{k} + (-\mathrm{k}) = 0$ as a $\hat{\mathfrak{z}}$-module, or in other words, that the $\hat{\mathfrak{z}}$-module $\mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2}$ factorizes to a $\mathfrak{z}$-module. As a consequence, the $\mathfrak{z}$-co-invariants of the tensor product *coincide* with its $\hat{\mathfrak{z}}$-co-invariants.

We can conclude that we are left with the task of finding the $\hat{\mathfrak{z}}$-co-invariants of $\mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2}$. Now it is not difficult at all to determine these co-invariants explicitly. Namely, every vector in $\mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2}$ is a finite sum of vectors $v_1 \otimes v_2$ with $v_1 \in \mathcal{H}_{\Lambda_1}$ and $v_2 \in \mathcal{H}_{\Lambda_2}$, and as a consequence without loss of generality we can take such vectors $v_1 \otimes v_2$ as representatives for the $\hat{\mathfrak{z}}$-co-invariants. Now since $\mathcal{H}_{\Lambda_1}$ is a highest weight module over $\hat{\mathfrak{z}}$, we can write $v_1 = x_- v_+$, where v_+ is a highest weight vector of $\mathcal{H}_{\Lambda_1}$ and $x_- \in \mathsf{U}(\hat{\mathfrak{z}}_-)$ with $\hat{\mathfrak{z}}_- \equiv \imath_0^{-1}(\mathfrak{g}_-)$. Moreover, as representatives of $\hat{\mathfrak{z}}$-co-invariants, we have the equivalence

$$v_1 \otimes v_2 = x_- v_+ \otimes v_2 \;\sim\; v_+ \otimes y_- v_2 \,, \tag{3.5}$$

where $y_- = (-1)^n \, x_n x_{n-1} \cdots x_2 x_1$ when $x_- = x_1 x_2 \cdots x_n$ with $x_i \in \hat{\mathfrak{z}}_-$. Without loss of generality we can therefore assume that $v_1 = v_+$, i.e. restrict our attention to representative vectors of the

[2] As a side remark, we mention that we can extend the automorphism $\omega := \imath_0 \circ (\imath_\infty)^{-1} = \imath_\infty \circ (\imath_0)^{-1}$, which acts as $\omega(\bar{x} \otimes t^n) = \bar{x} \otimes t^{-n}$ for $\bar{x} \in \bar{\mathfrak{g}}$ and $n \in \mathbb{Z}$ and as $\omega(K) = -K$, to an automorphism that includes the outer derivation on the centrally extended loop algebra, namely via $\omega(D) := -D$.

[3] Thus the horizontal part of the weight is $-\bar{\Lambda}_2^+$, which is the lowest weight of the finite-dimensional $\bar{\mathfrak{g}}$-module with highest weight $\bar{\Lambda}_2$ or, in other words, $-\bar{\Lambda}_2^+ = \bar{w}_{\max}(\bar{\Lambda}_2)$ with $\bar{w}_{\max}$ the longest element of the Weyl group of $\bar{\mathfrak{g}}$. Also, even though $\mathcal{H}_{\Lambda_2}$ as a $\hat{\mathfrak{z}}$-module is a lowest weight module and hence in particular not in the category $\mathcal{O}$, the distinction between the algebras $\mathfrak{g}$ and $\mathring{\mathfrak{g}}$ is again immaterial, because the module is integrable and restricted.

form $v_+ \otimes v_2$. Next we write $v_2 = y_+ v_-$, where $y_+ \in \mathsf{U}(\hat{\mathfrak{z}}_+)$ and v_- is the lowest weight vector of $\mathcal{H}_{\Lambda_2}$, regarded as a $\hat{\mathfrak{z}}$-module. Since any part of y_+ that is in the augmentation ideal of $\mathsf{U}(\hat{\mathfrak{z}}_+)$ annihilates v_+, we can use the same argument as before to conclude that the only possible representative for a $\hat{\mathfrak{z}}$-co-invariant is $v_+ \otimes v_-$. Finally we impose invariance under the Cartan subalgebra of $\hat{\mathfrak{z}}$, which amounts to the requirement that $v_+ \otimes v_-$ has weight zero, i.e. that

$$\Lambda_1 + (-\Lambda_2^+) = 0 \,. \tag{3.6}$$

Thus we conclude that the space of co-invariants is zero-dimensional unless $\Lambda_1 = \Lambda_2^+$, in which case it is one-dimensional and has $v_+ \otimes v_-$ as a (distinguished) representative. This result (and its derivation, too) is in complete analogy with the formula (2.25) for co-invariants of finite-dimensional semi-simple Lie algebras.

In short, we have shown that

$$\dim B_{\{\Lambda_1,\Lambda_2\}} = \delta_{\Lambda_1,\Lambda_2^+} \,. \tag{3.7}$$

Of course, this simple result can also be obtained by various other means. The reason why we presented this particular derivation is that it sets the stage for a similar analysis that can be performed for any arbitrary value m of insertion points.

4 An integral formula for $\dim B$

4.1 The space B as a two-point co-invariant

Let us now turn to the m-point situation, where $m > 2$. Our aim is to describe the space

$$B = \lfloor \mathcal{H}_{\vec{\Lambda}}^{(m)} \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})} \equiv \lfloor \bigotimes_{i=1}^{m} \mathcal{H}_{\Lambda_i} \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})} \tag{4.1}$$

of chiral blocks. Because of the independence of B of the choice of coordinates, we can assume that the first and mth insertion points p_1 and p_m are at $z_1 = 0$ and $z_m = \infty$, respectively. It can then be shown that there is a natural isomorphism

$$B \;\cong\; \lfloor \mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)} \otimes \mathcal{H}_{\Lambda_m} \rfloor_{\mathfrak{z}} \tag{4.2}$$

of vector spaces, where $\mathfrak{z}$ is the block algebra (3.1) corresponding to only two insertion points at zero and infinity, and where $\overline{\mathcal{H}}^{(m-2)}$ stands for the tensor product

$$\overline{\mathcal{H}}^{(m-2)} := \bigotimes_{i=2}^{m-1} \bar{\mathcal{H}}_{\bar{\Lambda}_i} \tag{4.3}$$

of finite-dimensional irreducible $\bar{\mathfrak{g}}$-modules. Here $\bar{\mathcal{H}}_{\bar{\Lambda}_i}$ denotes the irreducible highest weight module of $\bar{\mathfrak{g}}$ whose highest weight $\bar{\Lambda}_i$ is the horizontal projection of the highest weight Λ_i of the $\mathfrak{g}$-module $\mathcal{H}_{\Lambda_i}$. Furthermore, $\mathfrak{z}$ is defined to act on $\overline{\mathcal{H}}^{(m-2)}$ by evaluation in the obvious manner, i.e. $\bar{x}\otimes f \in \mathfrak{z}$ with $\bar{x} \in \bar{\mathfrak{g}}$ acts as $\sum_{i=2}^{m-1} f_{p_i}(0)\cdot \mathbf{1}\otimes \cdots \otimes \mathbf{1}\otimes \bar{R}_{\bar\Lambda_i}(\bar{x})\otimes \mathbf{1}\otimes \cdots \otimes \mathbf{1}$, where $\bar{R}_{\bar\Lambda_i}$ denotes the $\bar{\mathfrak{g}}$-representation carried by $\bar{\mathcal{H}}_{\bar\Lambda_i}$ and $f_{p_i}(0) = f(z_i)$ is the value of $f \in \mathcal{F}(\mathbb{P}^1_{(m)})$ at the insertion point. [4]

The proof of the isomorphism (4.2) between the space of chiral blocks and the space $\lfloor \mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)} \otimes \mathcal{H}_{\Lambda_m} \rfloor_{\mathfrak{z}}$ of $\mathfrak{z}$-co-invariants is not difficult, but a bit lengthy (compare also proposition 2.3. of [5]). Therefore we present here only an outline of the proof and defer most details to Appendix B.

Rather than proving the isomorphism (4.2) directly, we start from a somewhat more general setting. We consider two finite sets $P := \{p_1, p_2, \dots, p_n\}$ and $Q := \{q_1, q_2, \dots, q_l\}$ of pairwise distinct points of $\mathbb{P}^1$, where we assume that the set Q is not empty, while P may be empty. To each point we associate an integrable highest weight of $\mathfrak{g}$, and introduce the tensor product

$$\tilde{\mathcal{H}} := \tilde{\mathcal{H}}_{(1)} \otimes \tilde{\mathcal{H}}_{(2)} \qquad \text{with} \qquad \tilde{\mathcal{H}}_{(1)} := \bigotimes_{i=1}^{n} \bar{\mathcal{H}}_{\bar\Lambda_i}\,, \quad \tilde{\mathcal{H}}_{(2)} := \bigotimes_{j=1}^{l} \mathcal{H}_{\Lambda'_j} \tag{4.4}$$

of irreducible modules of the horizontal subalgebra $\bar{\mathfrak{g}}$ for the points in P and of irreducible modules of the affine Lie algebra $\mathfrak{g}$ for the points in Q, respectively. Finally, we fix an additional insertion point on $\mathbb{P}^1 \setminus (P \cup Q)$, with an associated integrable highest $\mathfrak{g}$-weight Λ. Without loss of generality we can assume that this insertion point is $z = \infty$.

Moreover, in addition to $\mathcal{H}_\Lambda$ we also consider the corresponding parabolic Verma module $\mathcal{P}_\Lambda = \mathsf{U}(\mathfrak{g}) \otimes_{\mathsf{U}(\mathfrak{p}^+)} \bar{\mathcal{H}}_{\bar\Lambda} \cong \mathsf{U}(\mathfrak{g}^-) \otimes \bar{\mathcal{H}}_{\bar\Lambda}$ as introduced in formula (2.14). As we will show in Appendix B by employing special properties of co-invariants of free modules and the behavior of exact sequences under the operation of taking co-invariants, we have the isomorphisms

$$\lfloor \tilde{\mathcal{H}} \otimes \bar{\mathcal{H}}_{\bar\Lambda} \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1\setminus Q)} \;\cong\; \lfloor \lfloor \tilde{\mathcal{H}} \otimes \mathcal{P}_\Lambda \rfloor_{\mathfrak{g}^-} \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1\setminus Q)} \;\cong\; \lfloor \tilde{\mathcal{H}} \otimes \mathcal{P}_\Lambda \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1\setminus(Q\cup\{\infty\}))} \tag{4.5}$$

and

$$\lfloor \tilde{\mathcal{H}} \otimes \mathcal{P}_\Lambda \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1\setminus(Q\cup\{\infty\}))} \;\cong\; \lfloor \tilde{\mathcal{H}} \otimes \mathcal{H}_\Lambda \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1\setminus(Q\cup\{\infty\}))}\,. \tag{4.6}$$

Combining these results, we learn that there is an isomorphism

$$\lfloor \tilde{\mathcal{H}} \otimes \mathcal{H}_\Lambda \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1\setminus(Q\cup\{\infty\}))} \;\cong\; \lfloor \tilde{\mathcal{H}} \otimes \bar{\mathcal{H}}_{\bar\Lambda} \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1\setminus Q)}\,. \tag{4.7}$$

Employing this isomorphism we replace successively all but the first and last [5] affine irreducible modules $\mathcal{H}_{\Lambda_i}$ that appear in the space (4.1) of co-invariants by the corresponding irreducible modules $\bar{\mathcal{H}}_{\bar\Lambda_i}$ of the horizontal subalgebra $\bar{\mathfrak{g}}$. This way one finally arrives at the isomorphism (4.2).

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[4] Note that the result (4.2) implies in particular that co-invariants taken with respect to different actions of the block algebra, corresponding to different sets of local coordinates, are isomorphic. Namely, after replacing the relevant affine module by the corresponding finite-dimensional module the only data entering the description of the block are the values of the functions $f \in \mathcal{F}(\mathbb{P}^1_{(m)})$ at the insertion point; manifestly these values do not depend on the choice of local coordinates.

[5] If we so wished, we could also replace one of the remaining $\mathfrak{g}$-modules $\mathcal{H}_{\Lambda_1}$ and $\mathcal{H}_{\Lambda_m}$ by the corresponding $\bar{\mathfrak{g}}$-module, thereby obtaining co-invariants with respect to the one-point block algebra $\bar{\mathfrak{g}}(\mathbb{P}^1\setminus\{\infty\}) = \bar{\mathfrak{g}}(\mathbb{C}) = \bar{\mathfrak{g}} \otimes \mathbb{C}[[z]] \cong \bar{\mathfrak{g}} \oplus \mathfrak{g}^+$. In contrast, according to the discussion in Appendix B, it is not possible to apply this manipulation also to *both* of the remaining $\mathfrak{g}$-modules.

4.2 Branching rules

Now we realize that in view of the isomorphism (4.2) we are actually in a situation very similar to the one investigated in section 3. The extension to the general case amounts to take properly into account the additional finite-dimensional $\bar{\mathfrak{g}}$-modules. We will argue that this can be achieved by means of suitable branching rules for completed tensor products involving finite-dimensional $\hat{\mathfrak{z}}$-modules. First we note that all elements of $\mathfrak{z}$ are represented on the tensor product $\overline{\mathcal{H}}^{(m-2)}$ of finite-dimensional $\bar{\mathfrak{g}}$-modules by evaluation at the respective insertion points $p_2, p_3, \dots, p_{m-1}$, so that no central term can arise in their Lie brackets. Therefore the action of $\mathfrak{z}$ on $\overline{\mathcal{H}}^{(m-2)}$ can be extended to an action of $\hat{\mathfrak{z}}$ with $\hat{K}$ represented by zero. We call the thus obtained finite-dimensional level-zero $\hat{\mathfrak{z}}$-module $\overline{\mathcal{H}}^{(m-2)}$, as well as any other $\hat{\mathfrak{z}}$-module that is obtained in an analogous manner from a finite-dimensional $\bar{\mathfrak{g}}$-module, an *evaluation module*. Evaluation modules are not restricted, hence not in the category $\mathcal{O}$, but the step operators of $\hat{\mathfrak{z}}$ still act locally nilpotently.

As a consequence, the tensor product $\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)}$ of $\hat{\mathfrak{z}}$-modules isn't an object in the category $\mathcal{O}$ either; but it is easily checked that this $\hat{\mathfrak{z}}$-module is still integrable (this fact is also used in the proof of (4.6) in Appendix B), and that the central element $\hat{K}$ acts as multiplication by k. Not surprisingly, it turns out to be rather difficult to compute the co-invariants of the module $\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)}$ directly. In order to determine the co-invariants, we would therefore like to replace $\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)}$ by a direct sum of irreducible highest weight modules of $\hat{\mathfrak{z}}$. To this end we work with a completion $(\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)})^-$ of $\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)}$. More specifically, we assume that there exists a suitable a completion such that $(\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)})^-$ becomes reducible and can be written as a direct sum

$$(\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)})^- \;\cong\; \mathcal{J} \oplus \bigoplus_{\ell} \mathcal{H}_{\mu_\ell}\,, \tag{4.8}$$

where each of the summands $\mathcal{H}_{\mu_\ell}$ is an integrable highest weight $\hat{\mathfrak{z}}$-module, while $\mathcal{J}$ is a direct sum of $\hat{\mathfrak{z}}$-modules which are irreducible, but are not weight modules. (Note that for the uncompleted module such a decomposition typically does not exist.)

Now the tensor product $(\bigoplus_\ell \mathcal{H}_{\mu_\ell}) \otimes \mathcal{H}_{\Lambda_m} \subseteq (\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)})^- \otimes \mathcal{H}_{\Lambda_m}$ is at level $\mathrm{k}+(-\mathrm{k}) = 0$ as a $\hat{\mathfrak{z}}$-module, and hence by repeating the arguments of section 3 one finds that its $\mathfrak{z}$-co-invariants are the same as its $\hat{\mathfrak{z}}$-co-invariants. The $\hat{\mathfrak{z}}$-co-invariants in turn can be computed in exactly the same manner as in the two-point situation of section 3. More precisely, among the integrable highest weight modules in the decomposition (4.8) only those modules contribute to the $\hat{\mathfrak{z}}$-co-invariants whose highest weight is Λ_m^+, and each of these yields precisely one independent co-invariant. In addition, the same analysis indicates that the submodule $\mathcal{J}$ of $(\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)})^-$ does not contribute any co-invariants, and also that the $\hat{\mathfrak{z}}$-co-invariants of $(\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)}) \otimes \mathcal{H}_{\Lambda_m}$ are the same as those of $(\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)})^- \otimes \mathcal{H}_{\Lambda_m}$.

Unfortunately, so far we cannot describe a topology on the module $\mathcal{H}_{\Lambda_1} \otimes \bar{\mathcal{H}}^{(m-2)}$ such that in the completion with respect to that topology the relation (4.8) holds. The existence of such a topology should therefore be regarded as a conjecture. In the sequel we will assume that this conjecture is valid; then the computation of the dimension $\dim B$ of the space (4.1) of co-invariants merely amounts to determining the integrable highest weight modules in the decomposition (4.8) of $\mathcal{H}_{\Lambda_1}\otimes\overline{\mathcal{H}}^{(m-2)}$. Rewriting the branching rule (4.8) as

$$\mathcal{H}_{\Lambda_1}\otimes\overline{\mathcal{H}}^{(m-2)} \;\cong\; \mathcal{J}\oplus\bigoplus_{\mu\in\mathrm{P_k}} N_{\Lambda_1;\bar\Lambda_2\ldots\bar\Lambda_{m-1};\mu}\,\mathcal{H}_\mu\,, \tag{4.9}$$

our arguments amount to the formula

$$\dim B_{\{\Lambda_i\},\{p_i\}} = N_{\Lambda_1;\bar\Lambda_2\ldots\bar\Lambda_{m-1};\Lambda_m^+}\,. \tag{4.10}$$

Notice that on the module $\mathcal{H}_{\Lambda_1}\otimes\bar{\mathcal{H}}^{(m-2)}$ (as on any other evaluation module of the affine Lie algebra, except for the trivial module) we do not have an action of the derivation in the Cartan subalgebra of the affine Lie algebra $\mathfrak{g}$. However, a posteriori, we can define such an action on a submodule of the completion of $\mathcal{H}_{\Lambda_1}\otimes\bar{\mathcal{H}}^{(m-2)}$, namely on the direct sum of the integrable highest weight modules $\mathcal{H}_\mu$ over the centrally extended block algebra $\hat{\mathfrak{z}}$. We will assume from now on that this has been done and that the eigenvalues of the derivation on the highest weight vectors are chosen in the standard way such that the characters furnish a module of SL(2,$\mathbb{Z}$). This, together with the assumption that the module $\mathcal{J}$ is not a weight module and hence does not contribute to characters, opens the possibility to compute the branching coefficients $N_{\Lambda_1;\bar\Lambda_2\ldots\bar\Lambda_{m-1};\mu}$ by manipulating characters, and indeed this will be achieved in the next subsections. Notice that the character of $\bigoplus_{\mu\in\mathrm{P_k}} N_{\Lambda_1;\bar\Lambda_2\ldots\bar\Lambda_{m-1};\mu}\,\mathcal{H}_\mu \subseteq (\mathcal{H}_{\Lambda_1}\otimes\overline{\mathcal{H}}^{(m-2)})^-$ as introduced in formula (4.9) is given by the product of the character χ_{Λ_1} of $\mathcal{H}_{\Lambda_1}$ and the $\bar{\mathfrak{g}}$-characters $\bar\chi_{\bar\Lambda_2}$ to $\bar\chi_{\bar\Lambda_{m-1}}$,

$$\chi_{\oplus_\mu N_{\Lambda_1;\bar\Lambda_2\ldots\bar\Lambda_{m-1};\mu}\mathcal{H}_\mu} = \chi_{\Lambda_1}(h)\cdot\bar\chi_{\bar\Lambda_2}\bar\chi_{\bar\Lambda_3}\cdots\bar\chi_{\bar\Lambda_{m-1}}(\bar h)\,, \tag{4.11}$$

where it is understood that the eigenvalues of the derivation are chosen as explained above.

4.3 A projection formula

Our task is now to find the integrable highest weight modules in the decomposition (4.8). In this subsection we present a projection formula which can be used to that effect. We start by introducing a certain linear functional P_Λ on the space of class functions φ on G, where G is the connected, simply connected compact real Lie group whose Lie algebra is the compact real form of the horizontal subalgebra $\bar{\mathfrak{g}}$ of $\mathfrak{g}$. We denote by T the maximal torus of G that corresponds to the chosen Cartan subalgebra $\bar{\mathfrak{g}}_\circ$ of $\bar{\mathfrak{g}}$. Thus the elements t of T are group elements of the form

$$t(\bar h) = \mathrm{E\scriptstyle XP}(2\pi\mathrm{i}\bar h) \quad\text{with}\;\; \bar h\in(\bar{\mathfrak{g}}_\circ)_{\mathbb{R}}\,, \tag{4.12}$$

and the normalized Haar measure $\mathrm{d}t$ on T is the flat measure $\int_T\mathrm{d}t = \prod_{i=1}^{\mathrm{rank}\,\bar{\mathfrak{g}}}(\int_0^1\mathrm{d}\bar h^i)$.

We then define, for each integrable weight Λ of $\mathfrak{g}$, the linear functional P_Λ by

$$\mathrm{P}_\Lambda[\varphi] := \int_T\mathrm{d}t\;\big(\mathcal{X}^{-1}\mathrm{e}^{-(\Lambda+\rho)}\big)(2\pi\mathrm{i}h)\cdot\varphi(\bar h)\,. \tag{4.13}$$

Here $\mathcal{X}$ is the 'universal Verma character' that appears in the Weyl–Kac character formula (2.15). Also, concerning the relation between elements of $\mathfrak{g}_\circ$ and of $\bar{\mathfrak{g}}_\circ$ and between elements of their dual spaces we adhere to the following conventions. The elements $\bar h \in \bar{\mathfrak g}_\circ$ and $h \in \mathfrak g_\circ$ are related by

$$h = \bar h - \tau L_0 + \varpi K \tag{4.14}$$

with $\tau, \varpi \in \mathbb{C}$, and the analogous decomposition of $\mathfrak g$-weights reads

$$\mu = \bar\mu + \mu^0 \Lambda_{(0)} + \nu\delta =: \bar\mu + \mu^\natural\,, \tag{4.15}$$

where $\Lambda_{(0)}$ is the zeroth fundamental $\mathfrak g$-weight and δ the null root of $\mathfrak g$. Thus the $\mathfrak g$-weight $\mu \in \mathfrak g_\circ^\star$ acts on $h \in \mathfrak g_\circ$ as

$$\mu(h) = \bar\mu(\bar h) + \mu^\natural(-\tau L_0 + \varpi K) = \bar\mu(\bar h) + \varpi\mu^0 + \nu\tau\,. \tag{4.16}$$

Notice that strictly speaking, φ has to be a functional on the full Cartan subalgebra of the affine Lie algebra $\mathfrak g$, including the derivation; we will not always explicitly display this dependence in the sequel.

Let us now consider the special case that φ is the character of a Verma module, $\varphi \equiv \varphi(\bar h;\tau,\varpi) = \mathcal{X}_{\Lambda'}(2\pi\mathrm{i}h)$; we get immediately

$$\begin{aligned} \mathrm{P}_\Lambda[\mathcal{X}_{\Lambda'}] &= \int_T \mathrm{d}t\, \big(\mathcal{X}^{-1}\mathrm{e}^{-(\Lambda+\rho)}\big)(2\pi\mathrm{i}h)\cdot\big(\mathcal{X}\mathrm{e}^{\Lambda'+\rho}\big)(2\pi\mathrm{i}h) \\ &= \int_T \mathrm{d}t\; \mathrm{e}^{\Lambda'-\Lambda}(2\pi\mathrm{i}h) \\ &= \mathrm{e}^{2\pi\mathrm{i}(\Lambda'-\Lambda)^\natural(-\tau L_0+\varpi K)}\,\delta_{\bar\Lambda,\bar\Lambda'} \equiv \mathrm{e}^{2\pi\mathrm{i}\varpi((\Lambda')^0-\Lambda^0)}\mathrm{e}^{2\pi\mathrm{i}\tau(\nu_{\Lambda'}-\nu_\Lambda)}\,\delta_{\bar\Lambda,\bar\Lambda'}\,. \end{aligned} \tag{4.17}$$

As a consequence, the operator P_Λ picks the Λ-isotypic component in the resolution of any module. In particular, we can analyze the irreducible characters, i.e. take $\varphi = \chi_\Lambda(2\pi\mathrm{i}h)$. By the Weyl–Kac character formula we know that $\chi_\Lambda = \sum_{w\in W}\epsilon(w)\,\mathcal{X}_{w(\Lambda+\rho)-\rho}$, and hence (4.17) tell us that

$$\begin{aligned} \mathrm{P}_\Lambda[\chi_{\Lambda'}] &= \sum_{w\in W}\epsilon(w)\,\mathrm{P}_\Lambda[\mathcal{X}_{w(\Lambda'+\rho)-\rho}] \\ &= \sum_{w\in W}\epsilon(w)\,\mathrm{e}^{2\pi\mathrm{i}[w(\Lambda'+\rho)-(\Lambda+\rho)]^\natural(-\tau L_0+\varpi K)}\,\delta_{\hat w(\bar\Lambda'+\bar\rho),\bar\Lambda+\bar\rho}\,. \end{aligned} \tag{4.18}$$

Here $\hat w$ denotes the induced affine action of $w \in W$ on the weight space $\bar{\mathfrak g}_\circ^\star$ of the horizontal subalgebra, i.e. the action of w on the horizontal part $\bar\mu$ of a $\mathfrak g$-weight $\mu \in \mathfrak g_\circ^\star$ yields the horizontal part of the w-transformed $\mathfrak g$-weight:

$$\hat w(\bar\mu) = \overline{w(\mu)}\,. \tag{4.19}$$

In the following, for notational simplicity we suppress the hat-symbol and just write $w(\bar\mu)$ for the horizontal projection $\overline{w(\mu)}$ of $w(\mu)$.

Next we restrict to the case of our interest, where both Λ and Λ' are integrable weights, and use the fact that the Weyl group W of $\mathfrak g$ acts freely on the interior of the Weyl alcoves. This property of W implies that in fact only a single Weyl group element, namely $w = id$, can give a non-zero contribution to the sum; as a consequence (4.18) reduces to

$$\mathrm{P}_{\Lambda'}[\chi_\Lambda] = \mathrm{e}^{2\pi\mathrm{i}(\Lambda-\Lambda')^\natural(-\tau L_0+\varpi K)}\cdot\delta_{\bar\Lambda,\bar\Lambda'}\,. \tag{4.20}$$

Moreover, we are only interested in weights Λ at some fixed value k of the level. Accordingly we now further restrict our attention to functionals $\mathrm{P}_{\Lambda'}$ for which Λ' is at level k as well; then $\bar\Lambda=\bar\Lambda'$ implies that also $\Lambda^0=\Lambda'^0$. Finally we require that the δ-component of the weights Λ and Λ' is already specified by the horizontal part of the weights (e.g. that it is given by $-\Delta_\Lambda=-(\bar\Lambda,\bar\Lambda+2\bar\rho)/4(\mathrm{k}+\mathrm{h})$, the natural value in conformal field theory; owing to the fact that for all $\nu\in\mathbb{C}$, $\mathcal{H}_\Lambda$ and $\mathcal{H}_{\Lambda+\nu\delta}$ are isomorphic as $\mathfrak{g}$-modules, this does not result in any loss of generality). Then the equality of $\bar\Lambda$ and $\bar\Lambda'$ also enforces equality of the δ-components, so it follows that in fact we not only have $\bar\Lambda=\bar\Lambda'$, but even $\Lambda=\Lambda'$.

Summarizing, we have proven that the projection formula

$$\mathrm{P}_{\Lambda'}[\chi_\Lambda] \equiv \int_T \mathrm{d}t\,\big(\mathcal{X}^{-1}\mathrm{e}^{-(\Lambda'+\rho)}\,\chi_\Lambda\big)(2\pi\mathrm{i}h) = \delta_{\Lambda,\Lambda'} \tag{4.21}$$

holds for all integrable $\mathfrak{g}$-weights Λ and Λ' at fixed level whose δ-components depend only on the horizontal part. More generally, acting with the operator P_Λ on the character of a direct sum of integrable highest weight modules provides us with the multiplicity of the module $\mathcal{H}_\Lambda$ in that sum. [6]

Note that when we interpret the parameter τ as a complex variable (rather than some formal indeterminate), in order for the character $\chi_\Lambda(2\pi\mathrm{i}h)$ to be a convergent power series in $\exp(2\pi\mathrm{i}\tau)$, it must be required that τ has positive imaginary part. However, the result (4.21) tells us e.g. that the expression

$$\mathrm{P}_{\Lambda'}[\chi_\Lambda] = \sum_{w\in W}\epsilon(w)\,\mathrm{e}^{2\pi\mathrm{i}[w(\Lambda+\rho)-(\Lambda'+\rho)]^\natural(-\tau L_0+\varpi K)}\int_T \mathrm{d}t\;\mathrm{e}^{2\pi\mathrm{i}(w(\bar\Lambda'+\bar\rho)-(\bar\Lambda+\bar\rho))(\bar h)} \tag{4.22}$$

is independent of τ, so that in the present context this restriction is in fact rather irrelevant. Similarly, in various formulæ that we will deal with below a priori the imaginary part of τ must be kept strictly positive in order that the affine characters, respectively the sums over the infinite Weyl group W, converge, and it is not guaranteed that the limit $\tau\to 0$ of the whole expression exists.

4.4 The integral formula

We now employ the formula (4.21) to count the number of integrable highest weight modules with highest weight Λ_m^+ in the completed tensor product $(\mathcal{H}_{\Lambda_1}\otimes\overline{\mathcal{H}}^{(m-2)})^-$. From our assumption (4.11) about the characters we deduce with the help of the projection formula (4.21) that the branching coefficients $N_{\Lambda_1;\bar\Lambda_2\ldots\bar\Lambda_{m-1};\mu}$ appearing introduced in equation (4.9) satisfy

$$N_{\Lambda_1;\bar\Lambda_2\ldots\bar\Lambda_{m-1};\Lambda_m^+} = \int_T \mathrm{d}t\,\big(\mathcal{X}^{-1}\mathrm{e}^{-\Lambda_m^+-\rho}\big)(2\pi\mathrm{i}h)\cdot\chi_{\Lambda_1}(2\pi\mathrm{i}h)\,\bar\chi_{\bar\Lambda_2}\bar\chi_{\bar\Lambda_3}\cdots\bar\chi_{\bar\Lambda_{m-1}}(2\pi\mathrm{i}\bar h)\,. \tag{4.23}$$

We now manipulate the right hand side of this identity as follows. We first insert the Weyl–Kac character formula (2.15) for χ_{Λ_1}; next we introduce a dummy summation $|\overline{W}|^{-1}\sum_{\bar w\in\overline{W}}$ and

[6] Incidentally, the identity (4.21) can also be used to derive an integral formula for branching functions for embeddings of affine Lie algebras. Such expressions for branching functions have also been obtained in [12,13].

substitute $w \mapsto \bar{w}^{-1} w$, where we consider the Weyl group $\overline{W}$ of $\bar{\mathfrak{g}}$ as canonically embedded in the Weyl group W of $\mathfrak{g}$; and finally we set $\bar{h} = \bar{w}^{-1}(\bar{h}')$ and use invariance of the measure dt under the transformation from $\bar{h}$ to $\bar{h}'$. This way we arrive at

$$\begin{aligned} N_{\Lambda_1;\bar{\Lambda}_2\ldots\bar{\Lambda}_{m-1};\Lambda_m^+} &= \int_T \mathrm{d}t\; \bar{\chi}_{\bar{\Lambda}_2}\bar{\chi}_{\bar{\Lambda}_3}\cdots\bar{\chi}_{\bar{\Lambda}_{m-1}}(2\pi\mathrm{i}\bar{h}) \sum_{w\in W} \epsilon(w)\,\mathrm{e}^{w(\Lambda_{m-1}+\rho)-(\Lambda_m^++\rho)}(2\pi\mathrm{i}h) \\ &= \frac{1}{|\overline{W}|} \int_T \mathrm{d}t\; \bar{\chi}_{\bar{\Lambda}_2}\bar{\chi}_{\bar{\Lambda}_3}\cdots\bar{\chi}_{\bar{\Lambda}_{m-1}}(2\pi\mathrm{i}\bar{h}) \sum_{w\in W}\sum_{\bar{w}\in\overline{W}} \epsilon(w\bar{w})\,\mathrm{e}^{\bar{w}^{-1}w(\Lambda_1+\rho)-\Lambda_m^+-\rho}(2\pi\mathrm{i}h) \\ &= \frac{1}{|\overline{W}|} \int_T \mathrm{d}t\; \bar{\chi}_{\bar{\Lambda}_2}\bar{\chi}_{\bar{\Lambda}_3}\cdots\bar{\chi}_{\bar{\Lambda}_{m-1}}(2\pi\mathrm{i}\bar{h}) \\ &\qquad \cdot \sum_{w\in W} \epsilon(w)\,\mathrm{e}^{w(\Lambda_1+\rho)}(2\pi\mathrm{i}h) \cdot \sum_{\bar{w}\in\overline{W}} \epsilon(\bar{w})\,\mathrm{e}^{-\bar{w}(\Lambda_m^++\rho)}(2\pi\mathrm{i}h)\,. \end{aligned} \tag{4.24}$$

Next we use the Weyl character formula for the characters $\bar{\chi}_{\bar{\Lambda}}$ of finite-dimensional $\bar{\mathfrak{g}}$-modules $\bar{\mathcal{H}}_{\bar{\Lambda}}$ [7] to obtain

$$\begin{aligned} N_{\Lambda_1;\bar{\Lambda}_2\ldots\bar{\Lambda}_{m-1};\Lambda_m^+} = \frac{1}{|\overline{W}|} \int_T \mathrm{d}t\; &\bar{\chi}_{\bar{\Lambda}_2}\bar{\chi}_{\bar{\Lambda}_3}\cdots\bar{\chi}_{\bar{\Lambda}_{m-1}}(2\pi\mathrm{i}\bar{h})\,\bar{\chi}_{\bar{\Lambda}_m^+}(-2\pi\mathrm{i}\bar{h}) \\ &\cdot \bar{\mathcal{X}}^{-1}(-2\pi\mathrm{i}\bar{h}) \sum_{w\in W} \epsilon(w)\,\mathrm{e}^{w(\Lambda_1+\rho)}(2\pi\mathrm{i}\bar{h})\,, \end{aligned} \tag{4.25}$$

where

$$\bar{\mathcal{X}} = \big(\sum_{\bar{w}\in\overline{W}} \epsilon(\bar{w})\,\mathrm{e}^{\bar{w}(\bar{\rho})}\big)^{-1} = \mathrm{e}^{-\bar{\rho}} \prod_{\bar{\alpha}>0} (1-\mathrm{e}^{-\bar{\alpha}})^{-1} \tag{4.26}$$

is the universal Verma character of $\bar{\mathfrak{g}}$, the horizontal analogue of $\mathcal{X}$ (2.16).

In a final step we rewrite (4.25) in the form

$$N_{\Lambda_1;\bar{\Lambda}_2\ldots\bar{\Lambda}_{m-1};\Lambda_m^+} = \frac{1}{|\overline{W}|} \int_T \mathrm{d}t\; \bar{\chi}_{\bar{\Lambda}_2}\bar{\chi}_{\bar{\Lambda}_3}\cdots\bar{\chi}_{\bar{\Lambda}_{m-1}}\bar{\chi}_{\bar{\Lambda}_m}(2\pi\mathrm{i}\bar{h}) \cdot \bar{\mathcal{X}}^{-1}(-2\pi\mathrm{i}\bar{h}) \sum_{w\in W} \epsilon(w)\,\mathrm{e}^{w(\Lambda_1+\rho)}(2\pi\mathrm{i}\bar{h})\,. \tag{4.27}$$

Here we employed the identity

$$\bar{\chi}_{\bar{\Lambda}^+}(\bar{h}) = \bar{\chi}_{\bar{\Lambda}}(-\bar{h}) \tag{4.28}$$

for characters of irreducible highest weight modules of $\bar{\mathfrak{g}}$. This relation follows by making the substitution $\bar{w} \mapsto \bar{w}\bar{w}_{\max}$, with $\bar{w}_{\max}$ the longest element of the Weyl group of $\bar{\mathfrak{g}}$, in the Weyl character formula, and by using the fact that this element acts as $\bar{w}_{\max}(\bar{\mu}) = -\bar{\mu}^+$ on $\bar{\mathfrak{g}}$-weights:

$$\bar{\chi}_{\bar{\Lambda}^+}(\bar{h}) = \frac{\sum_{\bar{w}\in\overline{W}} \epsilon(\bar{w})\mathrm{e}^{\bar{w}(\bar{\Lambda}^++\bar{\rho})(\bar{h})}}{\sum_{\bar{w}\in\overline{W}} \epsilon(\bar{w})\mathrm{e}^{\bar{w}(\bar{\rho})(\bar{h})}} = \frac{\epsilon(\bar{w}_{\max})\sum_{\bar{w}\in\overline{W}} \epsilon(\bar{w})\mathrm{e}^{-\bar{w}(\bar{\Lambda}+\bar{\rho})(\bar{h})}}{\epsilon(\bar{w}_{\max})\sum_{\bar{w}\in\overline{W}} \epsilon(\bar{w})\mathrm{e}^{-\bar{w}(\bar{\rho})(\bar{h})}} = \bar{\chi}_{\bar{\Lambda}}(-\bar{h})\,. \tag{4.29}$$

[7] In agreement with the remarks made above the final result, as displayed in formulæ (5.4) and (5.18) below, does not depend on τ and ϖ at all. Accordingly, for ease of notation we suppress the superficial dependence on these variables from now on.

5 From the integral formula to the modular S-matrix

5.1 $\dim B$ in terms of weight multiplicities

Our aim is now to deduce the Verlinde formula from the integral formula (4.27). This can be achieved by using information about the structure of the Weyl group W of $\mathfrak{g}$ or, more precisely, about the relation between W and the Weyl group $\overline{W}$ of the horizontal subalgebra $\bar{\mathfrak{g}}$ which is canonically embedded as a subgroup into W. There are essentially two different possibilities to implement this relationship. The first amounts to working with a special set $\mathring{W}$ of representatives of the coset of W by its subgroup $\overline{W}$. By definition, $\mathring{W}$ is that subset of W which consists of those representatives of the elements of $W/\overline{W}$ that have minimal length. One knows (see e.g. [14] and also [15, remark 8.1.]) that every element of $W/\overline{W}$ has a unique representative with this property, that for any integrable $\mathfrak{g}$-weight Λ and any $w\in W$ the $\bar{\mathfrak{g}}$-weight $w(\bar\Lambda+\bar\rho)-\bar\rho$ is dominant integral if and only if $w\in\mathring{W}$, and that each element $w\in W$ can be uniquely represented in the form $w=\bar w\mathring{w}$ with $\bar w\in\overline{W}$ and $\mathring{w}\in\mathring{W}$. Together with the Weyl character formula, which absorbs the summation over $\overline{W}$, it then follows that our result (4.25) can be rewritten as

$$\begin{aligned}\dim B \;&= \frac{1}{|\overline{W}|}\int_T\mathrm{d}t\;\bar\chi_{\bar\Lambda_2}\bar\chi_{\bar\Lambda_3}\cdots\bar\chi_{\bar\Lambda_{m-1}}\,\bar{\mathcal{X}}^{-1}(2\pi\mathrm{i}\bar h)\sum_{\mathring{w}\in\mathring{W}}\epsilon(\bar w\mathring{w})\,\bar\chi_{\mathring{w}(\bar\Lambda_1+\bar\rho)-\bar\rho}(2\pi\mathrm{i}\bar h)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\cdot\bar{\mathcal{X}}^{-1}(-2\pi\mathrm{i}\bar h)\,\bar\chi_{\bar\Lambda_m^+}(-2\pi\mathrm{i}\bar h)\\ &=\frac{1}{|\overline{W}|}\int_T\mathrm{d}t\;J(2\pi\mathrm{i}\bar h)\,\bar\chi_{\bar\Lambda_2}\bar\chi_{\bar\Lambda_3}\cdots\bar\chi_{\bar\Lambda_{m-1}}\bar\chi_{\bar\Lambda_m}(2\pi\mathrm{i}\bar h)\sum_{\mathring{w}\in\mathring{W}}\epsilon(\mathring{w})\,\bar\chi_{\mathring{w}(\bar\Lambda_1+\bar\rho)-\bar\rho}(2\pi\mathrm{i}\bar h)\,.\end{aligned}\tag{5.1}$$

Here we introduced the function

$$J(2\pi\mathrm{i}\bar h):=\bar{\mathcal{X}}^{-1}(2\pi\mathrm{i}\bar h)\,\bar{\mathcal{X}}^{-1}(-2\pi\mathrm{i}\bar h)=\prod_{\bar\alpha\in\bar\Delta_+}\left|\mathrm{e}^{\pi\mathrm{i}\bar\alpha(\bar h)}-\mathrm{e}^{-\pi\mathrm{i}\bar\alpha(\bar h)}\right|^2,\tag{5.2}$$

where the product extends over the set $\bar\Delta_+$ of positive $\bar{\mathfrak{g}}$-roots. This function can be recognized as the Jacobian factor that, together with $|\overline{W}|^{-1}$, appears in Weyl's integration formula that relates the integral of a class function over the whole group G to an integral over its maximal torus T. Thus we can rewrite (5.1) as an integral over G:

$$\dim B=\sum_{\mathring{w}\in\mathring{W}}\epsilon(\mathring{w})\int_G\mathrm{d}\mu_G\,\bar\chi_{\mathring{w}(\bar\Lambda_1+\bar\rho)-\bar\rho}\,\bar\chi_{\bar\Lambda_2}\bar\chi_{\bar\Lambda_3}\cdots\bar\chi_{\bar\Lambda_{m-1}}\bar\chi_{\bar\Lambda_m}\,,\tag{5.3}$$

where $\mathrm{d}\mu_G$ is the normalized Haar measure of G. From the orthogonality property of G-characters with respect to $\mathrm{d}\mu_G$ it then follows immediately that

$$\dim B=\sum_{\mathring{w}\in\mathring{W}}\epsilon(\mathring{w})\,\bar N_{\mathring{w}(\bar\Lambda_1+\bar\rho)-\bar\rho,\bar\Lambda_2\ldots\bar\Lambda_m}\,,\tag{5.4}$$

where $\bar N_{\bar\mu_1\bar\mu_2\ldots\bar\mu_l}$ denotes the number of singlets in the tensor product $\bar{\mathcal{H}}_{\bar\mu_1}\otimes\bar{\mathcal{H}}_{\bar\mu_2}\otimes\cdots\otimes\bar{\mathcal{H}}_{\bar\mu_l}$.

The right hand side of expression (5.4) actually constitutes a finite sum, even though $\mathring{W}$ is an infinite set. To see this, we observe that the $\bar{\mathfrak{g}}$-module $\bar{\mathcal{H}}_{\bar{\Lambda}_2}\otimes\cdots\otimes\bar{\mathcal{H}}_{\bar{\Lambda}_m}$ can be decomposed into a direct sum of finitely many irreducible $\bar{\mathfrak{g}}$-modules. According to the identity (2.25), the module $\bar{\mathcal{H}}_{\mathring{w}(\bar{\Lambda}_1+\bar{\rho})-\bar{\rho}}$ has to appear in this decomposition in order to give a non-zero contribution to $\bar{N}_{\mathring{w}(\bar{\Lambda}_1+\bar{\rho})-\bar{\rho},\bar{\Lambda}_2\ldots\bar{\Lambda}_m}$. Hence only finitely many $\mathring{w}$ can contribute and we obtain a manifestly integral expression for $\dim B$ in terms of multiplicities of invariant tensors of finite-dimensional $\bar{\mathfrak{g}}$-modules. (In the special case of $m=3$ this description of the WZW fusion rules is known as the Kac–Walton formula [7, 16].) As we will see in Subsection 6.3, the right hand side of (5.4) can also be interpreted as the Euler characteristic of a certain complex of $\bar{\mathfrak{g}}$-co-invariants.

If we so wish, we can proceed by expressing the number $\bar{N}_{\bar{\mu}_1\bar{\mu}_2\ldots\bar{\mu}_l}$ through weight multiplicities of finite-dimensional $\bar{\mathfrak{g}}$-modules. To this end one has to implement the so-called Racah–Speiser formula for the number of invariant tensors in a tensor product of finite-dimensional $\bar{\mathfrak{g}}$-modules. We find

$$\begin{aligned}\dim B \;&= \sum_{\mathring{w}\in\mathring{W}} \epsilon(\mathring{w}) \sum_{\bar{w}\in\overline{W}} \epsilon(\bar{w})\, \mathrm{mult}_{\bar{\Lambda}_2\ldots\bar{\Lambda}_{m-1}}(\bar{w}\mathring{w}(\bar{\Lambda}_1+\bar{\rho})-\bar{\rho}-\bar{\Lambda}_m) \\ &\equiv \sum_{w\in W} \epsilon(w)\, \mathrm{mult}_{\bar{\Lambda}_2\ldots\bar{\Lambda}_{m-1}}(w(\bar{\Lambda}_1+\bar{\rho})-\bar{\rho}-\bar{\Lambda}_m)\,,\end{aligned} \tag{5.5}$$

where $\mathrm{mult}_{\bar{\Lambda}_2\ldots\bar{\Lambda}_{m-1}}(\bar{\mu})$ denotes the multiplicity of the weight $\bar{\mu}$ in the tensor product module $\bar{\mathcal{H}}_{\bar{\Lambda}_1}\otimes\bar{\mathcal{H}}_{\bar{\Lambda}_2}\otimes\cdots\otimes\bar{\mathcal{H}}_{\bar{\Lambda}_{m-1}}$.

5.2 Resummation

Alternatively, we can also express $\dim B$ completely in terms of elements of the modular matrix S that describes the transformation properties of the characters of the $\mathfrak{g}$-modules $\mathcal{H}_\Lambda$ under the modular transformation $\tau\mapsto -1/\tau$. To this end, we recall that the affine Weyl group W can be regarded as the semi-direct product of $\overline{W}$ and of translations by elements of the coroot lattice $L^\vee$ of $\bar{\mathfrak{g}}$. We can therefore write the elements of the affine Weyl group W in (4.27) as pairs $w=(\bar{w};\bar{\beta})$ of elements $\bar{w}\in\overline{W}$ of the Weyl group of $\bar{\mathfrak{g}}$ and of coroot lattice elements $\bar{\beta}\in L^\vee$. Then the induced action (4.19) of $w\in W$ on the weight space of $\bar{\mathfrak{g}}$ is given by

$$w(\bar{\mu}) = \bar{w}(\bar{\mu}) + \mathrm{k}\bar{\beta}\,. \tag{5.6}$$

Also taking care of the shift in the effective level that results from the shift of the weights by the Weyl vector ρ (which has level h) in the Weyl group action, we can then rewrite (4.27) in the form

$$\begin{aligned}N_{\Lambda_1;\bar{\Lambda}_2\ldots\bar{\Lambda}_{m-1};\Lambda_m^+} \;&= \frac{1}{|\overline{W}|}\int_T \mathrm{d}t\; \bar{\chi}_{\bar{\Lambda}_2}\bar{\chi}_{\bar{\Lambda}_3}\cdots\bar{\chi}_{\bar{\Lambda}_{m-1}}\bar{\chi}_{\bar{\Lambda}_m}(2\pi\mathrm{i}\bar{h}) \\ &\qquad\cdot \bar{\mathcal{X}}^{-1}(-2\pi\mathrm{i}\bar{h}) \sum_{\bar{w}\in\overline{W}} \epsilon(\bar{w})\, \mathrm{e}^{\bar{w}(\bar{\Lambda}_1+\bar{\rho})}(2\pi\mathrm{i}\bar{h}) \sum_{\bar{\beta}\in L^\vee} \mathrm{e}^{(\mathrm{k}+\mathrm{h})\bar{\beta}}(2\pi\mathrm{i}\bar{h})\,.\end{aligned} \tag{5.7}$$

Relation (5.7) expresses the dimension of the space of chiral blocks as the integral of a certain function over a torus, namely the maximal torus of the Lie group G. We will now show that one can replace this integration by a finite summation over suitable $\bar{\mathfrak{g}}$-weights. This is achieved with the help of (a slightly generalized version of) Poisson resummation which states that for any function $f\colon (\bar{\mathfrak{g}}_\circ^\star)_{\mathbb{R}} \cong \mathbb{R}^{\mathrm{rank}\,\bar{\mathfrak{g}}} \to \mathbb{C}$ that is periodic with respect to $L^\vee$ and for any natural number l, the formula

$$\int_T \mathrm{d}t \sum_{\bar\beta\in L^\vee} f(\bar h)\,\mathrm{e}^{\bar\beta}(2\pi\mathrm{i} l\bar h) = |lL^\vee/L_{\mathrm{w}}| \sum_{\bar\lambda\in l^{-1}L_{\mathrm{w}}/L^\vee} f(\bar\lambda) \tag{5.8}$$

holds. Here as before $\mathrm{d}t$ is the normalized Haar measure on T and $\bar h\in\bar{\mathfrak{g}}_\circ$ is related to $t\in T$ by $t=\mathrm{E{\scriptsize XP}}(2\pi\mathrm{i}\bar h)$. Also, the integrand of the left hand side is single-valued on T because f is by assumption $L^\vee$-periodic, and because for different elements $\bar h_1,\, \bar h_2\in\bar{\mathfrak{g}}_\circ^\star$ which satisfy $\mathrm{E{\scriptsize XP}}(2\pi\mathrm{i}\bar h_1)=t=\mathrm{E{\scriptsize XP}}(2\pi\mathrm{i}\bar h_2)$ the values of $\bar\beta(\bar h_1)$ and $\bar\beta(\bar h_2)$ for $\bar\beta\in L^\vee$ differ by an integer, the exponential function is $L^\vee$-periodic as well.

To derive relation (5.8), we introduce the Fourier transform $\tilde f$ of f. The Fourier components are labelled by the lattice dual to $L^\vee$, i.e. by the weight lattice L_{w}:[8] $f(\bar h)=\sum_{\bar\lambda\in L_{\mathrm{w}}}\mathrm{e}^{2\pi\mathrm{i}\bar\lambda(\bar h)}\tilde f_{\bar\lambda}$. Then the left hand side of (5.8) can be written as

$$\int_T \mathrm{d}t \sum_{\bar\beta\in L^\vee}\sum_{\bar\lambda\in L_{\mathrm{w}}} \tilde f_{\bar\lambda}\,\mathrm{e}^{2\pi\mathrm{i}(\bar\lambda+l\bar\beta)(\bar h)} = \sum_{\bar\lambda\in L_{\mathrm{w}}} \tilde f_{\bar\lambda} \sum_{\bar\beta\in L^\vee}\int_T \mathrm{d}t\ \mathrm{e}^{2\pi\mathrm{i}(\bar\lambda+l\bar\beta)(\bar h)} = \sum_{\bar\lambda\in lL^\vee}\tilde f_{\bar\lambda}\,. \tag{5.9}$$

Here the last equality holds because the integral over the torus is non-zero only when the integrand is constant, which happens only when $\bar\lambda=-l\bar\beta$, i.e. when $\bar\lambda$ lies in l times the coroot lattice. On the other hand, for the right hand side of (5.8) we compute

$$\sum_{\bar\lambda\in l^{-1}L_{\mathrm{w}}/L^\vee} f(\bar\lambda) = \sum_{\bar\mu\in L_{\mathrm{w}}}\tilde f_{\bar\mu} \sum_{\bar\lambda\in l^{-1}L_{\mathrm{w}}/L^\vee} \mathrm{e}^{2\pi\mathrm{i}(\bar\lambda,\bar\mu)} = \Big|\frac{L_{\mathrm{w}}}{l\,L^\vee}\Big| \sum_{\bar\mu\in lL^\vee}\tilde f_{\bar\mu}\,, \tag{5.10}$$

where we use the fact that the sum over $\bar\lambda$ only gives contributions if the summand is constant and hence equal to one, which happens precisely if the inner product $(\bar\lambda,\bar\mu)$ is an integer. The resummation formula (5.8) now follows by equating relations (5.10) and (5.9).

Applying now the result (5.8) to formula (5.7) for $\dim B$, we get

$$\begin{aligned}\dim B \;&= \frac{1}{|\overline{W}|}\,\left|\frac{(\mathrm{k}{+}\mathrm{h})L^\vee}{L_{\mathrm{w}}}\right| \sum_{\bar\lambda\in(\mathrm{k}+\mathrm{h})^{-1}L_{\mathrm{w}}/L^\vee} \bar{\mathcal{X}}^{-1}(-\bar\lambda)\,\bar\chi_{\bar\Lambda_2}(\bar\lambda)\,\bar\chi_{\bar\Lambda_3}(\bar\lambda)\cdots\bar\chi_{\bar\Lambda_m}(\bar\lambda)\sum_{\bar w\in\overline{W}}\epsilon(\bar w)\,\mathrm{e}^{2\pi\mathrm{i}\bar w(\bar\Lambda_1+\bar\rho,\bar\lambda)}\\ &= \frac{1}{|\overline{W}|}\,\left|\frac{(\mathrm{k}{+}\mathrm{h})L^\vee}{L_{\mathrm{w}}}\right| \sum_{\bar\lambda\in(\mathrm{k}+\mathrm{h})^{-1}L_{\mathrm{w}}/L^\vee} J(\bar\lambda)\,\bar\chi_{\bar\Lambda_1}(\bar\lambda)\,\bar\chi_{\bar\Lambda_2}(\bar\lambda)\cdots\bar\chi_{\bar\Lambda_{m-1}}(\bar\lambda)\,\bar\chi_{\bar\Lambda_m}(\bar\lambda)\,,\end{aligned} \tag{5.11}$$

where in the second line we used once more the Weyl character formula.

[8] We identify $\bar{\mathfrak{g}}_\circ$ and the weight space $\bar{\mathfrak{g}}_\circ^\star$ via the non-degenerate invariant bilinear form.

5.3 Characters and the modular matrix S

The final step in establishing the Verlinde formula is now to express the result (5.11) in terms of the entries of the modular matrix S that governs [7] the transformation of the affine characters χ_Λ ($\Lambda \in \mathrm{P_k}$) under the modular transformation $\tau \mapsto -1/\tau$. This is possible because the generalized quantum dimensions, i.e. ratios of S-matrix elements, coincide with the characters of the horizontal subalgebra, evaluated at suitable arguments.

We first rewrite the summation in expression (5.11), which extends over the $\bar{\mathfrak{g}}$-weights $\bar\lambda \in ((\mathrm{k}+\mathrm{h})^{-1} L_\mathrm{w})/L^\vee \equiv L_\mathrm{w}/((\mathrm{k}+\mathrm{h})L^\vee)$, in such a way that only weights in the dominant Weyl alcove appear. The function J (5.2) is the square of a function that is odd under the Weyl group $\overline{W}$ and hence vanishes when $\bar h$ is on the boundary of some Weyl chamber. Owing to the $\overline{W}$-invariance of the characters and of J, it follows that each Weyl chamber gives the same contribution to the sum. Restricting the summation to those weights that belong to the dominant Weyl alcove therefore just amounts to cancelling the prefactor $1/|\overline{W}|$. In the summation we are then left with the weights of the form

$$\bar\lambda = \frac{\bar\Lambda + \bar\rho}{\mathrm{k}+\mathrm{h}}\,, \tag{5.12}$$

where the weights $\bar\Lambda + \bar\rho$ are the integral weights in the interior of the dominant Weyl alcove at level $\mathrm{k}+\mathrm{h}$. The weights $\bar\Lambda$ are thus the integral weights in the dominant Weyl alcove at level k, i.e. precisely the elements of the set $\mathrm{P_k}$ (2.9). Formula (5.11) can therefore be rewritten as

$$\dim B = \tfrac{1}{|\overline{W}|} \left| \tfrac{(\mathrm{k+h})L^\vee}{L_\mathrm{w}} \right| \sum_{\bar\Lambda \in \mathrm{P_k}} J(\tfrac{\bar\Lambda+\bar\rho}{\mathrm{k+h}})\, \bar\chi_{\bar\Lambda_1}(\tfrac{\bar\Lambda+\bar\rho}{\mathrm{k+h}})\, \bar\chi_{\bar\Lambda_2}(\tfrac{\bar\Lambda+\bar\rho}{\mathrm{k+h}}) \cdots \bar\chi_{\bar\Lambda_m}(\tfrac{\bar\Lambda+\bar\rho}{\mathrm{k+h}})\,. \tag{5.13}$$

A closed expression for the modular matrix S is provided by the Kac–Peterson formula

$$\begin{aligned} S_{\Lambda,\Lambda'} = {} & (-\mathrm{i})^{(\dim\bar{\mathfrak{g}}-\mathrm{rank}\,\bar{\mathfrak{g}})/2}\, |L_\mathrm{w}/L^\vee|^{-1/2} (\mathrm{k}+\mathrm{h})^{-\mathrm{rank}\,\bar{\mathfrak{g}}/2} \\ & \cdot \sum_{\bar w \in \overline{W}} \epsilon(\bar w)\, \exp\big[-\tfrac{2\pi\mathrm{i}}{\mathrm{k+h}}\, (\bar w(\bar\Lambda+\bar\rho), \bar\Lambda'+\bar\rho) \big]\,. \end{aligned} \tag{5.14}$$

By comparison with the Weyl character formula, it follows that the characters $\bar\chi_{\bar\Lambda_i}$ evaluated at the weights (5.12) satisfy

$$\bar\chi_{\bar\Lambda_i}(\tfrac{\bar\Lambda+\bar\rho}{\mathrm{k+h}}) = \Big(\frac{S_{\Lambda_i,\Lambda}}{S_{\mathrm{k}\Lambda_{(0)},\Lambda}} \Big)^{*}\,. \tag{5.15}$$

Similarly, together with the denominator identity one finds that

$$\begin{aligned} S_{\mathrm{k}\Lambda_{(0)},\Lambda} &= |L_\mathrm{w}/(\mathrm{k}+\mathrm{h})L^\vee|^{-1/2} \prod_{\bar\alpha>0} 2\sin\tfrac{(\bar\Lambda+\bar\rho,\bar\alpha)\pi}{\mathrm{k+h}} \\ &= \mathrm{i}^{-|\bar\Delta_+|}\, |(\mathrm{k}+\mathrm{h})L^\vee/L_\mathrm{w}|^{1/2} \prod_{\bar\alpha>0} \Big(\exp\big[\mathrm{i}\pi(\tfrac{(\bar\alpha,\bar\Lambda+\bar\rho)}{\mathrm{k+h}})\big] - \exp\big[-\mathrm{i}\pi(\tfrac{(\bar\alpha,\bar\Lambda+\bar\rho)}{\mathrm{k+h}})\big] \Big)\,, \end{aligned} \tag{5.16}$$

which in turn implies

$$|S_{\mathrm{k}\Lambda_{(0)},\Lambda}|^2 = \left| \tfrac{(\mathrm{k+h})L^\vee}{L^\mathrm{w}} \right| J(\tfrac{\bar\Lambda+\bar\rho}{\mathrm{k+h}})\,. \tag{5.17}$$

We now insert the identities (5.15) and (5.17) into formula (5.13) for $\dim B$. We then finally obtain (using also the fact that $S_{\mathrm{k}\Lambda_{(0)},\Lambda}$ and $\dim B$ are real)

$$\dim B_{\{\Lambda_i\},\{p_i\}} = \sum_{\bar\Lambda\in\mathrm{P_k}} (S_{\mathrm{k}\Lambda_{(0)},\Lambda})^2 \prod_{i=1}^{m} \frac{S_{\Lambda_i,\Lambda}}{S_{\mathrm{k}\Lambda_{(0)},\Lambda}}\,. \tag{5.18}$$

This is our desired result, the Verlinde formula that expresses the dimension $\dim B$ of spaces of genus zero chiral WZW blocks through the entries of the matrix S (5.14). Note that the right hand side of (5.18) is typically a complicated combination of complex numbers (actually all entries of S lie in a cyclotomic extension of the rationals), i.e. unlike the alternative expression on the right hand side of (5.4), it is not manifestly integral.

6 Homological aspects of the Verlinde formula

In this final section we add a few comments on general aspects of the Verlinde formula and present an interpretation of the Verlinde multiplicities $\dim B$ as expressed by formula (5.4) in terms of Euler characteristics of certain complexes of co-invariants.

6.1 The Verlinde formula

In the literature, derivations of the Verlinde formula have been described at various levels of rigor, ranging from more heuristic considerations to mathematically complete proofs. It seems fair to say that typically the more rigorous these proofs are, the smaller the number of situations is to which they apply. A conformal field theory deduction which does not use any specific properties of the chiral symmetry algebra of the theory, and hence applies to arbitrary rational conformal field theories, was given in [1,17,18]; it involves certain formal manipulations with chiral blocks and hence may be regarded as somewhat heuristic. All other proofs known to us involve (implicitly or explicitly) the block algebra $\bar{\mathfrak{g}}(U)$, and thereby the representation theory of affine Lie algebras; as a consequence they work exclusively for WZW conformal field theories. (The two kinds of approaches to the Verlinde formula also differ drastically in another aspect: those applicable to arbitrary conformal field theories only aim at establishing a relation between the dimensions of spaces of chiral blocks and the modular transformation properties of the characters, i.e. the zero point blocks at genus one, but they do not predict any concrete expressions for these modular transformations. In contrast, the proofs that only work in the WZW setting do provide such expressions, in particular the explicit form of the modular transformation matrix S; roughly speaking, they combine the information encoded in the Verlinde formula for general conformal field theories with the Kac–Peterson formula (5.14) for S.) Among these approaches which apply to the WZW case, there is one [19,20] that is based on the path-integral quantization of Chern–Simons gauge theories and is accordingly affected by the usual difficulties in setting up a rigorous quantization procedure, e.g. concerning the proper mathematical setting for path integrals. A related approach [21, 22] combines holomorphic quantization of Chern–Simons theories and surgery manipulations on three-manifolds. There is also an approach [23] which combines the isomorphism [24, 25] between the tensor categories of certain modules over affine Lie algebras $\mathfrak{g}$ at negative level and categories of modules over quantum groups $\mathsf{U}_q(\mathfrak{g})$ with q a root of unity with an isomorphism between categories of $\mathfrak{g}$-modules at positive and negative levels to deduce the Verlinde formula from the representation theory of quantum groups.

Another possibility is to exploit the fact that the spaces of chiral blocks are isomorphic to the spaces of holomorphic sections in line bundles over certain projective varieties, namely over moduli spaces of semi-stable principal bundles with structure group G, where G is the real, compact, connected and simply connected Lie group with Lie algebra $\bar{\mathfrak{g}}$. (In the mathematics literature, sometimes this isomorphism between chiral blocks and geometric objects, rather than an expression for their common dimension, is meant when the term 'Verlinde conjecture' is used.) One can attempt to analyze these spaces with various methods of traditional finite-dimensional algebraic geometry; this way several proofs have been established for the case of WZW theories based on $\bar{\mathfrak{g}} = \mathfrak{sl}(2)$, where the vector bundles have rank two and their moduli space can be described rather explicitly (see e.g. [26–28], as well as [2] [9] for a more detailed exposition and for more references). Finally, using the methods of algebraic geometry, proofs that work for $\mathfrak{sl}(N)$ with arbitrary N have been obtained in [4, 10, 5]; these arguments start from the description of the moduli space of flat connections over a curve in a double coset form. The proof of [4] which uses torsion-free sheaves even applies to all simple Lie algebras $\bar{\mathfrak{g}}$, except for $\bar{\mathfrak{g}} = F_4$, E_6, E_7 or E_8. While these proofs are formulated in a purely algebraic setting, a derivation where topological tools play an essential rôle was given in [32]; there the modules are completed to Hilbert spaces and the Verlinde formula is obtained from certain vanishing theorems for complexes of these Hilbert spaces (compare subsection 6.3 below).

The argument that we presented in sections 3 – 5 makes extensive use of the representation theory of affine Lie algebras and is in most aspects rather different from all those mentioned above. In particular, all our arguments work simultaneously for WZW theories based on arbitrary simple Lie algebras $\bar{\mathfrak{g}}$; [10] at no step in the derivation is there any need to distinguish between different cases that have to be treated separately. Furthermore, when combined with the results of [33] and [34], respectively, our methods should provide a possibility to characterize the spaces of chiral blocks and to prove the Verlinde formula not only for WZW theories, but also for two other classes of conformal field theories. The first of these classes is given by the so-called integer spin simple current extensions of WZW theories; these correspond to such WZW theories which in a Lagrangian setting are associated to non-simply connected group manifolds, while for ordinary WZW theories one always deals with the simply connected covering group of the simple Lie algebra $\bar{\mathfrak{g}}$. The second class of theories consists of all coset conformal field theories, including in particular those coset theories in which so-called field identification fixed points are present. As a final application we mention that the present representation theoretic approach, combined with the results obtained in [35, 36], should be helpful in verifying the conjectures made in [37] concerning the values of certain traces on the spaces of chiral blocks. (These traces have interesting applications to the construction of chiral blocks for integer spin

[9] Other sources for references are [29] and [30], and also the WWW pages
`http://www.ictp.trieste.it/~mblau/ver.html` and `http://www.desy.de/~jfuchs/Vfcb.html`.

[10] This applies likewise to the vanishing theorem in [32].

simple current extensions of WZW theories, to the classification of boundary conditions for conformal field theories [38], and in algebraic geometry to the derivation of a Verlinde formula for non-simply connected groups.)

6.2 Complexes of co-invariants

We will now present a homological interpretation of formula (5.4) which expresses the Verlinde multiplicities $\dim B$ as an alternating sum of non-negative integers. To this end we use the BGG resolution of an irreducible highest weight module of $\mathfrak{g}$ in terms of parabolic Verma modules (2.14) which are free $\mathfrak{g}^-$-modules. We introduce the (finite) direct sums

$$\mathcal{P}_\Lambda^{(j)} := \bigoplus_{\substack{\mathring{w}\in\mathring{W}\\ \ell(\mathring{w})=j}} \mathcal{P}_{\mathring{w}\,(\Lambda+\rho)-\rho} \tag{6.1}$$

of generalized Verma modules. Here $\ell(w)$ denotes the length of the Weyl group element w. Note that the modules $\mathcal{P}_\Lambda^{(j)}$ are still free $\mathfrak{g}^-$-modules, and that for every integrable $\mathfrak{g}$-weight Λ and every $\mathring{w}\in\mathring{W}$ the $\bar{\mathfrak{g}}$-weight $w(\bar\Lambda+\bar\rho)-\bar\rho$ is dominant integral. In particular we have

$$\mathcal{P}_\Lambda^{(j)} \;\cong\; \mathsf{U}(\mathfrak{g}^-)\otimes\big(\bigoplus_{\substack{\mathring{w}\in\mathring{W}\\ \ell(\mathring{w})=j}} \bar{\mathcal{H}}_{\mathring{w}\,(\bar\Lambda+\bar\rho)-\bar\rho}\big) = \mathsf{U}(\mathfrak{g}^-)\otimes\bar{\mathcal{H}}_\Lambda^{(j)} \tag{6.2}$$

as $\mathfrak{g}^-$-modules, where $\bar{\mathcal{H}}_\Lambda^{(j)}$ is the finite-dimensional $\bar{\mathfrak{g}}$-module

$$\bar{\mathcal{H}}_\Lambda^{(j)} := \bigoplus_{\substack{\mathring{w}\in\mathring{W}\\ \ell(\mathring{w})=j}} \bar{\mathcal{H}}_{\mathring{w}\,(\bar\Lambda+\bar\rho)-\bar\rho}\,. \tag{6.3}$$

The *parabolic BGG resolution* of an irreducible highest weight module $\mathcal{H}_\Lambda$ over $\mathfrak{g}$ with integrable highest weight Λ states [15] that there exists a semi-infinite exact sequence of $\mathfrak{g}$-modules with $\mathfrak{g}$-module homomorphisms of the form

$$\ldots\to\mathcal{P}_\Lambda^{(j)}\to\mathcal{P}_\Lambda^{(j-1)}\to\ldots\to\mathcal{P}_\Lambda^{(0)}\equiv\mathcal{P}_\Lambda\to\mathcal{H}_\Lambda\to 0\,. \tag{6.4}$$

Note that this complex is governed by the same set $\mathring{W}$ of representatives of the coset $W/\overline{W}$ that we encountered in Subsection 5.1. When we tensor the sequence (6.4) with the finite-dimensional $\bar{\mathfrak{g}}$-module $\overline{\mathcal{H}}^{(m-2)}\equiv\bigotimes_{i=1}^{m-1}\bar{\mathcal{H}}_{\bar\Lambda_i}$, we obtain a semi-infinite exact sequence

$$\begin{aligned}\ldots\to\overline{\mathcal{H}}^{(m-1)}\otimes\mathcal{P}_{\Lambda_m}^{(j)}\to\overline{\mathcal{H}}^{(m-1)}\otimes\mathcal{P}_{\Lambda_m}^{(j-1)}\to\ldots\\ \ldots\to\overline{\mathcal{H}}^{(m-1)}\otimes\mathcal{P}_{\Lambda_m}^{(1)}\to\overline{\mathcal{H}}^{(m-1)}\otimes\mathcal{P}_{\Lambda_m}\to\overline{\mathcal{H}}^{(m-1)}\otimes\mathcal{H}_{\Lambda_m}\to 0\,.\end{aligned} \tag{6.5}$$

We now observe that (by using the isomorphism (4.7) to replace also the first affine module in the description (4.2) of the chiral blocks by a finite-dimensional $\bar{\mathfrak{g}}$-module) the space B of chiral blocks is isomorphic to the co-invariants of $\overline{\mathcal{H}}^{(m-1)} \otimes \mathcal{H}_{\Lambda_m}$ with respect to the block algebra $\tilde{\mathfrak{g}}$ for a single insertion point. ($\tilde{\mathfrak{g}}$ is the algebra of $\bar{\mathfrak{g}}$-valued algebraic functions that are allowed to have a pole of finite order at a single distinguished point and is hence isomorphic to $\bar{\mathfrak{g}} \oplus \mathfrak{g}^-$.) We would like to combine this information with the exact sequence (6.5). To this end, we first observe that the spaces in the sequence (6.5) (except for the last two) carry the structure of $\tilde{\mathfrak{g}}$-modules which are free $\mathfrak{g}^-$-modules. On the other hand, taking co-invariants of an exact sequence does in general not produce another exact sequence, but only a complex. However, one can show (see Appendix A.2) that this procedure still constitutes a right-exact functor, so that the exact sequence (6.5) provides us, by taking co-invariants with respect to $\tilde{\mathfrak{g}}$, with a semi-infinite complex

$$\begin{aligned}&\ldots \to \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \mathcal{P}^{(j)}_{\Lambda_m} \rfloor_{\tilde{\mathfrak{g}}} \to \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \mathcal{P}^{(j-1)}_{\Lambda_m} \rfloor_{\tilde{\mathfrak{g}}} \to \ldots \\ &\qquad \ldots \to \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \mathcal{P}^{(1)}_{\Lambda_m} \rfloor_{\tilde{\mathfrak{g}}} \to \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \mathcal{P}_{\Lambda_m} \rfloor_{\tilde{\mathfrak{g}}} \to \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \mathcal{H}_{\Lambda_m} \rfloor_{\tilde{\mathfrak{g}}} \equiv B \to 0\end{aligned} \tag{6.6}$$

of spaces of $\tilde{\mathfrak{g}}$-co-invariants that is still exact at the two right-most positions.

Next we note that using the two fundamental facts (A.4) and (A.2) about co-invariants of free modules, one can derive the isomorphism

$$\begin{aligned}\lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \mathcal{P}_{\mu} \rfloor_{\tilde{\mathfrak{g}}} &\cong \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes (\mathsf{U}(\mathfrak{g}^-) \otimes \bar{\mathcal{H}}_{\bar{\mu}}) \rfloor_{\tilde{\mathfrak{g}}} \\ &\cong \lfloor \mathsf{U}(\mathfrak{g}^-) \otimes (\overline{\mathcal{H}}^{(m-1)} \otimes \bar{\mathcal{H}}_{\bar{\mu}}) \rfloor_{\mathfrak{g}^- \oplus \bar{\mathfrak{g}}} \cong \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \bar{\mathcal{H}}_{\bar{\mu}} \rfloor_{\bar{\mathfrak{g}}}\end{aligned} \tag{6.7}$$

of vector spaces. Hereby we can simplify the various spaces that occur in the complex (6.6), so that it can be written as a complex that, except for $B = \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \mathcal{H}_{\Lambda_m} \rfloor_{\tilde{\mathfrak{g}}}$, involves only $\bar{\mathfrak{g}}$-co-invariants:

$$\begin{aligned}&\ldots \to \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \bar{\mathcal{H}}^{(j)}_{\Lambda_m} \rfloor_{\bar{\mathfrak{g}}} \to \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \bar{\mathcal{H}}^{(j-1)}_{\Lambda_m} \rfloor_{\bar{\mathfrak{g}}} \to \ldots \\ &\qquad \ldots \to \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \bar{\mathcal{H}}^{(1)}_{\Lambda_m} \rfloor_{\bar{\mathfrak{g}}} \to \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \bar{\mathcal{H}}_{\Lambda_m} \rfloor_{\bar{\mathfrak{g}}} \to B \to 0\,.\end{aligned} \tag{6.8}$$

Thus we have finally arrived at a complex of $\bar{\mathfrak{g}}$-co-invariants which is governed by the subset $\mathring{W}$ of the Weyl group W of $\mathfrak{g}$. Moreover, by the same arguments as in the previous section, we deduce from formula (2.25) for $\bar{\mathfrak{g}}$-co-invariants that this semi-infinite complex is not really infinite, but actually constitutes a finite complex of $\bar{\mathfrak{g}}$-co-invariants, i.e. we even have

$$\begin{aligned}&0 \to \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \overline{\mathcal{H}}^{(j_{\max})}_{\bar{\Lambda}_m} \rfloor_{\bar{\mathfrak{g}}} \to \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \overline{\mathcal{H}}^{(j_{\max}-1)}_{\bar{\Lambda}_m} \rfloor_{\bar{\mathfrak{g}}} \to \ldots \\ &\qquad \ldots \to \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \overline{\mathcal{H}}^{(1)}_{\bar{\Lambda}_m} \rfloor_{\bar{\mathfrak{g}}} \to \lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \bar{\mathcal{H}}_{\bar{\Lambda}_m} \rfloor_{\bar{\mathfrak{g}}} \to B \to 0\end{aligned} \tag{6.9}$$

for some non-negative integer $j_{\max}$.

6.3 Verlinde multiplicities and Euler characteristics

Owing to the finiteness of the complex (6.9), we can express the dimension of the space B of chiral blocks as

$$\dim B = \mathrm{x} + \sum_{j=0}^{j_{\max}} (-1)^j \dim \big(\lfloor \overline{\mathcal{H}}^{(m-1)} \otimes \overline{\mathcal{H}}^{(j)}_{\bar{\Lambda}_m} \rfloor_{\bar{\mathfrak{g}}} \big) \tag{6.10}$$

through the Euler characteristic X of the complex (6.9) and an alternating sum of dimensions of spaces of $\bar{\mathfrak{g}}$-co-invariants. The latter coincide with the number of singlets in the respective $\bar{\mathfrak{g}}$-modules. Together with the identity $(-1)^{\ell(w)} = \epsilon(w)$ it then follows that the result (5.4) for the Verlinde multiplicities can be rephrased as the statement that the Euler characteristic X of the complex (6.9) vanishes.

Actually, we conjecture that not only $\mathrm{X} = 0$, but that the whole homology of the complex is zero, i.e. that (6.9) is in fact an *exact* sequence. Note that it is immediate that the sequence is indeed exact as long as the number of insertion points is $m \leq 2$. In these cases the only non-vanishing spaces in the complex (6.8) are those involving the modules $\mathcal{P}_{\Lambda_m}$ or $\mathcal{H}_{\Lambda_m}$ (for $m = 2$ this is easily seen by considerations similar to those that led to formula (3.7); the case $m = 1$ can be regarded as a special case of the $m = 2$ situation where the second insertion point carries the weight $\Lambda = \mathrm{k}\Lambda_{(0)}$). Exactness then already follows from the fact (compare Appendix A.2) that taking co-invariants constitutes a right-exact functor. Another situation where exactness can be established directly (though in a somewhat lengthy manner, by using the explicit form of the BGG maps and of the Clebsch–Gordan decomposition) is $m = 3$ for $\bar{\mathfrak{g}} = A_1$.

This conjecture about the vanishing of the homology of (6.9) has also been made, from a different perspective, in [31]. We do not know of any direct Lie algebraic proof of our conjecture. However, a similar result has been obtained in [32], based on a completion of the highest weight modules $\mathcal{H}_\Lambda$ to topological Hilbert spaces. The vanishing theorem of [32] refers to the cohomology of a complex that involves dual space of the space $\lfloor \mathcal{H}_{\vec{\Lambda}}^{(m)} \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})}$ of co-invariants considered here, i.e. the space of $\bar{\mathfrak{g}}(\mathbb{P}^1_{(m)})$-invariants in the algebraic dual $(\mathcal{H}_{\vec{\Lambda}}^{(m)})^\star$ of $\mathcal{H}_{\vec{\Lambda}}^{(m)}$. The vanishing statement for the homology should be related, of course, to the corresponding statement for the cohomology in [32]. Conversely, a proof of our conjectures concerning the structure of the completed tensor product module $(\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)})^-$ in subsection 4.2, and hence of the vanishing of the Euler number of the complex (6.9), might constitute a first step towards a more direct derivation of the vanishing theorem of [32]. In this context it could also be interesting to analyze the lowest weight socle of the algebraic dual $(\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)})^\star$ of $\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)}$ (compare appendix C).

We regard the results of [32] also as an indication that unitarity should be a crucial input for the vanishing of the whole homology. Indeed, this fits nicely with the observation [39] that when one inserts the modular matrix S for so-called admissible [40] $\mathfrak{g}$-modules (which appear in WZW theories at fractional level and are not unitarizable), one still obtains integers, which however can now also be negative,[11] and as already noted in [42], this suggests that in this situation the Verlinde multiplicities may still possess a homological interpretation. More specifically, note that when one deletes the space B in the complex (6.9) one obtains another complex whose Euler characteristic does not vanish any more but is given by $\dim B$; in the non-unitary case we would expect the presence of a similar complex whose Euler characteristic is still given by the Verlinde multiplicity, but whose homology is no longer concentrated at the last position.

The interpretation of such a complex and its relation to the true fusion coefficients (which are dimensions and therefore are manifestly non-negative, and which can for example be deduced

[11] The proper definition of S in these cases is, however, problematic; compare e.g. section 5 of [41].

by explicitly solving the constraints implied by null vector decoupling, in particular the Knizhnik–Zamolodchikov equation [42–45]) remains to be clarified. In this context the observation [39] that in many cases the Verlinde multiplicities are controlled by the integral parts of the weights, and hence by fusion rules of unitary theories, is particularly interesting. In accordance with general experience with Borel–Weil–Bott theory, it might find its explanation by the property that in the non-unitary case the homology is still concentrated at a single position of the complex, albeit not at the last one.

We finally remark that the traces of the action of certain outer automorphisms on the spaces of chiral blocks [37] are (possibly negative) integers. This suggests that they might find a natural interpretation as (possibly twisted) Euler characteristics as well. Such an interpretation would not only explain the rather surprising integrality properties of these traces, but may also be helpful for a proof of the conjectures in [37].

A Further properties of co-invariants

A.1 Co-invariants of free modules

When W is a *free* module over some Lie algebra $\mathfrak{h}$, i.e. when

$$W \;\cong\; \mathsf{U}(\mathfrak{h}) \otimes X \;\equiv\; \mathsf{U}(\mathfrak{h}) \otimes_{\mathbb{C}} X \tag{A.1}$$

for some $\mathbb{C}$-vector space X, then the tensor product $V \otimes W$ of W with any other $\mathfrak{h}$-module V is again a free $\mathfrak{h}$-module. More precisely [15, Prop. 1.7], there exists a natural isomorphism

$$V \otimes (\mathsf{U}(\mathfrak{h}) \otimes X) \;\cong\; \mathsf{U}(\mathfrak{h}) \otimes (V \otimes X) \tag{A.2}$$

of $\mathfrak{h}$-modules (on the right hand side, $V \otimes X$ is a tensor product of vector spaces); the isomorphism $\mathsf{U}(\mathfrak{h}) \otimes (V{\otimes}X) \stackrel{\cong}{\to} V \otimes (\mathsf{U}(\mathfrak{h}){\otimes}X)$ is given by $u \otimes (v{\otimes}x) \;\mapsto\; \sum_\ell u_\ell^1 v \otimes (u_\ell^2{\otimes}x)$, where $\sum_\ell u_\ell^1{\otimes}u_\ell^2 = \Delta(u)$ is the coproduct of $u \in \mathsf{U}(\mathfrak{h})$.

Moreover, by taking co-invariants the result (A.2) implies that there is a natural isomorphism

$$\lfloor V \otimes W \rfloor_{\mathfrak{h}} \;\cong\; V \otimes X \tag{A.3}$$

of vector spaces. In other words, the elements of the vector space $V \otimes X$ provide natural representatives for the co-invariants of $V \otimes W$.

Next we consider the situation that the Lie algebra $\mathfrak{h}$ is the semi-direct sum $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ of an ideal $\mathfrak{h}_1$ and a subalgebra $\mathfrak{h}_2$. Then $\lfloor V \rfloor_{\mathfrak{h}_1}$ is a $\mathsf{U}(\mathfrak{h}_2)$-module, and one can evaluate the space $\lfloor V \rfloor_{\mathfrak{h}}$ of co-invariants in a two-step procedure:

$$\lfloor V \rfloor_{\mathfrak{h}} = \lfloor \lfloor V \rfloor_{\mathfrak{h}_1} \rfloor_{\mathfrak{h}_2} . \tag{A.4}$$

Furthermore, when V is a free $\mathfrak{h}$-module, then equality (A.4) already holds when $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ with $\mathfrak{h}_1$ and $\mathfrak{h}_2$ subalgebras, i.e. it is not required that $\mathfrak{h}_1$ is an ideal.

In order to establish these statements, we first note that whenever $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ as a direct sum of vector spaces, then it follows from the Poincaré–Birkhoff–Witt theorem (upon choosing a basis of $\mathfrak{h}$ that is the union of bases $\{h^a_{(1)}\}$ of $\mathfrak{h}_1$ and $\{h^q_{(2)}\}$ of $\mathfrak{h}_2$ to obtain a suitably ordered basis of $\mathsf{U}(\mathfrak{h})$) that every $u \in \mathsf{U}(\mathfrak{h})$ can be written as

$$u = \xi\, \mathbf{1} + \sum_a \xi_a h^a_{(1)} + \sum_q \eta_q h^q_{(2)} + \sum_{a,q} \zeta_{a,q} h^a_{(1)} h^q_{(2)} + \dots\,, \tag{A.5}$$

and that this decomposition is unique. In particular, $\mathsf{U}^+(\mathfrak{h})$ decomposes as a vector space as

$$\mathsf{U}^+(\mathfrak{h}) = \mathsf{U}^+(\mathfrak{h}_1) \oplus \mathsf{U}(\mathfrak{h}_1)\, \mathsf{U}^+(\mathfrak{h}_2) . \tag{A.6}$$

Now in the special case when $\mathfrak{h}$ is the semi-direct sum $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ of an ideal $\mathfrak{h}_1$ and a subalgebra $\mathfrak{h}_2$, it is a direct consequence of $[\mathfrak{h}_1, \mathfrak{h}_2] \subseteq \mathfrak{h}_1$ that $\lfloor V \rfloor_{\mathfrak{h}_1}$ is a $\mathsf{U}(\mathfrak{h}_2)$-module. Together with (A.6) it then follows that for any $\mathfrak{h}$-module V one can evaluate the space $\lfloor V \rfloor_{\mathfrak{h}}$ of co-invariants in a two-step procedure as described in (A.4).

Furthermore, when $V \cong \mathsf{U}(\mathfrak{h})\otimes X$ is a free $\mathfrak{h}$-module, then the decomposition (A.5) of elements of $\mathsf{U}(\mathfrak{h})$ implies that every element of V can be written *uniquely* as

$$v = \mathbf{1}\otimes x + \sum_i u^+_{(1),i}\otimes x_i + \sum_j u^+_{(2),j}\otimes y_j + \sum_{i,j} \tilde u^+_{(1),i}\tilde u^+_{(2),j}\otimes x_{ij} \tag{A.7}$$

(all sums finite) with $x, x_i, y_j, x_{ij}\in X$, $u^+_{(1),i}, \tilde u^+_{(1),i}\in\mathfrak{h}_1$ and $u^+_{(2),i}, \tilde u^+_{(2),i}\in\mathfrak{h}_2$. It follows that a natural representative of the class of v in $\lfloor V\rfloor_{\mathfrak{h}}$ is given by $x\in X$. Moreover, when $\mathfrak{h}_1$ is actually a subalgebra of $\mathfrak{h}$, one can then consider the space of co-invariants $\lfloor V\rfloor_{\mathfrak{h}_1}$, and a natural representative of the class of v in $\lfloor V\rfloor_{\mathfrak{h}_1}$ is given by $\mathbf{1}\otimes x + \sum_j u^+_{(2),j}\otimes y_j$. Finally, when also $\mathfrak{h}_2$ is a subalgebra of $\mathfrak{h}$ and $\lfloor V\rfloor_{\mathfrak{h}_1}$ is an $\mathfrak{h}_2$-module, then we can take $\mathfrak{h}_2$-co-invariants of $\lfloor V\rfloor_{\mathfrak{h}_1}$, and a natural representative of the class of $\mathbf{1}\otimes x + \sum_j u^+_{(2),j}\otimes y_j$ is again x. Note that $\lfloor V\rfloor_{\mathfrak{h}_1}$ can indeed be endowed with the structure of an $\mathfrak{h}_2$-module, namely by identifying the action of $\mathfrak{h}_2$ on classes by its action on the distinguished representatives $\mathbf{1}\otimes x + \sum_j u^+_{(2),j}\otimes y_j$, i.e. by demanding that $h_2\in\mathfrak{h}_2$ acts on the $\mathsf{U}^+(\mathfrak{h}_1)$-class $[v]\in\lfloor V\rfloor_{\mathfrak{h}_1}$ of $v\in V$ by

$$h_2\,[v] := [h_2\otimes x + \sum_j h_2 u^+_{(2),j}\otimes y_j] \tag{A.8}$$

when v is decomposed as in (A.7). Using the uniqueness of that decomposition, it is straightforward to check that the prescription (A.8) yields a linear representation of $\mathsf{U}(\mathfrak{h}_2)$ on $\lfloor V\rfloor_{\mathfrak{h}_1}$.

In summary, we have shown that indeed, when V is a free $\mathfrak{h}$-module and $\mathfrak{h}=\mathfrak{h}_1\oplus\mathfrak{h}_2$ as a direct sum of vector spaces, then for the equality (A.4) of spaces of co-invariants to hold it is sufficient that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are subalgebras of $\mathfrak{h}$, while for arbitrary $\mathfrak{h}$-modules V one must in addition require that $\mathfrak{h}_1$ be an ideal of $\mathfrak{h}$.

A.2 Right-exactness

Here we demonstrate that the functor of taking co-invariants is right-exact. We first observe that whenever

$$\ldots\to V^{p+1}\xrightarrow{f^{p+1}} V^p \xrightarrow{f^p} V^{p-1}\to\ldots\to V^1\xrightarrow{f^1} V^0\xrightarrow{f^0} 0 \tag{A.9}$$

is a semi-infinite exact sequence of modules of an arbitrary Lie algebra $\mathfrak{h}$, then by taking co-invariants one obtains an analogous complex

$$\ldots\to \lfloor V^{p+1}\rfloor_{\mathfrak{h}}\xrightarrow{f^{p+1}_{\mathfrak{h}}} \lfloor V^p\rfloor_{\mathfrak{h}} \xrightarrow{f^p_{\mathfrak{h}}} \lfloor V^{p-1}\rfloor_{\mathfrak{h}}\to\ldots\to \lfloor V^1\rfloor_{\mathfrak{h}}\xrightarrow{f^1_{\mathfrak{h}}} \lfloor V^0\rfloor_{\mathfrak{h}}\xrightarrow{f^0_{\mathfrak{h}}} 0 \tag{A.10}$$

of vector spaces. To define this complex we note that the maps $f^p\colon V^p\to V^{p-1}$ are $\mathfrak{h}$-intertwiners. Therefore they give rise to linear maps $f^p_{\mathfrak{h}}\colon \lfloor V^p\rfloor_{\mathfrak{h}}\to\lfloor V^{p-1}\rfloor_{\mathfrak{h}}$ between the spaces of co-invariants with respect to $\mathfrak{h}$, which satisfy $f^p_{\mathfrak{h}}\circ\pi^p = \pi^{p-1}\circ f^p$, where for each p the map $\pi^p\colon V^p\to\lfloor V^p\rfloor_{\mathfrak{h}}$ is the canonical projection; thus they act as $f^p_{\mathfrak{h}}\colon [v^p]\mapsto[f^p(v^p)]$, where for $v\in V$ we denote by $[v]$ the equivalence class of v modulo $\mathsf{U}^+(\mathfrak{h})V$, and this action does not depend on the choice of representatives.

Let us verify that (A.10) is indeed a complex. If $[v^p]\in Im\, f^{p+1}_{\mathfrak{h}}$, then there exist a $w^{p+1}\in V^{p+1}$ as well as finitely many elements $w^p_j\in V^p$ and $x_j\in\mathfrak{h}$ such that v^p can be written as $v^p =$

$f^{p+1}(w^{p+1})+\sum_j x_j w_j^p$. The facts that the sequence (A.9) is exact and that f^p intertwines the $\mathfrak{h}$-action therefore imply that $f^p(v^p) = f^p(\sum_j x_j w_j^p) = \sum_j x_j f^p(w_j^p)$. It follows that $[f^p(v^p)] = 0$, so that $[v^p] \in Ker\, f_{\mathfrak{h}}^p$. Hence we have

$$Im\, f_{\mathfrak{h}}^{p+1} \subseteq Ker\, f_{\mathfrak{h}}^p\,, \tag{A.11}$$

or in other words, $f_{\mathfrak{h}}^p \circ f_{\mathfrak{h}}^{p+1} = 0$, for all $p \in \mathbb{Z}_{>0}$.

We can now show that the complex (A.10) is always *right-exact*, i.e. exact at its last two entries. (In general, there is however no reason why this complex should be exact also at the other positions.) We start at the right of the diagram. The map f^1 is surjective, so that we have $Im\,(f_{\mathfrak{h}}^1) = \lfloor V^0 \rfloor_{\mathfrak{h}}$. Since $Ker\, f_{\mathfrak{h}}^0 = \lfloor V^0 \rfloor_{\mathfrak{h}}$, this already shows exactness at $\lfloor V^0 \rfloor_{\mathfrak{h}}$. From the result (A.11) for general p we also already know that $Im\, f_{\mathfrak{h}}^2 \subseteq Ker\, f_{\mathfrak{h}}^1$. To show the converse, suppose that $[v^1] \in Ker\, f_{\mathfrak{h}}^1$. Then there exist a finite number of $x_j \in \mathfrak{h}$ and $w_j^0 \in V^0$ such that $f^1(v^1) = \sum_j x_j w_j^0$. Moreover, since f^1 is surjective, for each j we can write $w_j^0 = f^1(w_j^1)$ for some $w_j^1 \in V^1$. It follows that $f^1(v^1 - \sum_j x_j w_j^1) = 0$, and hence $v^1 - \sum_j x_j w_j^1 \in Ker\, f^1$. Since the original sequence (A.9) is exact at V^1, there then exists some $w^2 \in V^2$ such that $f^2(w^2) = v^1 - \sum_j x_j w_j^1$. Therefore $f_{\mathfrak{h}}^2([w^2]) = [v^1 - \sum_j x_j w_j^1] = [v^1]$, and hence $[v^1] \in Im\, f_{\mathfrak{h}}^2$. Thus we have shown that $Im\, f_{\mathfrak{h}}^2 = Ker\, f_{\mathfrak{h}}^1$, so that the sequence (A.10) of co-invariants is also exact at $\lfloor V^1 \rfloor_{\mathfrak{h}}$, as claimed.

B Proof of the isomorphisms (4.5) and (4.6)

Here we establish the isomorphisms (4.5) and (4.6). Important tools are provided by the results about co-invariants that were described in Appendix A.

We use the notations of subsection 4.1, and we pick a 'global' coordinate z of $\mathbb{P}^1$ such that the set Q of insertion points contains the point $z=0$. To derive (4.5), we first recall the isomorphism $\mathcal{P}_\Lambda \cong \mathsf{U}(\mathfrak{g}^-) \otimes \bar{\mathcal{H}}_{\bar\Lambda}$ (2.14). Owing to the general result (A.3), upon tensoring with the $\mathfrak{g}^-$-module $\tilde{\mathcal{H}}$ (4.4) this implies the natural isomorphism

$$\tilde{\mathcal{H}} \otimes \bar{\mathcal{H}}_{\bar\Lambda} \;\cong\; \lfloor \tilde{\mathcal{H}} \otimes \mathcal{P}_\Lambda \rfloor_{\mathfrak{g}^-}\,. \tag{B.1}$$

Note in particular that according to (B.1) the space $\lfloor \tilde{\mathcal{H}} \otimes \mathcal{P}_\Lambda \rfloor_{\mathfrak{g}^-}$ of co-invariants is a $\bar{\mathfrak{g}}(\mathbb{P}^1 \backslash Q)$-module. Next we observe that we can write the Lie algebra $\bar{\mathfrak{g}}(\mathbb{P}^1 \backslash (Q \cup \{\infty\}))$ as a vector space direct sum of the Lie algebra $\bar{\mathfrak{g}}(\mathbb{P}^1 \backslash Q)$ and the Lie algebra

$$\tilde{\mathfrak{g}}^- := \bar{\mathfrak{g}} \otimes z\,\mathbb{C}(\!(z)\!)\,, \tag{B.2}$$

i.e. that there is an isomorphism

$$\bar{\mathfrak{g}}(\mathbb{P}^1 \setminus (Q \cup \{\infty\})) \;\cong\; \bar{\mathfrak{g}}(\mathbb{P}^1 \setminus Q) \oplus \tilde{\mathfrak{g}}^- \tag{B.3}$$

of vector spaces, and moreover, that the two summands are in fact subalgebras of $\bar{\mathfrak{g}}(\mathbb{P}^1 \backslash (Q \cup \{\infty\}))$. Furthermore, a local coordinate around the additional insertion point ∞ is given by $t = 1/z$.

Thus via $\tilde{\mathfrak{g}}^-$ we have included functions with poles at this insertion point, or in other words, $\tilde{\mathfrak{g}}^-$ acts on the module $\mathcal{H}_\Lambda$ by elements of $\mathfrak{g}^-$ (this is of course the reason why we chose the superscript '$-$' in the notation $\tilde{\mathfrak{g}}^-$). Note that every vector of $\mathcal{H}_\Lambda$ is annihilated by all but a finite number of generators of $\tilde{\mathfrak{g}}^-$, so that we actually have to deal only with finite series; accordingly we will identify $\tilde{\mathfrak{g}}^-$ with $\mathfrak{g}^-$ from now on.

By taking co-invariants with respect to $\bar{\mathfrak{g}}(\mathbb{P}^1\backslash Q)$ in (B.1), we can then conclude that (4.5) is valid. When doing so, we just have to apply the identity (A.4); recall that for the validity of (A.4) it is sufficient that $\mathfrak{g}^-$ and $\bar{\mathfrak{g}}(\mathbb{P}^1\backslash Q)$ are subalgebras, since $\mathcal{P}_\Lambda$ is a free $\mathfrak{g}^-$-module.

In order to show (4.6) as well, we first introduce an action of $\mathfrak{g}^-$ in an obvious manner, i.e. analogous to the action (2.19) of the block algebra; thus on a factor $\mathcal{H}_{\Lambda'_j}$ of $\tilde{\mathcal{H}}_{(2)}$ the element $\bar{x}{\otimes}f\in\mathfrak{g}^-$ acts as $\bar{x}{\otimes}f_{q_j}(\zeta_j)$ with $f_{q_j}(\zeta_j)$ the Laurent series of f at q_j, while on a factor $\bar{\mathcal{H}}_{\bar{\Lambda}_i}$ of $\tilde{\mathcal{H}}_{(1)}$ it acts via evaluation, i.e. by $\bar{x}{\otimes}\tilde{f}_{p_i}(0)$ with $\tilde{f}_{p_i}(\zeta_i)$ the power series expansion of f at p_i. Note that this way the space $\tilde{\mathcal{H}}$ is *not* turned into a $\bar{\mathfrak{g}}(\mathbb{P}^1\backslash(Q\cup\{\infty\}))$-module. Indeed, the residue theorem that in the case of (2.19) forced the central terms to cancel now no longer implies such a cancellation, because the functions can have additional poles at ∞, so that in the Lie bracket of $\bar{x}{\otimes}f$ and $\bar{y}{\otimes}g$ ($\bar{x},\bar{y}\in\bar{\mathfrak{g}}$, $f,g\in\mathcal{F}(\mathbb{P}^1\backslash(Q\cup\{\infty\}))$ we are left over with a central term proportional to $-\mathrm{k}\,\kappa(\bar{x},\bar{y})\sum_{i=1}^m \mathrm{Res}_\infty(f\mathrm{d}g)$. (In physicists' terminology, this is formulated as follows. Every contour encircling all insertions points except for the one at infinity can be deformed to a contour around infinity. This contour, however, does not have the standard orientation, thus accounting for the minus sign of the level.) Thus the representation is only a projective one, and precisely as in the two-block case discussed in section 2 we have to work with a central extension $\hat{\bar{\mathfrak{g}}}(\mathbb{P}^1\backslash(Q\cup\{\infty\})$ of the block algebra $\bar{\mathfrak{g}}(\mathbb{P}^1\backslash(Q\cup\{\infty\})$.

Let us now denote by $\mathcal{K}_\Lambda$ the kernel of the canonical surjection π: $\mathcal{P}_\Lambda\to\mathcal{H}_\Lambda$. The space $\mathcal{K}_\Lambda$ is generated by a single primitive null vector w; in terms of the highest weight vector w_Λ of $\mathcal{P}_\Lambda$ it is given by

$$w=(E^{\bar{\theta}}{\otimes}t^{-1})^{\mathrm{k}-(\bar{\Lambda},\bar{\theta}^\vee)+1}w_\Lambda\,, \tag{B.4}$$

where by $E^{\pm\bar{\theta}}\in\bar{\mathfrak{g}}$ we denote the step operators in $\bar{\mathfrak{g}}$ that correspond to the highest $\bar{\mathfrak{g}}$-root $\bar{\theta}$ and to its negative, respectively. Now the fact that the element $E^{-\bar{\theta}}{\otimes}t$ of $\mathfrak{g}^+$ acts locally nilpotently on the space $\tilde{\mathcal{H}}$ (4.4) means that for each $v\in\tilde{\mathcal{H}}$ there exists a positive integer $M=M(v)$ such that $(E^{-\bar{\theta}}{\otimes}t)^M v=0$.[12] Further, by commuting the element $(E^{-\bar{\theta}}{\otimes}t)^M$ of $\mathsf{U}(\mathfrak{g})$ through $(E^{\bar{\theta}}{\otimes}t^{-1})^{M+\mathrm{k}-(\bar{\Lambda},\bar{\theta}^\vee)+1}$ and using the fact that the highest weight vector w_Λ of $\mathcal{P}_\Lambda$ is annihilated by $\mathfrak{g}^+$, it follows that the vector $w':=(E^{-\bar{\theta}}{\otimes}t)^M(E^{\bar{\theta}}{\otimes}t^{-1})^{M+\mathrm{k}-(\bar{\Lambda},\bar{\theta}^\vee)+1}w_\Lambda$ is a non-zero multiple of w. This means that there is a non-zero vector $\tilde{w}\in\mathcal{P}_\Lambda$ such that w can be written as $w=(E^{-\bar{\theta}}{\otimes}t)^M\tilde{w}$. Just as on $\tilde{\mathcal{H}}$, the algebra $\bar{\mathfrak{g}}(\mathbb{P}^1\backslash(Q\cup\{\infty\})$ only acts projectively on $\mathcal{K}_\Lambda$, but the central element of the extension $\hat{\bar{\mathfrak{g}}}(\mathbb{P}^1\backslash(Q\cup\{\infty\})$ now acts with value k rather than $-$k.

It follows that for each vector $v\in\tilde{\mathcal{H}}$ we have, as an element of $\tilde{\mathcal{H}}\otimes_{\mathsf{U}(\bar{\mathfrak{g}}(\mathbb{P}^1\backslash(Q\cup\{\infty\})))}\mathcal{P}_\Lambda\equiv$

[12] Note that the expression $(E^{-\bar{\theta}}{\otimes}t)^M v\equiv(E^{-\bar{\theta}}{\otimes}z^{-1})^M v$ is an element of $\bar{\mathfrak{g}}(\mathbb{P}^1\backslash(Q\cup\{\infty\})$ only if $z^{-1}\in\mathcal{F}(\mathbb{P}^1\backslash(Q\cup\{\infty\}))$. This is indeed satisfied because we have $0\in Q$. It is here that our assumption about Q being non-empty enters.

$\lfloor \tilde{\mathcal{H}} \otimes \mathcal{P}_\Lambda \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1\setminus(Q\cup\{\infty\}))}$, the identity

$$v \otimes w = v \otimes (E^{-\bar\theta} \otimes t)^M \tilde{w} = (-1)^M \left((E^{-\bar\theta} \otimes t)^M v \right) \otimes \tilde{w} = 0 \,. \tag{B.5}$$

Since the vector w generates the kernel $\mathcal{K}_\Lambda$ as a (projective) $\bar{\mathfrak{g}}(\mathbb{P}^1\setminus(Q\cup\{\infty\}))$-module, this result holds analogously for every other element of $\tilde{\mathcal{H}} \otimes \mathcal{K}_\Lambda$ as well. It follows that the image of $\tilde{\mathcal{H}} \otimes \mathcal{K}_\Lambda$ under the surjection $id \times \pi$: $\tilde{\mathcal{H}} \otimes \mathcal{P}_\Lambda \to \tilde{\mathcal{H}} \otimes \mathcal{H}_\Lambda$ is zero in $\lfloor \tilde{\mathcal{H}} \otimes \mathcal{H}_\Lambda \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1\setminus(Q\cup\{\infty\}))}$. In short, the kernel $\mathcal{K}_\Lambda$ does not contain any co-invariants:

$$\lfloor \tilde{\mathcal{H}} \otimes \mathcal{K}_\Lambda \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1\setminus(Q\cup\{\infty\}))} = 0 \,. \tag{B.6}$$

Now we have a short exact sequence $0 \to \mathcal{K}_\Lambda \xrightarrow{\imath} \mathcal{P}_\Lambda \xrightarrow{\pi} \mathcal{H}_\Lambda \to 0$, which when tensored with the space $\tilde{\mathcal{H}}$ yields another exact sequence

$$0 \to \tilde{\mathcal{H}} \otimes \mathcal{K}_\Lambda \to \tilde{\mathcal{H}} \otimes \mathcal{P}_\Lambda \to \tilde{\mathcal{H}} \otimes \mathcal{H}_\Lambda \to 0 \,. \tag{B.7}$$

Upon taking co-invariants with respect to the Lie algebra $\bar{\mathfrak{g}}(\mathbb{P}^1\setminus(Q\cup\{\infty\}))$, this provides us with a complex

$$0 = \lfloor \tilde{\mathcal{H}} \otimes \mathcal{K}_\Lambda \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1\setminus(Q\cup\{\infty\}))} \to \lfloor \tilde{\mathcal{H}} \otimes \mathcal{P}_\Lambda \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1\setminus(Q\cup\{\infty\}))} \to \lfloor \tilde{\mathcal{H}} \otimes \mathcal{H}_\Lambda \rfloor_{\bar{\mathfrak{g}}(\mathbb{P}^1\setminus(Q\cup\{\infty\}))} \to 0 \,. \tag{B.8}$$

The right-exactness of the functor of taking co-invariants that was described in subsection A.2 now tells us that (B.8) is in fact an exact sequence. From (B.6) we can therefore conclude that the isomorphism (4.6) is valid, as claimed.

C On the lowest weight socle of $\left(\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)}\right)^\star$

The *socle* of a module is by definition the linear hull of all its irreducible submodules; it is in fact a direct sum of irreducible modules. Here we are concerned with the *highest*, respectively *lowest weight socle* $soc_\pm\mathcal{H}$ of a module $\mathcal{H}$ over a Lie algebra $\mathfrak{h}$, defined as the linear hull of all its irreducible highest (respectively lowest) weight submodules, which is a direct sum of irreducible highest (respectively lowest) weight modules. The situation of our interest is the one where $\mathfrak{h}$ is an affine Lie algebra and $\mathcal{H} = (\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)})^\star$, with $\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)}$ the tensor product of the level-k integrable highest weight module $\mathcal{H}_{\Lambda_1}$ and the level-0 evaluation module $\overline{\mathcal{H}}^{(m-2)}$, which was considered in subsection 4.2 (where $\mathfrak{h}$ was realized as the centrally extended block algebra $\hat{\mathfrak{z}}$).

Recall that we describe the space B of chiral blocks as the space $\lfloor V \rfloor_\mathfrak{h}$ of co-invariants of the tensor product

$$V = (\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)}) \otimes \tilde{\mathcal{H}}_{\Lambda_m} \tag{C.1}$$

of $\mathcal{H}_{\Lambda_1} \otimes \overline{\mathcal{H}}^{(m-2)}$ with an irreducible lowest weight $\mathfrak{h}$-module $\tilde{\mathcal{H}}_{\Lambda_m}$ at level $-$k. Its dual space $B^\star$ then coincides with the space $(V^\star)^\mathfrak{h}$ of $\mathfrak{h}$-invariants (singlets) in the algebraic dual $V^\star$ of V, i.e. the elements of $(V^\star)^\mathfrak{h}$ are in one-to-one correspondence to functions on the co-invariants. This holds because the kernel of every $\psi \in (V^\star)^\mathfrak{h}$ contains the submodule $\mathsf{U}^+(\mathfrak{h})(V)$ of V, so that by setting $\hat\psi([v]) := \psi(v)$ for each $\psi \in (V^\star)^{\mathfrak h}$ one defines a linear function on the space $\lfloor V\rfloor_{\mathfrak h}$ of co-invariants, and vice versa.

One can even show that the module $\mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)}$ has trivial highest and lowest weight socles. In contrast, every element in the dual $\lfloor V\rfloor_{\mathfrak h}^\star$ gives rise to a submodule of $(\mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)})^\star$ that is isomorphic to the irreducible lowest weight module $\tilde{\mathcal H}_{\Lambda_m}$. Indeed, given any $\hat\psi \in \lfloor V\rfloor_{\mathfrak h}^\star$, we can define for every $v \in \tilde{\mathcal H}_{\Lambda_m}$ the linear function $\psi_v \in (\mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)})^\star$ by

$$\psi_v(w) := \hat\psi([w\otimes v]) \tag{C.2}$$

for all $w \in \mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)}$. One can check that the action of $\mathfrak g$ on $\tilde{\mathcal H}_{\Lambda_m}$ precisely reproduces the action of $\mathfrak g$ on the subspace

$$(\mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)})^\star_{(\hat\psi)} := \{\psi_v \in (\mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)})^\star \mid v \in \tilde{\mathcal H}_{\Lambda_m}\} \tag{C.3}$$

of $(\mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)})^\star$, and vice versa. In other words, the subspace $(\mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)})^\star_{(\hat\psi)}$ is isomorphic to $\tilde{\mathcal H}_{\Lambda_m}$ as an $\mathfrak h$-module. Moreover, by construction this correspondence between $\hat\psi$ and $(\mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)})^\star_{(\hat\psi)}$ is one-to-one, so $(\mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)})^\star$ contains the direct sum

$$\bigoplus_{\hat\psi\in\lfloor V\rfloor_{\mathfrak h}^\star} (\mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)})^\star_{(\hat\psi)} \;\cong\; \lfloor V\rfloor_{\mathfrak h}^\star \otimes \tilde{\mathcal H}_{\Lambda_m} \tag{C.4}$$

as a submodule. Note that this is indeed a *direct* sum (because the modules $(\mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)})^\star_{(\hat\psi)}$ are minimal non-zero submodules of $(\mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)})^\star$), which shows in particular that

$$[soc_-(\mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)})^\star : \tilde{\mathcal H}_{\Lambda_m}] \;=\; \dim \lfloor V\rfloor_{\mathfrak h}\,. \tag{C.5}$$

Clearly, the relation (C.5) implies that

$$[(\mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)})^\star : \tilde{\mathcal H}_{\Lambda_m}] \;\ge\; \dim \lfloor V\rfloor_{\mathfrak h}\,. \tag{C.6}$$

This inequality gets strengthened to strict equality iff the lowest weight socle series of $(\mathcal H_{\Lambda_1}\otimes\overline{\mathcal H}^{(m-2)})^\star$ terminates after its first term. By showing that such a termination occurs under appropriate conditions on the evaluation module $\overline{\mathcal H}^{(m-2)}$ (so as to implement the fact that it originates from integrable $\mathfrak h$-modules), one would obtain an alternative possibility to determine the branching coefficient $N_{\Lambda_1;\bar\Lambda_2\ldots\bar\Lambda_{m-1};\Lambda_m^+}$.

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