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# Lectures on Heterotic–Type I Duality*

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**Abstract**

We present a review of heterotic–type I string duality. In particular, we discuss the effective field theory of six- and four-dimensional compactifications with $N > 1$ supersymmetries. We then describe various duality tests by comparing gauge couplings, $N = 2$ prepotentials, as well as higher-derivative F-terms.

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We present a review of heterotic–type I string duality. In particular, we discuss the effective field theory of six- and four-dimensional compactifications with $N > 1$ supersymmetries. We then describe various duality tests by comparing gauge couplings, $N = 2$ prepotentials, as well as higher-derivative F-terms.

## 1. INTRODUCTION

In the last couple of years, there has been remarkable progress in understanding non-perturbative aspects of string theories. The new ingredient is the discovery of various duality symmetries; these, in particular, invert the coupling constant and exchange perturbative states with non-perturbative solitonic excitations. Since inversion of the coupling involves the Planck constant $\hbar$, duality symmetries can be realized only at the quantum level. Once the dual models are identified, non-perturbative dynamics of one model can be studied by means of the standard perturbative techniques in the other model.

Dualities are known under the names of S, T and U, depending on whether they invert the string coupling in the target space-time, or the coupling of the underlying two-dimensional field theory of the string world-sheet, or exchange these two, respectively [1]. One of the main consequences of string dualities is that all known, apparently distinct, superstring theories correspond to different perturbative expansions of the same underlying theory, the so-called M-theory [2], whose low-energy limit is eleven-dimensional supergravity. In fact, there are five consistent superstring theories in ten dimensions: two type II theories of closed strings, IIA and IIB, which have two supersymmetries in ten dimensions, of opposite and same chirality, respectively; two heterotic closed string theories with $N = 1$ supersymmetry and $SO(32)$ or $E_8\times E_8$ gauge groups; and the type I theory of open and closed strings with $N = 1$ supersymmetry and $SO(32)$ gauge group [3].

It was already known for quite some time that the two type II theories, as well as the two heterotic ones, are related by T-duality, which is an exact perturbative symmetry after compactifying to lower dimensions. For instance, upon compactification to nine dimensions on a circle of radius $R$, type IIA and type IIB theories become equivalent by inverting $R$:

$$R \overset{T}{\longleftrightarrow} \frac{\alpha'}{R}\,, \tag{1}$$

where $\alpha'$ is the Regge slope. In four dimensions, the two theories are also identified by mirror symmetry with respect to the internal six-dimensional Calabi–Yau manifold. T-duality can then be used to compute world-sheet instanton corrections by using algebraic geometry in the mirror (dual) theory [4]. Similarly, the two heterotic theories are equivalent but T-duality is more complicated due to the presence of Wilson lines.

It has recently been realized that, in ten dimensions, there are two fundamental non-perturbative S-dualities. On the one hand, the heterotic and type I $SO(32)$ theories are equivalent by inverting the string coupling [2,5]:

$$\lambda \overset{S}{\longleftrightarrow} \frac{1}{\lambda}\,. \tag{2}$$

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We recall that the string coupling is given by the vacuum expectation value (vev) of the dilaton field: $\lambda = \langle e^{\phi}\rangle$. On the other hand, type IIB is self-dual under an $SL(2;\mathbf{Z})$ S-duality [6] acting on the complex field $\chi + i\lambda$, where $\chi$ is the Ramond–Ramond (RR) scalar. In this way, the two $N = 2$ and three $N = 1$ theories are separately connected. Their mutual connection is believed to arise from U-duality via M-theory [2,7]. In fact, the coupling constants of the type IIA and heterotic $E_8 \times E_8$ strings can be identified with the radius of the eleventh dimension of the M-theory compactified on $S^1$ and $S^1/\mathbf{Z}_2$, respectively:

$$\lambda = (R_{11}/l_P)^{3/2}\,, \tag{3}$$

where $l_P$ is the Planck length in eleven dimensions.

The purpose of these notes is to describe the main features, consequences and tests of heterotic–type I string duality. In Section 2, we give a brief introduction to type I strings, D-branes and orientifolds. In Section 3, we derive the main duality conjectures in dimensions 10, 6 and 4 by arguments based on the effective field theory. In Section 4, we discuss duality in six dimensions with $N = 1$ supersymmetry, while in Section 5, we study four-dimensional compactifications with $N = 2$ supersymmetry. In Section 6, we present general results for the one-loop corrections to the $N = 2$ prepotential and vector moduli metrics. We then perform various duality tests in the Higgs (Section 7.1) and Coulomb (Section 7.2) phase of $(T^4/\mathbf{Z}_2)\times T^2$ compactifications. Finally, in Section 8, we compare the infinite series of couplings corresponding to higher-derivative F-terms.

## 2. TYPE I STRINGS AND D-BRANES

Type I strings arise as world-sheet orbifolds (orientifolds) of the world-sheet left–right-symmetric IIB theory by modding it out with respect to the involution $\Omega$ that exchanges left- and right-movers [8,9]. The projection in the "untwisted" closed string sector introduces unoriented closed strings and has the effect of symmetrizing the Neveu–Schwarz–Neveu–Schwarz (NSNS) sector and antisymmetrizing the RR sector. As a result, the number of supersymmetries is reduced by half. In ten dimensions, the invariant (bosonic) massless states are the dilaton and graviton of the NSNS sector and the 2-index antisymmetric tensor of the RR sector.

Open strings appear as type IIB strings that close up to an $\Omega$ reflection, in a way similar to "twisted" states in orbifold constructions:

$$\Omega X(\sigma,\tau) \equiv X(-\sigma,\tau) = X(\sigma,\tau)\,. \tag{4}$$

Indeed, the mode expansion of an $\Omega$-invariant free string satisfying Eq. (4) is

$$\begin{aligned} X(\sigma,\tau) &= x + 2\alpha' p\tau \\ &+ i\sqrt{2\alpha'}\sum_{m\neq 0}\frac{\alpha_m}{m}\cos(m\sigma)e^{-im\tau}\,, \end{aligned} \tag{5}$$

which is the usual open string with Neumann boundary conditions $\partial_\sigma X|_{\sigma=0,\pi} = 0$. Open strings carry at their ends Chan–Paton charges that give rise to a gauge group. From the orientifold point of view, they play the role of fixed points. Their multiplicity is constrained by the tadpole cancellation which replaces (oriented) closed string modular invariance. In ten dimensions, one finds 32 real charges leading to an $SO(32)$ gauge group.

The modern approach to type I theory emphasizes the role of p-brane solitons of the corresponding type II string theory. In particular, the multiplicity of (complex) charges corresponds to the number of the so-called Dp-branes where open strings can end. Dp-branes provide sources for the RR type II (p+1)-potentials [10]. An open string with one end (say at $\sigma = 0$) on a Dp-brane has Neumann boundary conditions for $X^\mu$ with $\mu = 0, 1, \ldots$,p and Dirichlet boundary conditions along the perpendicular directions, $\partial_\tau X^\mu|_{\sigma=0} = 0$ for $\mu = \mathrm{p}+1, \ldots, 9$. These conditions fix the position of the string end in the space orthogonal to the p-brane, allowing it to move freely only in the directions parallel to the p-brane. According to D-brane terminology, strings with one end on a Dp-brane and the other one on a Dq-brane are called pq strings.

Type IIB has even-form RR potentials that can couple to Dp-branes with p odd. The $\Omega$ projection leaves invariant 2 mod 4 forms coupled to D1-, D5- and D9-branes only. In ten dimensions, Lorentz invariance allows only D9-branes with multiplicity 16 fixed by the tadpole cancellation. However after compactification to six or four dimensions, D5-branes can also appear.

Following the discussion above, the 1-loop partition function receives contributions from four genus-one surfaces: the torus ($\mathcal{T}$), the Klein bottle ($\mathcal{K}$), the annulus ($\mathcal{A}$) and the Möbius strip ($\mathcal{M}$). It takes the generic form:

$$Z_{1-\mathrm{loop}} = \frac{1}{2}(\mathcal{T}+\mathcal{K}) + \frac{1}{2}(\mathcal{A}+\mathcal{M}) \ , \tag{6}$$

where the two terms in brackets correspond to closed (untwisted) and open (twisted) states. In particular, $\mathcal{K}$ and $\mathcal{M}$ describe the propagation of closed and open unoriented strings, imposing the $\Omega$ projection in the two sectors of the theory.

T-duality can be incorporated into the open string framework with the main effect of interchanging Neumann (N) with Dirichlet (D) boundary conditions. In fact, imposing NN conditions in some compact direction $X \equiv X + 2\pi R$ yields

$$X(\sigma,\tau) = X_L(\tau+\sigma) + X_R(\tau-\sigma) \ , \tag{7}$$

and gives the expression (5) with the quantized momentum $p = n/R$. In analogy with closed strings, T-duality transforms $X_L \to X_L$, $X_R \to -X_R$, and inverts the radius $R$ to $\tilde{R} = \alpha'/R$ (1). It follows that $X \to \tilde{X}$, with the dual coordinate

$$\begin{aligned} \tilde{X}(\sigma,\tau) &= X_L(\sigma+\tau) - X_R(\sigma-\tau) \ , \\ &= \tilde{x} + 2nR\sigma + i\sqrt{2\alpha'}\sum_{l\neq 0}\frac{\alpha_l}{l}\sin l\sigma \ . \end{aligned} \tag{8}$$

We see that $\tilde{X}(0,\tau) = \tilde{x}$ and $\tilde{X}(\pi,\tau) = \tilde{x}+2\pi nR$, which means that the string is wrapped $n$ times around the circle and has Dirichlet boundary conditions on a D-brane at $\tilde{X} = \tilde{x}$. Thus a Dp-brane compactified on a circle is exchanged under T-duality with a D(p-1)-brane obtained by wrapping around the compact dimension.

Furthermore, under T-duality, $\Omega$ is mapped to $\tilde{\Omega} = \Omega\mathcal{R}$ where $\mathcal{R}$ is the $\mathbf{Z}_2$ orbifold transformation $X \to -X$. Indeed, starting from Neumann boundary conditions for the compact coordinate $X$ (7) and performing a T-duality transformation, one finds that the dual coordinate $\tilde{X}$ (8) is antiperiodic under the exchange of left- and right-movers.

Finally, T-duality exchanges Wilson lines of the gauge fields associated with Neumann boundary conditions with the D-brane positions in the compactified space. In fact, in the presence of Wilson lines $a_{i,j}$ corresponding to an NN (traceless) gauge group generator $T_{(i,j)} = \mathrm{diag}(0\dots 1_i \dots -1_j \dots 0)$, on a circle, the compact momentum is shifted as:

$$\frac{n}{R} \to \frac{n + a_i - a_j}{R} \ . \tag{9}$$

Using the same line of reasoning as before, it is easy to see that the dual coordinate satisfies $\tilde{X}(\pi,\tau) - \tilde{X}(0,\tau) = 2\pi(n + a_i - a_j)R$. This shows that up to a common shift, $a_i$'s represent positions of the open string ends, or equivalently the location of D-branes.

## 3. EFFECTIVE FIELD THEORIES

Effective Lagrangians, and most of all their dependence on the dilaton fields, provide a powerful guide into various duality symmetries. Let $\phi_n$ denote the dilaton field in $n$ non-compact dimensions. Focusing on the Einstein and Yang–Mills kinetic terms of the heterotic $SO(32)$ and type I string effective Lagrangians we have

$$\mathcal{L} = e^{-2\phi_{10}}\left\{\frac{1}{2}R + \frac{1}{4}F^2 \left| \begin{matrix} 1 \\ e^{\phi_{10}} \end{matrix} \right. + \dots\right\} , \tag{10}$$

where the heterotic and type I gauge kinetic factors are assembled together as upper and lower entries, respectively. These Lagrangian terms contain also implicit dimensionful factors, which are all equal to 1 in the units of $\alpha'$.

After compactification from 10 to $D \le 10$ dimensions on the same manifold of volume $V_{10-D}$,

$$\mathcal{L} = e^{-2\phi_{10}}V_{10-D}\left\{\frac{1}{2}R + \frac{1}{4}F^2 \left| \begin{matrix} 1 \\ e^{\phi_{10}} \end{matrix} \right. + \dots\right\} . \tag{11}$$

The above expressions allow the following identification of dilatons in dimension $D$:

$$e^{-2\phi_D} \equiv e^{-2\phi_{10}}V_{10-D} \ , \tag{12}$$

in terms of which we have

$$\mathcal{L} = e^{-2\phi_D}\left\{\frac{1}{2}R + \frac{1}{4}F^2 \left| \begin{matrix} 1 \\ e^{\phi_D}V_{10-D}^{1/2} \end{matrix} \right. + \dots\right\} . \tag{13}$$

Finally, we go to the Einstein frame by rescaling the metric $g_{\mu\nu} \to g_{\mu\nu} e^{4\phi_D/(D-2)}$ to obtain

$$\mathcal{L} = \frac{1}{2}R + \frac{1}{4}F^2 \left| \begin{matrix} e^{\frac{-4}{D-2}\phi_D} \\ e^{\frac{D-6}{D-2}\phi_D} V_{10-D}^{1/2} \end{matrix} \right. + \ldots \qquad (14)$$

For $D = 10$, the identification of the two Lagrangians implies $\phi_{10}^H = -\phi_{10}^I$; the two string theories are equivalent under S-duality. In each case, since the gravitational coupling constant is $\kappa_{10}^2 = e^{2\phi_{10}}\alpha'^4$ [2,5],

$$\lambda_{10} \leftrightarrow 1/\lambda_{10} \quad ; \quad \alpha' \leftrightarrow \lambda_{10}\alpha' \,. \qquad (15)$$

A derivation of type I and heterotic theories from eleven dimensions provides more insight into their duality. One can see that M-theory compactified on $S^1/\mathbf{Z}_2 \times S^1$ and $S^1 \times S^1/\mathbf{Z}_2$ gives rise to the heterotic and type I strings compactified on $S^1$ [7]. Here, $\mathbf{Z}_2$ acts as the reflection on the eleventh and tenth coordinate, respectively, changing also the sign of the 3-form. Moreover in the second case, one can show that the $\mathbf{Z}_2$ symmetry is identical to $\Omega\mathcal{R}$ which, as shown in Section 2, is T-dual to the world-sheet involution of type IIB, allowing construction of type I strings from type IIA. The corresponding identifications are [7]:

$$\lambda^I = \frac{R_{11}^M}{R_{10}^M} = 1/\lambda^H \,, \qquad (16)$$

$$R_{10}^I = \frac{1}{R_{10}^M\sqrt{R_{11}^M}} \quad , \quad R_{10}^H = \frac{1}{R_{11}^M\sqrt{R_{10}^M}} \,, \qquad (17)$$

where $R_{10,11}^M$, $R_{10}^I$ and $R_{10}^H$ are the compact radii in the respective theories. Thus, S-duality is now recovered from U-dualities of M-theory compactified on a torus whose complex structure (16) is identified with the coupling constants.

For $D = 6$, one derives from Eq. (14) a U-duality relation $e^{-2\phi_6^H} = V_4^I$. Moreover, using Eqs. (12) and (15), one obtains the same identification with the indices $H$ and $I$ exchanged, hence

$$V_4 \leftrightarrow 1/\lambda_6^2 \,. \qquad (18)$$

For $D = 4$, Eq. (14) implies a weak–weak coupling duality with $e^{-2\phi_4^H} = e^{-\phi_4^I}(V_6^I)^{1/2}$. Using Eqs. (12) and (15), one also obtains $V_6^H = e^{-3\phi_4^I}(V_6^I)^{-1/2}$ and similar relations with the indices $H$ and $I$ exchanged. Finally, we get [11]

$$\lambda_4^2 \leftrightarrow \lambda_4 V_6^{-1/2} \quad ; \quad V_6 \leftrightarrow \lambda_4^{-3} V_6^{-1/2} \,. \qquad (19)$$

It is interesting that the ten-dimensional strong–weak coupling duality between heterotic and type I strings leads in lower dimensions to relations that hold when both theories are weakly coupled and thus, can be tested at the perturbative level. These relations can also be used to study non-perturbative effects in one theory by using the perturbative description of the dual theory. For instance, world-sheet instanton contributions of the characteristic magnitude $e^{-V_4^{1/2}/\alpha'}$ or $e^{-V_6^{1/3}/\alpha'}$ are mapped through the relations (18) and (19) to non-perturbative effects of order $e^{-1/\lambda_6}$ and $e^{-1/\lambda_4}$, respectively. Since $\lambda_4^H$ is also the gauge coupling, such world-sheet instantons on the type I side translate on the heterotic theory to stringy non-perturbative effects of order $e^{-1/g}$, which are much stronger than ordinary field-theoretical instantons.

## 4. TYPE I–HETEROTIC DUALITY IN SIX DIMENSIONS

The first non-trivial test of heterotic–type I duality is for six-dimensional vacua with $N = 1$ supersymmetry, obtained upon compactification of the ten-dimensional theories on $K3$ manifolds [12].

We start by recalling some basic facts about $D = 6$, $N = 1$ supersymmetry. Besides the gravitational multiplet, which contains 12 bosonic and 12 fermionic degrees of freedom, namely the metric, an anti-self-dual 2-index antisymmetric tensor and the gravitino (with 9, 3 and 12 physical components respectively), there are three different matter multiplets: the tensor ($T$), the vector ($V$) and the hyper ($H$), containing $4 + 4$ physical states. These are a self-dual 2-form and a real scalar (for $T$), a vector (for $V$) and two complex scalars (for $H$), together with a Weyl spinor. Cancellation of the gravitational anomaly in six dimensions requires the number of massless matter multiplets $n_T$, $n_V$ and $n_H$ to satisfy the following constraint:

$$29n_T + n_H - n_V = 273\,. \qquad (20)$$

Perturbative heterotic spectra always contain one tensor multiplet whose scalar component is identified with the six-dimensional dilaton $\phi_6^H$. All other moduli belong to hypermultiplets, since vector multiplets contain no scalars. Moreover, Eq. (20) fixes the difference between hypers and vectors, $n_H - n_V = 244$. On the other hand, perturbative type I strings may contain several tensor multiplets. Therefore in order to make perturbative duality tests possible, we will restrict ourselves to type I vacua with $n_T = 1$. In contrast to the heterotic case, however, the type I dilaton $\phi_6^I$ belongs to a hypermultiplet, in agreement with the duality relation (18).

At a generic point of the moduli space, all massless hypermultiplets are neutral under the gauge group. Supersymmetry requires that hypermultiplets decouple from the interactions of vectors and tensors, so that the effective action splits into two terms: $S_{eff} = S_{T,V} + S_H$, where $S_{T,V}$ depends on tensors and vectors only, while $S_H$ describes the interactions of hypers. The important point is that the type I string coupling is given by a vev of a hypermultiplet, while the heterotic coupling is determined by a vev of a tensor multiplet. It follows that $S_{T,V}$ receives no quantum corrections on the type I side, while $S_H$ is determined classically on the heterotic side.

Here, we focus only on $S_{T,V}$, which is easier to analyze. In fact, its form is almost completely fixed by supersymmetry and anomaly cancellation. In the Einstein frame, the bosonic part of the Lagrangian becomes [13]:

$$\begin{aligned}\mathcal{L}^{(6)}_{T,V} \;&=\; \frac{1}{2}R + \frac{1}{2}\left(\frac{\partial\omega}{\omega}\right)^2 + \frac{1}{4}(v\omega + v'\omega^{-1})F^2 \\ &+\; \frac{1}{16}\omega^2(dB - v\Omega)^2 + \frac{v'}{16}B\wedge F\wedge F, \qquad (21)\end{aligned}$$

where $\omega$ is the scalar of the tensor multiplet, $B$ is the 2-form potential, $F$ denotes the field-strengths of gauge fields with $\Omega$ the corresponding Chern–Simons terms, $v$ and $v'$ are gauge group-dependent constants, and an implicit summation over gauge fields is understood.

On the heterotic side, $\omega \equiv e^{-\phi_6^H}$, therefore $v$ is given classically by the Kač–Moody level $k$ of the underlying affine Lie algebra, while $v'$ represents a one-loop string correction. The difference $\Delta v'$ between two gauge groups coincides with the corresponding difference of $N = 2$ $\beta$-functions that one obtains upon further compactification to $D = 4$ on a $T^2$ [14]. This can be compared with the result of the dual type I theory, where $\omega \equiv V_{K3}^{1/2}$, and both $v$ and $v'$ are determined at the tree level.

Gauge fields with $v = 0$ are non-perturbative from the heterotic point of view and correspond to parallel 5-branes on the type I side. On the other hand, when $vv' < 0$ for some gauge field, there is a point in the tensor moduli space where the gauge coupling blows up. It has been argued that such a singularity is resolved by the appearance of tensionless strings [15], generated on the type I side by type IIB 3-branes wrapping around a collapsing 2-cycle of $K3$. Note finally that if there is a $D = 6$, $N = 1$ type II vacuum with only one antisymmetric tensor, $v'$ will be given at the tree level and $v$ at one loop, as required by heterotic–type II S-duality in six dimensions.

We conclude this Section by describing one of the simplest $N = 1$ heterotic–type I dual pairs in six dimensions. They are obtained by compactifying the ten-dimensional theories on the $K3$ orbifold $T^4/\mathcal{R}$, where $\mathcal{R}$ is the $\mathbf{Z}_2$ transformation $X^m \to -X^m$ for $m = 6,7,8,9$.

On the heterotic side, the $\mathbf{Z}_2$ projection reduces the supersymmetry to $N = 1$ and breaks the gauge group $SO(32)$ down to $U(16)$. From the original $SO(32)$ gauge multiplet there remain also two massless hypermultiplets transforming in the $\mathbf{120}_{1/2}$ representation. In addition, there are 4 untwisted neutral hypermultiplets parametrizing the moduli of $T^4$, as well as 16 twisted ones transforming in the $\mathbf{16}_{1/4}$ representation. Of course, there is also the standard $N = 1$ supergravity multiplet together with the tensor dilaton multiplet.

This model is dual to the type I theory compactified on the same $T^4/\mathcal{R}$ which can be constructed from type IIB by modding it out by the orientifold group $\{1, \Omega, \mathcal{R}, \Omega\mathcal{R}\}$ [8,16]. Then, besides the 16 D9–branes, tadpole cancellation requires the presence of 16 D5–branes. The massless spectrum of this $K3$ orientifold model is as follows. In the $\Omega$-twisted sector, open strings satisfy 99 boundary conditions and the $SO(32)$ gauge group is broken by the $\mathcal{R}$ projection to $U(16)_9$. In the $\Omega\mathcal{R}$-twisted sector, one finds 55 strings with both endpoints fixed on $T^4/\mathcal{R}$, and an additional 5-brane gauge group of rank 16. The maximal gauge group is $U(16)_5$ which appears when all 5-branes are located at the same fixed point of the orbifold. There are also two massless hypermultiplets transforming in the $\mathbf{120}_{1/2}$ representation of $U(16)_5$, which parametrize the relative distances between the D5-branes, as mentioned in Section 2. When the 5-branes are pulled apart the gauge group is broken, and the maximum breaking occurs when each 5-brane is located at a separate fixed point; the gauge group is then $U(1)_5^{16}$ and there is no massless matter. There are also 95 + 59 strings which give 16 massless hypermultiplets in the $\mathbf{16}_{1/4}$ representation of $U(16)_9$ [or 1 in the $(\mathbf{16}_{1/4}, \mathbf{16}_{1/4})$ of $U(16)_9 \times U(16)_5$]. Finally, the closed string sector contains the $N = 1$ six-dimensional supergravity multiplet, one tensor multiplet and 20 hypermultiplets, 4 of them from the $\mathcal{R}$-untwisted sector and 16 from the $\mathcal{R}$-twisted one. Out of the four scalar components of each twisted hypermultiplet, three are coming from the symmetrized NSNS sector while the fourth one originates by Poincaré duality from a RR 4-form potential that is equivalent to a scalar in six dimensions.

The above type I massless spectrum is the same as in the heterotic model with the exception of the $U(1)_5^{16}$ gauge multiplets which are non-perturbative on the heterotic side, and 16 hypermultiplets from the $\mathcal{R}$-twisted closed string sector. However, all $U(1)_5$'s are broken due to anomalous couplings of their gauge field strengths with the RR 4-forms, $C_4^i \wedge F_2^i$ [17]. As a result these gauge multiplets become massive, absorbing all 16 twisted hypermultiplets, and the massless spectra of the two theories become identical. The effective Lagrangian that describes the interactions of the 9-brane gauge multiplets with the tensor multiplet is of the form (21) with $v = 1$ and $v' = 0$. In Section 7, we will compactify the above dual pair on $T^2$ and discuss some non-trivial duality tests in four dimensions.

## 5. TYPE I–HETEROTIC DUALITY IN FOUR DIMENSIONS

When heterotic and type I theories are compactified on $K3 \times T^2$, one obtains $N = 2$ supersymmetric models in four dimensions with $n_V + 3$ vector multiplets and $n_H$ hypermultiplets. The three additional vector multiplets describe the universal sector which arises from the single tensor multiplet and the Kaluza-Klein reduction of the graviton multiplet on $T^2$. More precisely, the vector field components correspond to three linear combinations of $G_{\mu 4}$, $G_{\mu 5}$, $B_{\mu 4}$ and $B_{\mu 5}$, while the fourth combination can be identified with the graviphoton of the $N = 2$ supergravity multiplet. Similarly, their scalar components are combinations of the three $T^2$ metric components $G_{44}$, $G_{55}$ and $G_{45}$, the internal antisymmetric tensor $B_{45}$, the four-dimensional dilaton $\phi_4$, the $K3$ volume $V$, and the universal axion $\alpha$ dual to the antisymmetric tensor $B_{\mu\nu}$. In addition, there are $2n_V$ scalar components of the six-dimensional gauge fields $a_{4,5}$ that correspond to the Wilson lines on $T^2$. The way these fields are arranged into $N = 2$ multiplets is different in the two theories.

We first discuss the case of vanishing Wilson lines. On the heterotic side, the $K3$ volume belongs to a hypermultiplet while the remaining six scalars form the well-known complex fields:

$$S_H = \alpha + ie^{-2\phi_4} \quad , \quad T = B_{45} + i\sqrt{G} \qquad U = (G_{45} + iG^{1/2})/G_{44} \, , \tag{22}$$

where $G = G_{44}G_{55} - G_{45}^2$. On the type I side, the six-dimensional string dilaton $\phi_6 = \phi_4 + (1/2) \ln \sqrt{G}$ remains in a hypermultiplet. A straightforward dimensional reduction of the six-dimensional Lagrangian (21) gives [11]

$$\mathcal{L}_V^{(4)} = \frac{1}{2}R + \frac{1}{4}(v\mathrm{Im}S_I + v'\mathrm{Im}S')F^2 + \ldots \tag{23}$$

where

$$\begin{aligned} S_I &= \alpha + ie^{-\phi_4}G^{1/4}V^{1/2} \, , \\ S' &= B_{45} + ie^{-\phi_4}G^{1/4}V^{-1/2} \, . \end{aligned} \tag{24}$$

It is easy to see that with the above definitions and with $U$ the same as in Eq. (22), $S_I$, $S'$ and $U$ have diagonal kinetic terms and form the scalar components of the three vector multiplets.

As seen in Eq. (22) the heterotic dilaton belongs to a vector multiplet, which implies that the hypermultiplet moduli space remains the same as in six dimensions and does not receive any quantum corrections. On the other hand, on the type I side, the four-dimensional dilaton $\phi_4$ is a "mixture" of vector and hypermultiplet components, so that the string coupling $e^{\phi_4} = e^{\phi_6}(\mathrm{Im}S_I\mathrm{Im}S')^{-1/4}$. Hence, both hyper and vector sectors can receive quantum corrections once the string coupling combines with other fields to form the full scalar supermultiplet components.

Wilson lines can be turned on along the flat directions of the potential corresponding to the Cartan directions of the gauge group. They can be diagonalized to $a^i_{4,5}$, $i = 1 \ldots r$, where $r$ is the rank, and generically break the gauge group to its abelian Cartan subgroup $U(1)^r$. In the presence of Wilson lines, the $N = 2$ special coordinates for the type I theory become:

$$A_i = a^i_4 - a^i_5 U\,,\ \ S_I = S_I|_{A=0} + \sum_i \frac{v'_i}{2} a^i_5 A_i\,,$$
$$S' = S'|_{A=0} + \sum_i \frac{v_i}{2} a^i_5 A_i\,. \quad (25)$$

Similarly, in the heterotic theory, $A_i$ is defined as above, the $T$ modulus is redefined as $S'$, while $S_H$ remains the same as in Eq. (22).

In terms of these fields, the tree-level prepotentials that determine the interactions of $N = 2$ vector multiplets in the two theories can be read off from the six-dimensional action (21) and take the form [11]:

$$\begin{aligned} \mathcal{F}^I_{\rm tree} &= S_I S' U - \frac{1}{2}\sum_i (v_i S_I + v'_i S')A_i^2 \\ \mathcal{F}^H_{\rm tree} &= S_H T U - \frac{1}{2}\sum_i v_i S_H A_i^2\,. \end{aligned} \quad (26)$$

It follows that the heterotic–type I duality mapping inherited from Eq. (18) in $D = 6$, is [11]

$$S_H \leftrightarrow S_I,\ \ T \leftrightarrow S',\ \ U \leftrightarrow U,\ \ A_i \leftrightarrow A_i\,. \quad (27)$$

Let us now discuss quantum corrections. As already mentioned above, both prepotentials can be modified quantum mechanically. Perturbative corrections are restricted by continuous Peccei–Quinn symmetries. On the heterotic side, there is a Peccei–Quinn symmetry associated with the universal axion $\alpha = \mathrm{Re}S_H$, which is dual to the antisymmetric tensor. On the type I side, besides a similar symmetry associated with $\mathrm{Re}S_I$, there is a second symmetry due to the fact that $\mathrm{Re}S' = B_{45}$ originates from the RR sector. These symmetries, together with analyticity, imply that the perturbative corrections must be independent of $S_{I,H}$ and $S'$, hence they are entirely due to one-loop effects.

The general forms of the respective prepotentials are:

$$\begin{aligned} \mathcal{F}^I(S,S',U,A) &= \mathcal{F}^I_{\rm tree} + f^I(U,A) + \ldots \\ \mathcal{F}^H(S,T,U,A) &= \mathcal{F}^H_{\rm tree} + f^H(T,U,A) + \ldots \end{aligned} \quad (28)$$

where $f^{I,H}$ are the one-loop corrections and dots represent non-perturbative contributions. In Eq. (28) and everywhere below we drop the subscripts I and H referring to $S$, in view of the identification (27). Since the Peccei–Quinn symmetries are broken by instanton effects to discrete shifts of axions, the non-perturbative corrections can be expanded in integer powers of $e^{2\pi i S}$ whose magnitude is suppressed by the instanton action.

As already mentioned before, heterotic–type I duality includes regimes that are weakly coupled on both sides. It is possible to compare prepotentials (as well as other quantities that will be discussed later) already at the perturbative level, in the appropriate limits of moduli parameters. First of all, in order to reach the perturbative limit on type I side, one has to take the limit of large $\mathrm{Im}S$ and $\mathrm{Im}S'$. This is mapped via Eq. (27) to the region of large $\mathrm{Im}S$ and $\mathrm{Im}T$ which corresponds to a weakly coupled heterotic theory in the limit of "large" $T^2$ torus compactification, provided that the limit is taken with $\mathrm{Im}S > \mathrm{Im}S'$. Thus the perturbative heterotic prepotential reproduces its type I counterpart in the region of $K3$ volume $V > 1$, cf. Eq. (24). More precisely, from the expansions (28), one obtains

$$f^H(T,U,A) \stackrel{T\to i\infty}{\longrightarrow} -\frac{1}{2}\sum_i v_i' T A_i^2 + f^I(U,A), \quad (29)$$

with the fields mapped as in Eq. (27). Note that a part of the one-loop heterotic prepotential is mapped into the tree-level type I term proportional to $v_i'$, while the remaining part reproduces the type I one-loop correction.

It is also interesting to study the other perturbative branch of the type I theory with $V < 1$, which is mapped in the heterotic theory to the non-perturbative region $\mathrm{Im}T > \mathrm{Im}S \to \infty$. These two regions are related by type I T-duality $V \to 1/V$ which corresponds on the heterotic side to the non-perturbative $S \leftrightarrow T$ exchange. If in addition there exists a type IIA description of the same model, the exact prepotential can be determined classically on the type II side, and this region can be probed directly, providing a perturbative test of type I–type II duality that is non-perturbative from the heterotic point of view.

## 6. ONE-LOOP CORRECTIONS

The standard way of deriving the one-loop prepotential in heterotic theory [18,19] is to extract it from the one-loop corrections to gauge couplings:

$$\frac{4\pi^2}{g^2} = \frac{\pi}{2}\mathrm{Im}S + \Delta^H\,, \quad (30)$$

where $\Delta^H$ is the threshold function [20]. Its moduli dependence is governed by [18]

$$\partial_U\partial_{\bar U}\Delta^H = bK^{H(0)}_{U\bar U} + 4\pi^2 K^{H(1)}_{U\bar U}, \quad (31)$$

where $b$ is the beta-function coefficient, $K^{H(0)}_{U\bar U} = -1/(U-\bar U)^2$ is the tree-level moduli metric and $K^{H(1)}_{U\bar U}$ is the one-loop correction. The first term in the r.h.s. of Eq. (31) depends on the gauge group while the second one is a universal, gauge-group–independent contribution. The latter can be used to extract the one-loop prepotential $f^H$.

A direct string computation gives $\Delta^H$ as an integral over the complex Teichmüller parameter $\tau = \tau_1 + \tau_2$ of the world-sheet torus in its fundamental domain [18]:

$$\Delta^H = -\frac{1}{2}\int\frac{d^2\tau}{\tau_2}\bar\eta^{-2}\mathrm{Tr}_R F(-)^F\big(Q^2 - \frac{1}{2\pi\tau_2}\big), \quad (32)$$

where $\eta$ is the Dedekind eta function, $Q$ is the gauge group generator, and the trace is over the Ramond sector of the internal $(2,0)$ superconformal theory with $U(1)$-charge operator $F$. After taking $\partial_U\partial_{\bar U}$ derivatives, one finds that the two terms in the trace give rise to the two terms in the r.h.s. of Eq. (31), respectively. Using the properties of the underlying superconformal field theory that describes $K3\times T^2$ compactifications, one can rewrite the integrand as a sum over $N=2$ BPS states [21]. In particular, the one-loop Kähler metric reads:

$$K^{H(1)}_{U\bar U} = -\frac{1}{8\pi^2}\int\frac{d^2\tau}{\tau_2}\partial_U\partial_{\bar U} \quad (33)$$
$$\Big(\sum_{\substack{\text{BPS}\\ \text{hypermultiplets}}} - \sum_{\substack{\text{BPS}\\ \text{vectormultiplets}}}\Big) e^{i\pi\tau M_L^2} e^{-i\pi\bar\tau M_R^2}\,,$$

where $M_L$ and $M_R$ denote the contribution to the masses from the left- and right-movers, respectively. In this formula, the supergravity multiplet is counted as a vector multiplet.

One-loop threshold corrections have also been studied in type I theory [22], however their structure is different from the heterotic case. The function $\Delta^I$ contains the group-dependent contribution only, proportional to the beta function. The universal term is absent, which means that it is automatically absorbed into the definition of $\mathrm{Im}S$. Hence another procedure is needed to compute the one-loop Kähler metrics. It turns out that the quantity to be examined is the Planck mass [11]. Unlike the heterotic case, the Einstein term receives a one-loop correction:

$$e^{-2\phi}R \stackrel{\text{1-loop}}{\longrightarrow} \big(e^{-2\phi} + \frac{\delta}{\sqrt{G}}\big)R. \quad (34)$$

The relation between the function $\delta$ and the Kähler metric is[2]:

$$K^{I(1)}_{U\bar U} = \frac{1}{16\pi\mathrm{Im}S'}\partial_U\partial_{\bar U}\delta\,. \quad (35)$$

The computation of the one-loop correction to the Planck mass in type I theory has been presented in [11]. The function $\delta$ can be determined

[2]The present definition of $\delta$ differs from its original definition in Ref. [11] by a factor of $\sqrt{G}$, so that $\delta$ becomes independent of $\sqrt{G}$.

from a physical amplitude involving one modulus and two gravitons, and receives contributions from the annulus and Möbius strip diagrams for open strings, and from the Klein bottle for closed strings. The result can be expressed as an integral over the real modular parameter of these one-loop surfaces:

$$\frac{\delta}{\sqrt{G}} = -\frac{1}{\pi}\int_0^{'\infty}\frac{dt}{t^2}(\mathcal{I}_{\mathcal{A}}Z_{\mathcal{A}}+\mathcal{I}_{\mathcal{M}}Z_{\mathcal{M}}+4\mathcal{I}_{\mathcal{K}}Z_{\mathcal{K}}),\quad(36)$$

where $\mathcal{I}_\sigma$ are indices associated to $K3$ which count the open string spinors (propagating on $\sigma = \mathcal{A}, \mathcal{M}$) and closed string RR bosons (propagating on $\sigma = \mathcal{K}$) weighted with the fermion-parity operator $(-)^{F_{\rm int}}$, while $Z_\sigma$ denote the corresponding $T^2$ partition functions. The prime in the integral indicates that the quadratic divergence in the ultraviolet limit $t \to 0$ has been subtracted, as dictated by the tadpole cancellation [22]. There is no need however for such a regularization at the level of the Kähler metric, where the divergence disappears after taking $\partial_U\partial_{\bar U}$ derivatives, c.f. Eq. (35).

The above result for the Kähler metric generalizes the heterotic expression obtained from Eq. (32) to the type I case. It can be reexpressed in a form similar to (33) as a sum over $N = 2$ BPS states that originate only from the massless modes in six dimensions [11]:

$$\begin{aligned}\frac{1}{\sqrt{G}}\partial_U\partial_{\bar U}\delta &= -\frac{2}{\pi}\int_0^\infty\frac{dt}{t^2}\partial_U\partial_{\bar U} \qquad (37)\\ &\Bigg(\sum_{\substack{\rm BPS\\ \rm hypermultiplets}} - \sum_{\substack{\rm BPS\\ \rm vectormultiplets}}\Bigg)e^{-\pi t M^2/2},\end{aligned}$$

where the masses $M$ come from the momentum in the internal $T^2$. The same formula gives type I threshold corrections after inserting the operator $tQ^2$ inside the sum.

## 7. EXAMPLE OF TYPE I–HETEROTIC DUALITY

In this Section, we discuss an example of a dual pair with $N = 2$ supersymmetry in four dimensions, obtained by compactifying on $T^2$ the six-dimensional $K3$ orbifold models described in Section 4. More precisely, we will analyze two cases. The first one corresponds to the Higgs phase, in which the six-dimensional gauge group $U(16)_9$ is completely broken by giving appropriate vev's to the charged hypermultiplets. One thus obtains $n_T = 1$, $n_V = 0$ and $n_H = 244$ consistently with the anomaly cancellation condition (20) in $D = 6$. Upon compactification to $D = 4$ one finds the so-called STU model which contains 3 massless vector multiplets ($S$, $T$ and $U$ in the heterotic notation) and 244 neutral hypermultiplets. The second case corresponds to the Coulomb phase obtained by turning on Wilson lines on $T^2$. $U(16)_9$ is then broken to $U(1)^{16}$ and all charged hypermultiplets become massive. The resulting four-dimensional massless spectrum contains 19 vector multiplets and 4 neutral hypermultiplets.

### 7.1. Higgs Phase

On the heterotic side, the STU model has been studied extensively in the past because it admits also a type II description [23]. In fact, it can be obtained by compactifying the ten-dimensional type IIA theory on the Calabi–Yau manifold described by the weighted hypersurface of degree 24, $WP_{1,1,2,8,12}(24)$, with Hodge numbers $h_{(1,1)} = 3$ and $h_{(1,2)} = 243$. Since the type II dilaton belongs to a hypermultiplet, on the type II side the exact prepotential can be determined at the classical level. Therefore, this model provides an example of type II–heterotic–type I triality which can be tested in appropriate limits at both perturbative and non-perturbative levels.

Starting from the type II prepotential $\mathcal{F}^{II}$ with the field identification of $S, T, U$ guided by the $K3$ fibration, and taking the limit Im$S \to \infty$ one can perform a perturbative test of heterotic–type II duality [23,24]:

$$\mathcal{F}^{II}(S,T,U) \overset{S\to i\infty}{\longrightarrow} STU + f^H(T,U)\,, \qquad (38)$$

where the two terms in the r.h.s. coincide with the tree-level and the one-loop contributions to the heterotic prepotential. A non-perturbative test [23] can also be done by taking the zero-slope limit along the conifold singularity of the Calabi–Yau manifold, reproducing the Seiberg–Witten prepotential of the rigid $SU(2)$ $N = 2$ super-Yang–Mills theory whose perturbative limit is described by the heterotic model on the $T = U$ line of enhanced symmetry.

In order to test heterotic–type I duality, we first recall that the heterotic $S$, $T$ and $U$ fields are mapped to $S$, $S'$ and $U$ defined in Eq. (24). As explained in Section 5, a perturbative test can be performed by taking the limit of $\mathrm{Im}T \to \infty$ of the one-loop heterotic prepotential and comparing it with the type I counterpart.

The one-loop type I prepotential can be reconstructed from the Kähler metric given by Eqs. (35,37). In the model under consideration, Eq. (37) yields

$$\begin{aligned} \partial_U \partial_{\bar{U}} \delta \quad &= \quad -\frac{2\sqrt{G}}{\pi} \int_0^\infty \frac{dt}{t^2}\,(244-4) \times \\ &\qquad \partial_U \partial_{\bar{U}} \sum_{p\in\Gamma_2} e^{-\pi t|p|^2/2} \,, \end{aligned} \tag{39}$$

where $\Gamma_2$ is the $T^2$ momentum lattice:

$$p = \frac{m_4 - m_5 U}{\sqrt{2G^{1/2}\mathrm{Im}U}} \,, \tag{40}$$

with integer $m_{4,5}$. After performing the $t$-integration, Eq. (35) gives [11]:

$$K^{I(1)}_{U\bar{U}} = -\frac{15}{\pi^4}\frac{1}{\mathrm{Im}S'} \sum_{m_4,m_5}{}' \frac{1}{|m_4 - m_5 U|^4} \,, \tag{41}$$

where the prime means $(m_4, m_5) \neq (0,0)$.

The one-loop prepotential can now be determined by the standard $N=2$ formula [19]:

$$\begin{aligned} \partial^3_U f^I(U) \quad &= \quad 2\,\mathrm{Im}S'\,\partial^2_U \partial_{\bar{U}} (U-\bar{U})^3 K^{I(1)}_{U\bar{U}} \\ &= \quad 4\,E_4(U) \,, \end{aligned} \tag{42}$$

with the function

$$E_4(U) = \frac{45}{\pi^4} \sum_{m_4,m_5}{}' \frac{1}{(m_4 - m_5 U)^4} \,. \tag{43}$$

This result agrees with the heterotic prepotential [19] in the limit $\mathrm{Im}T \to \infty$:

$$\partial^3_U f^H(U, \mathrm{Im}T \to \infty) = 4\,E_4(U) = \partial^3_U f^I(U) \,. \tag{44}$$

Note that the perturbative type I computation is valid not only in the region $\mathrm{Im}S > \mathrm{Im}S' \to \infty$ ($K3$ volume $V > 1$), but also in the region $\mathrm{Im}S' > \mathrm{Im}S \to \infty$ ($V < 1$). This is due to the symmetry of the present type I model under T-duality, $V \leftrightarrow 1/V$, which exchanges 9-branes and 5-branes. Although the region of $V < 1$ cannot be reached by means of perturbative expansion from the heterotic side, it can be reached from the type II side by taking the limit $\mathrm{Im}T \to \infty$ before the limit $\mathrm{Im}S \to \infty$. Since the exact prepotential of the STU model, as evaluated on the type IIA side, is known to be symmetric under the exchange of $S$ and $T$ [23], such an order of limits gives $\mathcal{F}^{II} \to STU + f^I(U)$, the same as in the other order.

### 7.2. Coulomb Phase

In the Coulomb phase, we find it more convenient to test duality by a direct examination of threshold corrections to the gauge couplings of $U(1)^{15}$ associated with the unbroken Cartan generators of $SU(16)_9$ [25]. After combining Eqs. (31) and (35) with Eq. (29), we see that duality predicts the following large $\mathrm{Im}T$ expansion of the heterotic threshold corrections:

$$\begin{aligned} \Delta^H(T,U,A) \quad &= \quad \frac{\pi}{2} v' \mathrm{Im}T + \Delta^I(U,A) \\ &\qquad + \frac{\pi}{4\mathrm{Im}T}\delta(U,A) + \ldots \,, \end{aligned} \tag{45}$$

up to exponentially suppressed corrections. Recall from the discussion of the six-dimensional model in Section 5 that $v' = 0$.

The type I quantities $\Delta^I(U,A)$ and $\delta(U,A)$ have been discussed for generic models in Section 6. In order to apply Eqs. (36) and (37) to the model under consideration, we first determine the quantities $\mathcal{I}_\sigma Z_\sigma$ for various surfaces $\sigma = \mathcal{A}, \mathcal{M}, \mathcal{K}$. In the 99 open string sector,

$$\begin{aligned} \mathcal{I}_{\mathcal{A}} Z_{\mathcal{A}} \quad &= \quad -2 \sum_{a^I + a^J + \Gamma_2} s_{IJ} e^{-\pi t|p|^2/2} \\ \mathcal{I}_{\mathcal{M}} Z_{\mathcal{M}} \quad &= \quad -2 \sum_{2a^I + \Gamma_2} e^{-\pi t|p|^2/2} \,, \end{aligned} \tag{46}$$

where for convenience, we introduced the index $I \equiv i$ or $\bar{\imath}$, with $i$ and $\bar{\imath}$ running over the $\mathbf{16}$ and $\overline{\mathbf{16}}$ of $SU(16)$, respectively, and $a^{\bar{\imath}} \equiv -a^i$. The matrix $s_{IJ}$ represents the action of the orbifold group $\mathcal{R}$ on the Chan–Paton charges: $s_{IJ} = -1$ or 1, depending on whether $I$ and $J$ belong to the same or conjugate representations of $SU(16)$, respectively. For open string surfaces with boundaries, the momenta of the $T^2$ lattice (40) are shifted in a self-explanatory way by the Wilson lines of the

9-brane group, according to Eq.(9); for instance:

$$a^I + \Gamma_2 \; : m_{4,5} \to m_{4,5} + a^I_{4,5} \,. \qquad (47)$$

Similarly, in the 95+59 sectors:

$$\mathcal{I}_{\mathcal{A}} Z_{\mathcal{A}} = 2\times 16 \sum_{a^I+\Gamma_2} e^{-\pi t|p|^2/2}\,, \quad \mathcal{I}_{\mathcal{M}} = 0\,. \qquad (48)$$

There should be no contribution from the 55 and closed string sectors since the 5-brane $U(1)^{16}$ gauge supermultiplets become massive by absorbing 16 twisted hypermultiplets from the closed string sector, as explained in Section 4. However, for a generic position of 5-branes, the 55 sector contributes

$$\mathcal{I}_{\mathcal{A}} Z_{\mathcal{A}} + \mathcal{I}_{\mathcal{M}} Z_{\mathcal{M}} = -4\times 16 \sum_{\Gamma_2} e^{-\pi t|p|^2/2}\,, \qquad (49)$$

while the closed string sector gives

$$\mathcal{I}_{\mathcal{K}} Z_{\mathcal{K}} = 16 \sum_{\Gamma_2} e^{-\pi t|p|^2/2}\,, \qquad (50)$$

due to the RR components of the twisted hypermultiplets. Note that these two contributions cancel each other.

Inserting the above results (46-50) into Eq. (36), one obtains [25]:

$$\begin{aligned}\delta \;&=\; \frac{2\sqrt{G}}{\pi}\int_0^{'\infty}\frac{dt}{t^2} \qquad (51)\\ &\left\{\sum_{a^I+a^J+\Gamma_2} s_{IJ} \;+\; \sum_{2a^I+\Gamma_2} -\,16 \sum_{a^I+\Gamma_2}\right\} e^{-\pi t|p|^2/2}\,.\end{aligned}$$

A similar formula, obtained by inserting the charge operator $tQ^2$ in the sum, gives the type I threshold corrections [22]:

$$\begin{aligned}\Delta^I(U,A_i) = -\frac{1}{8}\int_0^{'\infty}\frac{dt}{t}\Bigg\{&\sum_{a^I+a^J+\Gamma_2} s_{IJ}(q_I+q_J)^2 + \\ &\sum_{2a^I+\Gamma_2}(2q_I)^2 -\; 16\sum_{a^I+\Gamma_2} q_I^2\Bigg\} e^{-\pi t|p|^2/2}, \qquad (52)\end{aligned}$$

where $q_i$ are the $U(1)$ charges associated to SU(16) Cartan generators in the **16** representation and $q_{\bar{\imath}} = -q_i$.

We now turn to the heterotic side and determine the large Im$T$ behavior of $\Delta^H$ given by the general formula of Eq. (32). Specifying to the model under consideration:

$$\begin{aligned}\Delta^H(T,U,A) = -\frac{1}{8}\int\frac{d^2\tau}{\tau_2}\sum_s Z_s(\bar\tau)\times \\ \sum_{p_L,p_R\in\Gamma_s^{(2,18)}}\left(Q^2 - \frac{1}{2\pi\tau_2}\right) e^{i\pi\tau|p_L|^2} e^{-i\pi\bar\tau|p_R|^2}, \qquad (53)\end{aligned}$$

where the first sum is over the $N=2$ sectors of the $T^4/\mathbf{Z}_2$ orbifold, $s=(P,A),(A,P),(A,A)$, where $P$ and $A$ denote periodic and antiperiodic boundary conditions, respectively. $Z_s$ are the corresponding partition functions for all right-movers except for the 18 right-moving momenta that are included in the $\Gamma_s^{(2,18)}$ lattices. In fact, the moduli dependence is due entirely to these lattices. In particular, the left-moving (complex) momenta are:

$$p_L = \frac{1}{\sqrt{2\mathrm{Im}T\mathrm{Im}U}}(m_4' - m_5'U - n_5T - n_4TU), \qquad (54)$$

with integer $n_{4,5}$ and the momentum numbers shifted appropriately to $m_{4,5}'$ by the 16 Wilson lines.

The limit Im$T\to\infty$ can be taken in the following steps [25]. First, one observes that the contributions of all winding modes are exponentially suppressed, hence one can restrict the lattice summation to $n_4=n_5=0$. Next, by making the change of variable

$$\tau_2 = \frac{t}{4}\mathrm{Im}T\ , \qquad (55)$$

one can easily show that the integration domain becomes the strip $t\ge 4/\mathrm{Im}T$, $-1/2\le\tau_1\le 1/2$, up to exponentially small corrections in Im$T$. Finally, the $\tau_1$ integration selects the states that originate from the massless modes only in $D=6$.

The leading contribution diverges linearly in Im$T$, reflecting the quadratic ultraviolet divergence of the integral in the region $t\to 0$, where Im$T$ acts as a regulator. The coefficient of the divergence determines $v'$ of Eq. (45). It is given by the leading term of the expression (53) Poisson resummed in the $T^2$ momentum numbers, with the $\Gamma_s^{(2,18)}$ lattices degenerating to the moduli-independent $\Gamma_s^{16}$:

$$v' = -\frac{1}{4\pi}\int\frac{d^2\tau}{\tau_2^2}\sum_s Z_s(\bar\tau) \sum_{p_R\in\Gamma_s^{16}}\left(Q^2-\frac{1}{2\pi\tau_2}\right)e^{-i\pi\bar\tau|p_R|^2} \quad (56)$$

The above result coincides with the one-loop threshold correction of the six-dimensional heterotic theory. The integral (56) can be shown to vanish in the model under consideration, in agreement with the fact that $v'=0$ on the type I side, as explained in Section 4.

The subleading contribution is $T$-independent and comes entirely from the $Q^2$ part of Eq. (53). It coincides with the type I expression for $\Delta^I$, Eq. (52), after using the condition $\sum_i a^i_{4,5}=0$ for the Wilson lines corresponding to $SU(16)$ Cartan generators. Similarly, the term proportional to $1/\tau_2$ inside the bracket in Eq. (53) reproduces the third term in Eq. (45), of order $1/\mathrm{Im}T$, with $\delta$ given in Eq. (51).

It is interesting to trace the origin of individual type I one-loop contributions to the heterotic side. The contributions of the 99 sector (on the annulus and Möbius strip), corresponding to the first two terms in Eqs. (51,52), originate on the heterotic side from the untwisted (P,A) orbifold sector, while the contribution of the 95+59 sectors (on the annulus), corresponding to the third term, originate from the twisted (A,P)+(A,A) orbifold sectors.

## 8. HIGHER-DERIVATIVE F-TERMS

We now consider a class of higher-derivative F-terms in the effective actions which, in $N=2$ superfield formalism, take the form

$$I_g = \mathcal{F}_g W^{2g}\ , \quad (57)$$

with integer $g\geq 0$. Here, $W$ is the Weyl superfield

$$W_{\mu\nu} = F^-_{\mu\nu} - R^-_{\mu\nu\lambda\rho}\theta^1\sigma_{\lambda\rho}\theta^2+\dots\ , \quad (58)$$

which is anti-self-dual in the Lorentz indices. $F^-_{\mu\nu}$ and $R^-_{\mu\nu\lambda\rho}$ are the (anti-self-dual) graviphoton field strength and Riemann tensor, respectively. The couplings $\mathcal{F}_g$ are holomorphic sections of degree $2-2g$ of the vector moduli space, up to a holomorphic anomaly for $g\geq 1$ [26]. They generalize the well known prepotential $\mathcal{F}\equiv\mathcal{F}_0$: $\mathcal{F}_g$ ($g\geq 1$) determines ($2g$+2)-derivative couplings of two gravitons and $(2g-2)$ graviphotons together with all interactions related by supersymmetry [27].

On the type II side, $\mathcal{F}_g$ is determined entirely at genus $g$, while on the heterotic and type I sides these couplings arise at one loop (with additional tree-level contributions to $\mathcal{F}_0$ and $\mathcal{F}_1$). The corresponding perturbative expansions in the two theories are [25]:

$$\mathcal{F}_1^H = 4\pi\mathrm{Im}S + f_1^H(T,U,A)\,,$$
$$\mathcal{F}_g^H = \mathcal{F}_g^H(T,U,A) \qquad (g\geq 2)\,, \quad (59)$$

$$\mathcal{F}_1^I = 4\pi(\mathrm{Im}S + v_1'\mathrm{Im}S') + f_1^I(U,A)\,,$$
$$\mathcal{F}_g^I = \mathcal{F}_g^H(U,A) \qquad (g\geq 2)\,. \quad (60)$$

In the example of dual pair discussed in Section 7, $v_1'=1$, as required by the symmetry under type I T-duality.

The standard way of encoding $\mathcal{F}_g$'s is to combine them in the generating function [26]:

$$\mathcal{F}(\lambda) = \sum_{g=1}^{\infty} g^2\lambda^{2g}\mathcal{F}_g\,. \quad (61)$$

Heterotic–type I duality predicts the following asymptotic expansion in the large $\mathrm{Im}T$ limit:

$$\mathcal{F}^H(\lambda;S,T,U,A) = 4\pi\lambda^2(\mathrm{Im}S+\mathrm{Im}T) + \mathcal{F}^I(\lambda;U,A) + \frac{2\pi\lambda^2}{\mathrm{Im}T}\delta(U,A)+\dots \quad (62)$$

The heterotic function $\mathcal{F}^H(\lambda)$ has been computed in Refs. [28,29,25] and is given by an integral similar to (53) with the operator inside brackets replaced by $-(\lambda^2/4\pi^2)\frac{d^2}{d\tilde\lambda^2}G^H(\tilde\lambda,\tau)$, where $G^H$ is the partition function of space-time coordinates in the presence of a (anti-self-dual) graviphoton field strength background $\lambda$, and $\tilde\lambda\equiv\bar p_L\tau_2\lambda/\sqrt{2\,\mathrm{Im}T\mathrm{Im}U}$.

The type I generating function $\mathcal{F}^I(\lambda)$ receives two types of contributions. The first one can be obtained from the expression for the threshold corrections by a procedure similar to the heterotic case, and arises from the world-sheet surfaces $\mathcal{A}$, $\mathcal{M}$ and $\mathcal{K}$. The second one arises from the $N=4$ supersymmetric type IIB sector propagating on the world-sheet torus; it contributes to $\mathcal{F}_1^I$ only. The sum of the two is [30,25]:

$$\mathcal{F}^I(\lambda) = \frac{\lambda^2}{64\pi^2}\int_0^{'\infty}\frac{dt}{t}(\mathcal{I}_\mathcal{A}Z_\mathcal{A}+\mathcal{I}_\mathcal{M}Z_\mathcal{M}+4\mathcal{I}_\mathcal{K}Z_\mathcal{K}) \times\frac{d^2}{d\lambda^2}\left(\frac{\lambda\pi}{\sin\pi\tilde{\lambda}}\right)^2 + \frac{\lambda^2}{8}\int_0^{'\infty}\frac{dt}{t}(\mathcal{I}_\mathcal{T}Z_\mathcal{T}-\mathcal{I}_\mathcal{K}Z_\mathcal{K})\ , \tag{63}$$

where the indices $\mathcal{I}_\sigma$ and partition functions $Z_\sigma$ have been defined in Section 6, with the exception of $\mathcal{I}_\mathcal{T}$ and $Z_\mathcal{T}$ which are the Witten index (equal to 24 for $K3$) and the torus partition function, respectively. In contrast to the first integrand which is reduced to a sum over $N=2$ BPS states as in Section 6, the second term $(\mathcal{I}_\mathcal{T}Z_\mathcal{T}-\mathcal{I}_\mathcal{K}Z_\mathcal{K})$ receives contributions only from the $\mathcal{R}$-untwisted $N=4$ sector of the theory.

Note that the integration in Eq. (63) is infrared singular at $t\to\infty$. The divergence is proportional to $\lambda^2$ and affects $\mathcal{F}_1$ only, reproducing the trace anomaly of the effective field theory. Unlike the case of gauge couplings, it is not regulated by non-vanishing Wilson lines.

Specializing to the $K3$ orbifold model discussed in Section 7, in the Higgs phase we have:

$$\mathcal{F}^I(\lambda) = \frac{\lambda^2}{32\pi^2}\int_0^{'\infty}\frac{dt}{t}\left\{240\frac{d^2}{d\lambda^2}\left(\frac{\lambda\pi}{\sin\pi\tilde{\lambda}}\right)^2 + 4\pi^2(24-16)\right\}\sum_{p\in\Gamma_2}e^{-\pi t|p|^2/2}\ . \tag{64}$$

Similarly, in the Coulomb phase, using Eqs. (46-50) and

$$\mathcal{I}_\mathcal{T}Z_\mathcal{T} = 24\sum_{\Gamma_2}e^{-\pi t|p|^2/2}\ , \tag{65}$$

we find:

$$\mathcal{F}^I(\lambda) = \frac{\lambda^2}{32\pi^2}\int_0^{'\infty}\frac{dt}{t}\left\{32\pi^2\sum_{\Gamma_2} - \left(\sum_{a^I+a^J+\Gamma_2}s_{IJ}+\sum_{2a^I+\Gamma_2}-16\sum_{a^I+\Gamma_2}\right) \times\frac{d^2}{d\lambda^2}\left(\frac{\pi\lambda}{\sin\pi\tilde{\lambda}}\right)^2\right\}e^{-\pi t|p|^2/2}\ , \tag{66}$$

where we used the same notation as in Eq. (51), and $\tilde{\lambda}\equiv\bar{p}t\lambda/\sqrt{32G^{1/2}\mathrm{Im}U}$.

It is easy to see [25,30], by following the same steps as those described in Section 7 for the threshold corrections, that the above type I expressions (64) and (66) do appear in the $\mathrm{Im}T\to\infty$ limit of the heterotic (one-loop) generating function $\mathcal{F}^H(\lambda,T,U,A)$, in agreement with Eq. (62). The correspondence of individual type I and heterotic terms is as before, with the additional torus contribution to $\mathcal{F}_1^I$ [the first term on the r.h.s. of Eq. (66)] originating from the (P,A) orbifold sector on the heterotic side. The leading term $4\pi\lambda^2\mathrm{Im}T$ in Eq. (62) reflects the quadratic ultraviolet divergence of $\mathcal{F}_1^H$. The subleading term of order $1/\mathrm{Im}T$ arises also from $\mathcal{F}_1^H$, and corresponds to the "universal" part of gravitational threshold corrections, reproducing $\delta$ of Eqs. (39) and (51).

## REFERENCES


1. For recent reviews, see:
   J.H. Schwarz, hep-th/9607067 and 9607201;
   J. Polchinski, Rev. Mod. Phys. 68 (1996) 1245, hep-th/9607050;
   R. Dijkgraaf, hep-th/9703136.
2. E. Witten, Nucl. Phys. B 443 (1995) 85, hep-th/9503124.
3. M.B. Green, J.H. Schwarz and E. Witten, "Superstring Theory" (Cambridge U. Press, 1987).
4. P. Candelas, X. de la Ossa, A. Font, S. Katz and D.R. Morrison, Nucl. Phys. B 416 (1994) 481, hep-th/9308083.
5. J. Polchinski and E. Witten, Nucl. Phys. B 460 (1996) 525, hep-th/9510169.
6. C.M. Hull and P.K. Townsend, Nucl. Phys. B 438 (1995) 109, hep-th/9410167.
7. P. Horava and E. Witten, Nucl. Phys. B 460 (1996) 506, hep-th/9510209.
8. A. Sagnotti, in "Non-Perturbative Quantum Field Theory", G. Mack et al., eds. (Perga-

mon Press, Oxford, 1988) p. 521;
M. Bianchi and A. Sagnotti, Nucl. Phys. B 361 (1991) 519.
9. P. Horava, Nucl. Phys. B 327 (1989) 461.
10. J. Polchinski, Phys. Lett. B 75 (1995) 4724, hep-th/9510017; for a review, see:
J. Polchinski, hep-th/9611050.
11. I. Antoniadis, C. Bachas, C. Fabre, H. Partouche and T.R. Taylor, Nucl. Phys. B 489 (1997) 160, hep-th/9608012.
12. M. Berkooz, R.G. Leigh, J. Polchinski, J.H. Schwarz, N. Seiberg and E. Witten, Nucl. Phys. B 475 (1996) 15, hep-th/9605184.
13. A. Sagnotti, Phys. Lett. B 294 (1992) 196.
14. G. Aldazabal, A. Font, L.E. Ibáñez and F. Quevedo, Phys. Lett. B 380 (1996) 33.
15. N. Seiberg and E. Witten, Nucl. Phys. B 471 (1996) 121, hep-th/9603003.
16. E.G. Gimon and J. Polchinski, Phys. Rev. D 54 (1996) 1667, hep-th/9601038.
17. M. Douglas, hep-th/9512077;
M. Douglas and G. Moore, hep-th/9603167.
18. I. Antoniadis, E. Gava and K.S. Narain, Nucl. Phys. B 383 (1992) 109;
I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, Nucl. Phys. B 407 (1993) 706;
I. Antoniadis and T.R. Taylor, in: "String Theory, Quantum Gravity and Unification of Fundamental Interactions", M. Bianchi et al., eds. (World Scientific, 1993) p.6, hep-th/9301033.
19. I. Antoniadis, S. Ferrara, E. Gava, K.S. Narain and T.R. Taylor, Nucl. Phys. B 447 (1995) 35, hep-th/9504034;
B. de Wit, V. Kaplunovsky, J. Louis and D. Lüst, Nucl. Phys. B 451 (1995) 53.
20. L.J. Dixon, V.S. Kaplunovsky and J. Louis, Nucl. Phys. B 355 (1991) 649.
21. J.A. Harvey and G. Moore, Nucl. Phys. B 463 (1996) 315, hep-th/9510182.
22. C. Bachas and C. Fabre, Nucl. Phys. B 476 (1996) 418, hep-th/9605028.
23. S. Kachru and C. Vafa, Nucl. Phys. B 450 (1995) 69, hep-th/9508155;
A. Klemm, W. Lerche and P. Mayr, Phys. Lett. B 357 (1995) 313, hep-th/9506112.
24. I. Antoniadis, S. Ferrara, E. Gava, K.S. Narain and T.R. Taylor, Nucl. Phys. B (Proc. Suppl.) 46 (1996) 162;
V. Kaplunovsky, J. Louis and S. Theisen, Phys. Lett. B 357 (1995) 71.
25. I. Antoniadis, H. Partouche and T.R. Taylor, hep-th/9703076.
26. M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Nucl. Phys. B 405 (1993) 279, hep-th/9302103; Commun. Math. Phys. 165 (1994) 311, hep-th/9309140.
27. I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, Nucl. Phys. B 413 (1994) 162, hep-th/9307158.
28. I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, Nucl. Phys. B 455 (1995) 109, hep-th/9507115.
29. J.F. Morales and M. Serone, Nucl. Phys. B 481 (1996) 389, hep-th/9607193.
30. M. Serone, Phys. Lett. B 395 (1997) 42, hep-th/9611017;
J.F. Morales and M. Serone, hep-th/9703049.