CERN-TH/97-41
hep-th/9707053

A CLASS OF NON-SINGULAR GRAVI-DILATON BACKGROUNDS [1]

A. Buonanno[(a,b)], M. Gasperini[(c)] and C. Ungarelli[(a,b)]

[(a)] *Theory Division, CERN, CH-1211 Geneva 23, Switzerland*
[(b)] *Istituto Nazionale di Fisica Nucleare, Sezione di Pisa, Pisa, Italy*
[(c)] *Dipartimento di Fisica Teorica, Università di Torino,*
Via P. Giuria 1, 10125 Turin, Italy

Abstract

We present a class of static, spherically symmetric, non-singular solutions of the tree-level string effective action, truncated to first order in α'. In the string frame the solutions approach asymptotically (at $r \to 0$ and $r \to \infty$) two different anti-de Sitter configurations, thus interpolating between two maximally symmetric states of different constant curvature. The radial-dependent dilaton defines a string coupling which is everywhere finite, with a peak value that can be chosen arbitrarily small so as to neglect quantum-loop corrections. This example stresses the possible importance of finite-size α' corrections, typical of string theory, in avoiding space-time singularities.

To appear in **Mod. Phys. Lett. A**

CERN-TH/97-41
March 1997

[1] *This essay received an "Honorable Mention" from the Gravity Research Foundation, 1997 — Ed.*

It has often been conjectured, soon after the appearance of the first paper [1] discussing string corrections to the Schwarzschild metric, that the higher-derivative terms (the so-called α' corrections), appearing at next-to-leading order in the string effective action, should regularize the curvature singularity present at the origin in the Schwarzschild solution [2, 3]. It has been shown, in particular, that the singularity may indeed disappear when the delta-function of the effective point-like source, supporting the solution at the origin, is smeared out by the effect of α' corrections [4].

To support the expected "smoothing" of short-distance divergences, due to the fundamental cut-off scale of string theory, we will discuss in this paper static and spherically symmetric solutions of the string effective action in vacuum, with no sources (nor cosmological term) included in the action, but with higher-derivative terms included up to first order in α'

In general relativity it is well known that all non-trivial spherically symmetric solutions in vacuum are singular at the origin, and that the only solution that is regular everywhere is the trivial Minkowski manifold. For the lowest order string effective action all the non-trivial solutions are also singular. In this paper we will show that, when we add higher-derivative corrections to first order in α', the action admit instead non-trivial solutions in which the curvature is bounded everywhere, and asymptotically approaches two constant finite values at $r \to 0$ and $r \to \infty$.

We shall work in the string frame, where the gravi-dilaton effective action of critical string theory, at tree level in the string-loop expansion but including first-order α' corrections, can be written in the form [5]:

$$S = \int d^4x \sqrt{-g} e^{-\phi} \left[-R - (\nabla\phi)^2 + \frac{k\alpha'}{4} \left(R^2_{GB} - (\nabla\phi)^4 \right) \right]. \tag{1}$$

Here ϕ is the dilaton field, $R^2_{GB} \equiv R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$ is the Gauss–Bonnet invariant, and $k = 1, 1/2$ for the bosonic and heterotic string, respectively (conventions: $g_{\mu\nu} = (+ - - -)$, $R_{\mu\nu\alpha}{}^{\beta} = \partial_\mu \Gamma_{\nu\alpha}{}^{\beta} - ...$, and $R_{\nu\alpha} = R_{\mu\nu\alpha}{}^{\mu}$). Note that the particular field redefinition that we have chosen eliminates higher-than-second derivatives from the field equations, but necessarily introduces dilaton-dependent α' corrections in the effective action (the Gauss–Bonnet term, by itself, may represent the complete first-order α' corrections only in the conformally related Einstein frame [5]).

Looking for static and spherically symmetric solutions, parametrized by

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - \mathcal{R}^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$
$$\nu = \nu(r), \quad \lambda = \lambda(r), \quad \mathcal{R} = \mathcal{R}(r), \quad \phi = \phi(r), \tag{2}$$

the effective action (after integration by parts) can be rewritten as:

$$S = 4\pi \int dr e^{\frac{\nu}{2} - \frac{\lambda}{2} - \phi} \Big[\mathcal{R}^2\phi'^2 - \mathcal{R}^2\nu'\phi' - 4\mathcal{R}\mathcal{R}'\phi' + 2\mathcal{R}'^2 + 2\mathcal{R}\mathcal{R}'\nu' + 2e^{\lambda} +$$
$$+ k\alpha'\nu'\phi'\left(e^{-\lambda}\mathcal{R}'^2 - 1\right) - k\frac{\alpha'}{4}e^{-\lambda}\mathcal{R}^2\phi'^4\Big] \tag{3}$$

(a prime denotes differentiation with respect to r). By varying $\lambda, \nu, \mathcal{R}$ and ϕ, and imposing the radial gauge $\mathcal{R} = r$, we obtain respectively the equations (we put $k = 1$ for simplicity)

$$-\frac{1}{2}L + 2e^{\lambda} - \alpha'\nu'\phi' e^{-\lambda} + \frac{\alpha'}{4} r^2\phi'^4 e^{-\lambda} = 0, \tag{4}$$

$$\left[\phi'' + \left(\frac{\nu'}{2} - \frac{\lambda'}{2} - \phi'\right)\phi'\right]\left[-r^2 + \alpha'\left(e^{-\lambda} - 1\right)\right] +$$
$$+2r\left(\frac{\nu'}{2} - \frac{\lambda'}{2} - \phi'\right) - \phi'\left(2r + \alpha'\lambda' e^{-\lambda}\right) + 2 - \frac{L}{2} = 0, \tag{5}$$

$$2r\nu'' + 2\alpha' e^{-\lambda}\left(\nu''\phi' + \nu'\phi'' - \nu'\phi'\lambda'\right) - 2r\left(\phi'^2 - \nu'\phi'\right) +$$
$$\left(\frac{\nu'}{2} - \frac{\lambda'}{2} - \phi'\right)\left(-4r\phi' + 4 + 2r\nu' + 2\alpha'\nu'\phi' e^{-\lambda}\right) - 4r\phi'' + \frac{\alpha'}{2} r\phi'^4 e^{-\lambda} = 0, \tag{6}$$

$$4r\phi' + 2r^2\phi'' - 4 - \nu'\left(2r + \alpha'\lambda' e^{-\lambda}\right) + \nu''\left[-r^2 + \alpha'\left(e^{-\lambda} - 1\right)\right]$$
$$-\alpha' e^{-\lambda}\left(-\lambda' r^2\phi'^3 + 2r\phi'^3 + 3r^2\phi'^2\phi''\right) + L +$$
$$\left(\frac{\nu'}{2} - \frac{\lambda'}{2} - \phi'\right)\left[r^2\left(2\phi' - \nu'\right) - 4r + \alpha'\nu'\left(e^{-\lambda} - 1\right) - \alpha' r^2\phi'^3 e^{-\lambda}\right] = 0, \tag{7}$$

where

$$L(r) = r^2\phi'^2 - \phi'\left(r^2\nu' + 4r\right) + 2 + 2r\nu' + 2e^{\lambda} + \alpha'\nu'\phi'\left(e^{-\lambda} - 1\right) - \frac{\alpha'}{4} r^2\phi'^4 e^{-\lambda}. \tag{8}$$

Of these four equations, only three are independent. The first one, following from the variation of λ, does not contain second derivatives and can be used as a constraint on the initial conditions.

For $\phi = \text{const}$ and $\alpha' = 0$ the only non-trivial solution of the above system is the well known singular Schwarzschild metric, $e^{\nu} = e^{-\lambda} = 1 - 2m/r$. For $\phi' \neq 0$ and $\alpha' \neq 0$, on the contrary, there are non-trivial solutions with no curvature singularities at the origin. We can easily check that the above equations are indeed satisfied, at $r = 0$, by

$$\lambda(0) = 0, \quad \lambda'(0) = 0, \quad \nu'(0) = 0, \quad \nu''(0) = 2/\alpha', \quad \phi'(0) = \text{const} \neq 0, \tag{9}$$

corresponding to a non-zero but finite value of the curvature invariants at the origin. By using the Taylor expansion, and imposing the initial conditions (9), we then find a class of solutions that around $r = 0$ are approximated by

$$\begin{aligned} &\phi = \phi(0) + r\phi'(0) + \frac{r^2}{2}\phi''(0) + \mathcal{O}(r^3), \\ &\nu = \nu(0) + \frac{r^2}{\alpha'} + \mathcal{O}(r^3), \qquad\qquad \lambda = -\frac{r^2}{\alpha'} + \mathcal{O}(r^3) \end{aligned} \tag{10}$$

where $\phi(0)$ and $\nu(0)$ are left arbitrary, while ϕ' and ϕ'' are determined by the field equations as (in units $\alpha' = 1$):

$$\phi'(0) = -1.414..., \qquad\qquad \phi''(0) = -0.4166... \tag{11}$$

(there is also a non-trivial solution with $\phi'(0) > 0$, but in that case the singularity appears at a finite distance from the origin).

It is interesting to note that, for small enough r, the metric part of the solution (10) approximates an anti-de Sitter background with

$$e^{-\lambda} = 1 + \Lambda r^2, \qquad\qquad e^{\nu} = e^{\nu_0}\left(1 + \Lambda r^2\right), \tag{12}$$

and cosmological parameter $\Lambda = 1/\alpha'$ (the constant $\nu_0 = \nu(0)$ can be absorbed by rescaling the time coordinate). Such a metric parametrizes a maximally symmetric manifold, whose constant curvature invariants are determined by Λ as

$$R^2_{\mu\nu\alpha\beta} = 24\Lambda^2, \qquad\qquad R^2_{\mu\nu} = 36\Lambda^2, \qquad\qquad R^2 = 144\Lambda^2. \tag{13}$$

The computation of the Ricci tensor around the origin, for the approximated solution (10), gives in fact $R_\mu^\nu(0) = (3/\alpha')\delta_\mu^\nu$, in agreement with eq. (13) for $\Lambda = 1/\alpha'$.

In the opposite limit $r \to \infty$, the equations of motion (4)–(7) are again asymptotically satisfied by a maximally symmetric anti-de Sitter manifold, but with a different value of the effective cosmological constant. By setting

$$\phi = \phi_\infty - \gamma \log r, \qquad \nu = \nu_\infty + 2\log r, \qquad \lambda = -\log\Lambda - 2\log r \tag{14}$$

(where ν and λ approximate the metric (12) for $\Lambda r \gg 1$), the field equations (4)–(7) are in fact reduced, for $r \to \infty$, to a system of four algebraic equations:

$$-3 - 3\gamma - \frac{1}{2}\gamma^2 + 3\alpha'\Lambda\gamma + \frac{3}{8}\alpha'\Lambda\gamma^4 = 0, \tag{15}$$

$$3 + 2\gamma + \frac{1}{2}\gamma^2 - 2\alpha'\Lambda\gamma - \alpha'\Lambda\gamma^2 + \frac{1}{8}\alpha'\Lambda\gamma^4 = 0, \tag{16}$$

$$12 + 8\gamma + 2\gamma^2 - 8\alpha'\Lambda\gamma - 4\alpha'\Lambda\gamma^2 + \frac{1}{2}\alpha'\Lambda\gamma^4 = 0, \tag{17}$$

$$-12 + 6\alpha'\Lambda - 6\gamma - \gamma^2 + 3\alpha'\Lambda\gamma^3 + \frac{3}{4}\alpha'\Lambda\gamma^4 = 0, \tag{18}$$

which are not, of course, all independent (the left-hand side of eqs. (16) and (17) are indeed proportional), and which provide non-trivial solutions for the two unknown Λ and γ. Discarding negative values of γ, since we are looking for configurations with decreasing dilaton, we are then left with two possible pairs of real solutions:

$$\alpha'\Lambda = 1.820..., \qquad \gamma = 1.079...,$$

$$\alpha'\Lambda = 2.194..., \qquad \gamma = 0.8258.... \tag{19}$$

In the limit $r \to \infty$ the solution (14) satisfies the condition of maximally symmetric manifold (eq. (13)), with the asympotic value of the curvature fixed by the numerical values (19). The two constant parameters of the solution, ϕ_∞ and ν_∞, can be chosen so as to continuously match this anti-de Sitter configuration to the other one approaching the origin. A possible example of a solution that is everywhere regular (with no event horizons) and smoothly interpolates between the two asymptotic states of constant curvature, has beeen obtained by integrating numerically eqs. (4)–(7), and is illustrated by the following four figures (plotted for $\alpha' = 1$).

In Fig. 1 we show the logarithmic derivative of the time and radial components of the metric tensor, which display the typical anti-de Sitter behaviour of eq. (12), $\nu' = -\lambda' = 2\Lambda r/(1+\Lambda r^2)$, with the only difference that the effective cosmological parameter Λ is slightly different in the two regimes $r \ll \sqrt{\alpha'}$ and $r \gg \sqrt{\alpha'}$.

In Fig. 2 we show that the string coupling $g_s = e^{\phi/2}$ is decreasing for $r \to \infty$, and that its first and second derivatives are everywhere bounded. The normalization of ϕ at $r = 0$ is fixed by an arbitrary integration constant; it can always be chosen in such a way that $g_s < 1$ everywhere. The figure corresponds to the particular case $\phi(0) = -1$.

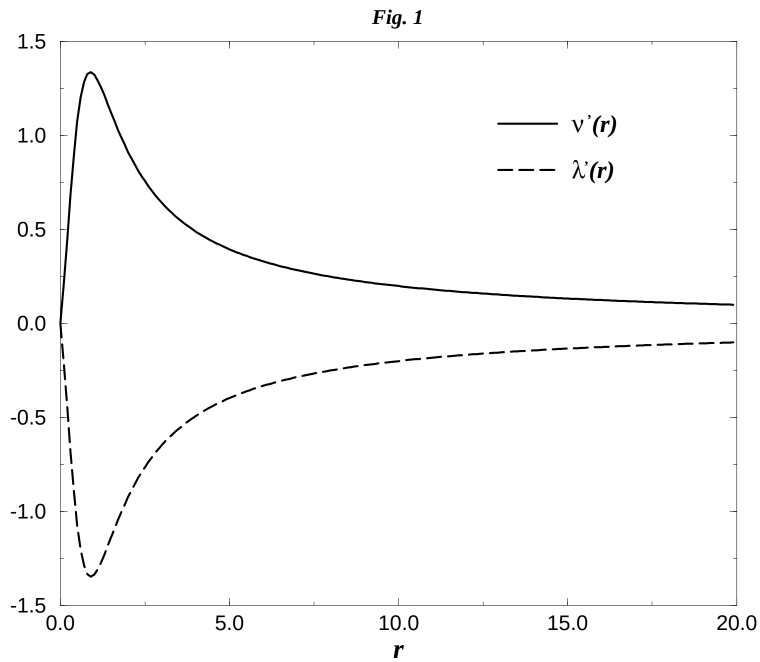

Figure 1: *Radial behaviour of the logarithmic derivatives of the metric tensor.*

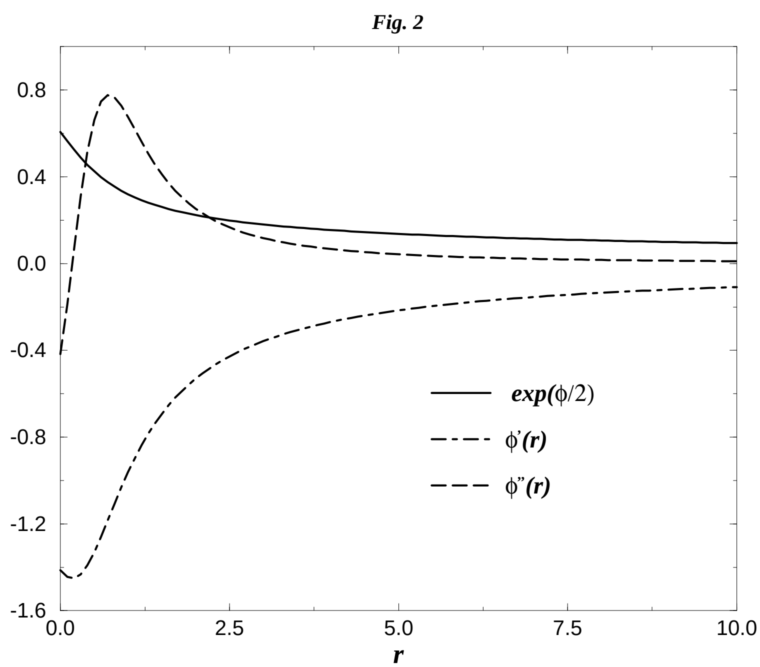

Figure 2: *Radial behaviour of the string coupling* $g_s = e^{\phi/2}$*, and of the derivatives of the dilaton field.*

Fig. 3

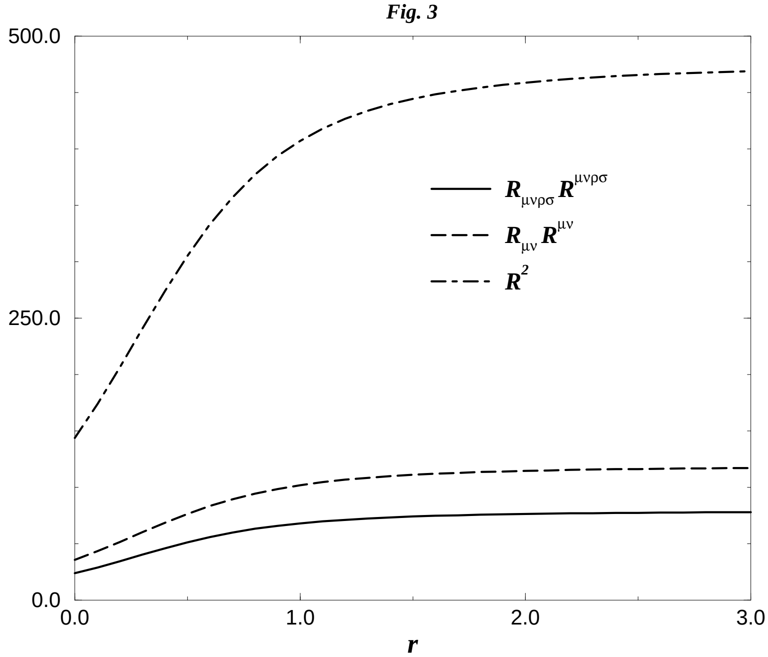

Figure 3: *Radial behaviour of the curvature invariants.*

In Fig. 3 we plot the curvature invariants $R^2_{\mu\nu\alpha\beta}$, $R^2_{\mu\nu}$ and R^2. Their radial dependence interpolates between the constant values determined, according to eq. (13), by an effective cosmological term $\Lambda = 1/\alpha'$ at $r \to 0$ and $\Lambda = 1.82/\alpha'$ at $r \to \infty$ (the latter corresponding to the first solution of eq. (19)).

In Fig. 4 we show that the ratios $R^2_{\mu\nu\alpha\beta}/R^2$ and $R^2_{\mu\nu}/R^2$ approach, asymptotically, the constant values $1/6 = 0.166$ and $1/4 = 0.25$ respectively, which are typical of maximally symmetric manifolds according to eq. (13).

In conclusion, a few comments are in order. This class of backgrounds is certainly interesting as a class of regular solutions of a higher-derivative model of gravity. For what concerns string theory, however, it is presently unclear whether such solutions can be extended to all orders in α', to represent a non-trivial zero of the corresponding sigma-model β-functions. The truncation of the action is indeed unmotivated in a string theory context, when the background reaches curvature scales of order one in string units, like in the example studied in this paper. Also, as discussed in the cosmological case [6], the existence of solutions that are everywhere regular may be a property that depends on the particular field redefinition adopted, until the α' expansion of the string effective action is truncated

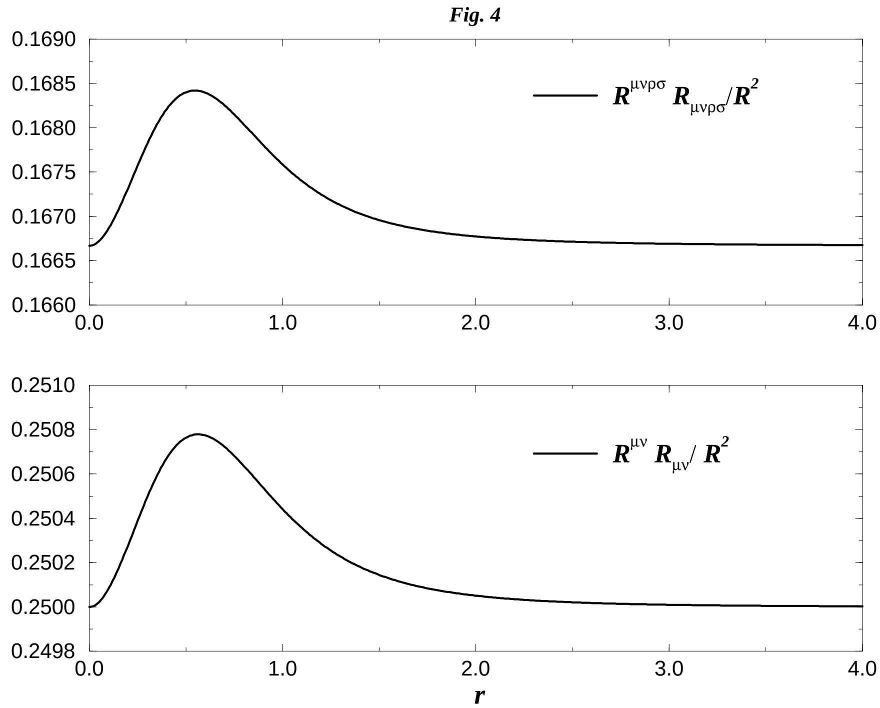

Figure 4: *Radial interpolation between the two asymptotic anti-de Sitter configurations.*

at a given finite order. This ambiguity can only be resolved by a solution corresponding to a true conformal field theory, exact to all orders in α'. The investigation of this aspect of the problem is postponed to future works.

Nevertheless, the existence of regular solutions in the perturbative regime of the quantum-loop expansion ($g_s \ll 1$), seems to support the expectation that space-time singularities may be regularized, already at a "classical" level, by the finite-size "stringy" α' corrections.

It is important to stress, finally, that constant curvature configurations may generally appear, asymptotically, when higher-derivative terms are added to a lowest-order action. Such configurations, however, are in general disconnected from the origin by one (or more) curvature singularities, appearing at finite values of r. By contrast, for the action (1) in which the higher-derivative corrections appear precisely in the form dictated by string theory, the regular asymptotic regimes can be smoothly connected, and there are solutions in which the curvature is everywhere bounded. In this sense the action (1) automatically implements the "limiting-curvature" hypothesis [7], often invoked to regularize space-time singularities.

Acknowledgements

We are grateful to Gabriele Veneziano for many useful conversations. A. B. and C. U. are partially supported by University of Pisa.

References

[1] C. G. Callan, R. C. Myers and M. J. Perry, Nucl. Phys. B311 (1988) 673.

[2] B. Harms and Y. Leblanc, Ann. of Phys. 242 (1995) 265.

[3] A. A. Tseytlin, *Black holes and exact solutions in string theory* (hep-th/9410008), in *Proc. of the 3rd Course of the Int. School of Astrophysics D. Chalonge*, Erice, 1994, eds. N. Sanchez and A. Zichichi (World Scientific, Singapore, 1995), p. 311.

[4] A. A. Tseytlin, Phys. Lett. B 363 (1995) 223.

[5] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. B293 (1987) 385.

[6] M. Gasperini, M. Maggiore and G. Veneziano, Nucl. Phys. B494, 315 (1997).

[7] V. F. Mukhanov and R. Brandenberger, Phys. Rev. Lett. 68 (1992) 1969.