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Ellipticity of boundary value problems is characterized in terms of the Calderon projector. The presence of topological obstructions for the chiral Dirac operator under local boundary conditions in even dimension is discussed. Functional determinants for Dirac operators on manifolds with boundary are considered.

The functional determinant for a Dirac operator on a bidimensional disk, in the presence of an Abelian gauge field and subject to global boundary conditions of the type introduced by Atiyah-Patodi-Singer, is evaluated. The relationship with the index theorem is also commented.

INTRODUCTION

The wide application of functional determinants in Quantum and Statistical Physics is by now a well known fact. In order to evaluate one-loop effects, one faces to the necessity of defining a regularized determinant for elliptic differential operators, among which the Dirac first order one plays a central role. An interesting related problem is the modification of physical quantities due to the presence of boundaries. The study of boundary effects has lately received much attention, both in mathematics and physics, since it is of importance in many different situations [1-9], like index theorems for manifolds with boundary, effective models for strong interactions, quantum cosmology and application of QFT to statistical systems, among others (see [10] for a recent review).

In previous work [11,12], we studied elliptic Dirac boundary problems in the case of local boundary conditions. In particular, we developed for this case a scheme for evaluating determinants from the knowledge of the associated Green's function, based on Seeley's theory of complex powers [13].

Another type of boundary conditions extensively studied in the literature are global ones, of the type introduced by Atiyah, Patodi and Singer (APS) [14] in connection with the index theorem for manifolds with boundaries (see [15,10] for a review.) Other motivation for considering these global (or spectral) conditions is the presence of topological obstructions for the chiral Dirac operator under local boundary conditions (although this restriction no longer holds when considering the whole Dirac operator [11].)

ELLIPTIC BOUNDARY PROBLEMS AND REGULARIZED DETERMINANTS

Elliptic differential operators

Let D be a linear differential operator of order ω in a region Ω of \mathbf{R}^ν ,

$$D = \sum_{|\alpha| \leq \omega} a_\alpha(x) (-i\partial_x)^\alpha \quad (1)$$

(where $\alpha = (\alpha_1, \dots, \alpha_\nu)$, $|\alpha| = \alpha_1 + \dots + \alpha_\nu$, and the coefficients $a_\alpha(x) \in C^\infty$). Its *symbol* at $x \in \Omega$ is a polynomial in $\xi \in \mathbf{R}^\nu$ of degree ω defined by

$$\sigma(D)(x, \xi) = \sum_{|\alpha| \leq \omega} a_\alpha(x) \xi^\alpha. \quad (2)$$

The *principal symbol* of D is the part of $\sigma(D)(x, \xi)$ homogeneous of degree ω in ξ ,

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$$\sigma_\omega(D)(x, \xi) = \sum_{|\alpha|=\omega} a_\alpha(x) \xi^\alpha. \quad (3)$$

An operator D is *elliptic* at x if $\sigma_\omega(D)(x, \xi)$ is invertible $\forall \xi \neq 0$.

If D is elliptic in a compact region Ω then, for $|\xi| > 0$,

$$|\sigma_\omega(D)(x, \xi)| \geq \text{constant} \cdot |\xi|^\omega > 0, \quad \forall x \in \Omega, \quad (4)$$

since both sides are homogeneous of degree ω , and $a_\alpha(x) \in C^\infty$.

For example, in \mathbf{R}^2 , the operator $D = -i(\partial_1 + i\partial_2)$ is elliptic, since $\xi_1 + i\xi_2 = 0 \Rightarrow \xi = 0$. The Laplacian, $\nabla = (\partial_1)^2 + (\partial_2)^2$ is also elliptic.

Pseudodifferential operators

Given $f(x) \in \mathcal{S}(\mathbf{R}^\nu)$, the Schwartz space, and its Fourier transform $\hat{f}(\xi)$, the action of D on f can be expressed as

$$Df(x) = \frac{1}{(2\pi)^\nu} \int e^{ix \cdot \xi} \sigma(D)(x, \xi) \hat{f}(\xi) d^\nu \xi. \quad (5)$$

More generally, given a smooth function $\sigma(D)(x, \xi)$, with at most polynomial growth in ξ , such that for any α and β

$$|\partial_\xi^\alpha \partial_x^\beta \sigma(D)(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{\omega - \alpha}, \quad (6)$$

for some constants $C_{\alpha, \beta}$ (with ω not necessarily a positive integer), (5) defines a *pseudo-differential* operator D of order ω .

A pseudodifferential operator whose symbol decreases faster than any power of ξ is called *infinitely smoothing*. Two pseudodifferential operators are said to be equivalent if they differ by an infinitely smoothing operator. This equivalence allows for the introduction of asymptotic expansions of symbols.

The basic operation in symbol calculus corresponds to the composition of operators, and is given by

$$(\sigma_1 \cdot \sigma_2)(x, \xi) = \sigma_1(x, \xi) e^{-i \frac{\partial \leftarrow}{\partial \xi_\mu} \frac{\partial \rightarrow}{\partial x^\mu}} \sigma_2(x, \xi). \quad (7)$$

The Calderón projector and Elliptic boundary problems

We will be concerned with boundary value problems associated to first order elliptic operators

$$D : C^\infty(M, E) \rightarrow C^\infty(M, F), \quad (8)$$

where M is a bounded closed domain in \mathbf{R}^ν with smooth boundary ∂M , and E and F are k -dimensional complex vector bundles over M .

In general, such differential operators have a closed range of finite codimension, but an infinite-dimensional space of solutions,

$$\text{Ker}(D) = \{\varphi(x) / D\varphi(x) = 0, x \in M\}. \quad (9)$$

Hence, to get a well defined problem, we have to restrict the class of admissible sections. The natural way of doing this is by imposing boundary conditions which exclude *almost all* solutions of the operator, leaving only a finite-dimensional kernel.

In a collar neighborhood of ∂M in M , we will take coordinates $\bar{x} = (x, t)$, with t the inward normal coordinate and x local coordinates for ∂M (that is, $t > 0$ for points in $M \setminus \partial M$ and $t = 0$ on ∂M), and conjugate variables $\bar{\xi} = (\xi, \tau)$.

One of the most suitable tools for studying boundary problems is the Calderón projector Q [16]. For the case we are interested in, D of order 1 as in (8), Q is a (not necessarily orthogonal) projection

$$Q : L^2(\partial M, E_{/\partial M}) \rightarrow \{T\varphi / \varphi \in \text{Ker}(D)\}, \quad (10)$$

being $T : C^\infty(M, E) \rightarrow C^\infty(\partial M, E_{/\partial M})$ the trace map.

As shown in [16], Q is a zero-th order pseudo differential operator, and its principal symbol $q(x; \xi)$, depends only on the principal symbol of D , $\sigma_1(D)$.

Given any fundamental solution $K(\bar{x}, \bar{y})$ of D , the projector Q can be constructed in the following way: for $f \in C^\infty(\partial M, E_{/\partial M})$, one gets $\varphi \in \text{Ker}(D)$ by means of a Green formula involving $K(\bar{x}, \bar{y})$, and takes the limit of φ for $\bar{x} \rightarrow \partial M$.

Although Q is not uniquely defined, since one can take any fundamental solution K of D to construct it, the image of Q and its principal symbol $q(x; \xi)$ are independent of the choice of K [16].

We find it enlightening to compute the principal symbol of the Calderón projector for the Dirac operator

$$D(A) = i \not{\partial} + \not{A} = \sum_{\mu=0}^{\nu-1} \gamma_\mu \left(i \frac{\partial}{\partial x_\mu} + A_\mu \right), \quad (11)$$

where $\{A_\mu, \mu = 0, \dots, \nu - 1\}$ is the gauge field. In the present case, k is the dimension of the Dirac spinors in \mathbf{R}^ν , $k = 2^{\lfloor \nu/2 \rfloor}$.

Let $K(\bar{x}, \bar{y})$ be a fundamental solution of the Dirac operator $D(A)$ in a neighborhood of the region $M \subset \mathbf{R}^\nu$, i.e.

$$D^\dagger(A)K^\dagger(\bar{x}, \bar{y}) = \delta(\bar{x} - \bar{y}). \quad (12)$$

We can write

$$K(\bar{x}, \bar{y}) = K_0(\bar{x}, \bar{y}) + R(\bar{x}, \bar{y}) \quad (13)$$

where $K_0(\bar{x}, \bar{y})$ is the fundamental solution of $i \not{\partial}$ vanishing at infinity,

$$K_0(\bar{x}, \bar{y}) = -i \frac{\Gamma(\nu/2)}{2 \pi^{\nu/2}} \frac{(\bar{x} - \bar{y})}{|\bar{x} - \bar{y}|^\nu}, \quad (14)$$

and $|R(\bar{x}, \bar{y})|$ is $O(1/|\bar{x} - \bar{y}|^{\nu-2})$ for $|\bar{x} - \bar{y}| \sim 0$.

For f a smooth function on ∂M ,

$$Qf(x) = -i \lim_{\bar{x} \rightarrow \partial M} \int_{\partial M} K(\bar{x}, y) \not{n} f(y) d\sigma_y, \quad (15)$$

where $\not{n} = \sum_l \gamma_l n_l$, and $n = (n_l)$ is the unitary outward normal vector on ∂M . Note that, if $f = T\varphi$, with $\varphi \in \text{Ker}(D)$, the Green formula yields $Qf = f$, as required.

From (13), (14) and (15) one gets

$$Qf(x) = \frac{1}{2}f(x) - i P.V. \int_{\partial M} K_0(x, y) \not{n} f(y) d\sigma_y - i \int_{\partial M} R(x, y) \not{n} f(y) d\sigma_y. \quad (16)$$

To calculate the principal symbol of Q , we write the second term in the r.h.s. of (16) in local coordinates on ∂M ,

$$-i P.V. \int_{\mathbf{R}^{\nu-1}} \frac{\Gamma(\nu/2)}{2 \pi^{\nu/2}} \frac{(x-y)_j}{|x-y|^\nu} \gamma_j \gamma_n f(y) dy = \frac{1}{2} \gamma_j \gamma_n \mathbf{R}_j(f)(x), \quad (17)$$

where $\mathbf{R}_j(f)$ is the j -th Riesz transform of f . The symbol of the operator in (17) is (see for example [17])

$$\frac{1}{2} i \gamma_j \gamma_n \frac{\xi_j}{|\xi|} = \frac{1}{2} i \frac{\not{\xi}}{|\xi|} \not{n}. \quad (18)$$

The last term in the r.h.s. of (16) is a pseudodifferential operator of order ≤ -1 , because of the local behavior of $R(x, y)$, and then it does not contribute to the calculus of the principal symbol we are carrying out. Then, coming back to global coordinates, we finally obtain

$$q(x; \xi) = \frac{1}{2}(Id_{k \times k} + i \frac{\xi}{|\xi|} \not{x}). \quad (19)$$

Note that

$$\begin{aligned} q(x; \xi) q(x; \xi) &= q(x; \xi) \\ \text{tr } q(x; \xi) &= k/2, \end{aligned} \quad (20)$$

and consequently $\text{rank } q(x; \xi) = k/2$.

In particular, for $\nu = 2$ and the γ -matrices given by

$$\begin{aligned} \gamma_0 = \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \gamma_1 = \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \gamma_5 = -i\gamma_0\gamma_1 &= \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned} \quad (21)$$

we obtain

$$q(x; \xi) = \begin{pmatrix} H(\xi) & 0 \\ 0 & H(-\xi) \end{pmatrix} \quad (22)$$

$\forall x \in \partial M$, with $H(\xi)$ the Heaviside function.

According to Calderón [16], elliptic boundary conditions can be defined in terms of $q(x; \xi)$, the principal symbol of the projector Q .

Definition 1:

Let us assume that the rank of $q(x; \xi)$ is a constant r (as is always the case for $\nu \geq 3$ [16]).

A zero-th order pseudo differential operator

$$B : [L^2(\partial M, E_{\partial M})] \rightarrow [L^2(\partial M, G)], \quad (23)$$

with G an r dimensional complex vector bundle over ∂M , gives rise to an *elliptic boundary condition* for a first order operator D if,

$$\forall \xi : |\xi| \geq 1, \quad \text{rank}(b(x; \xi) q(x; \xi)) = \text{rank}(q(x; \xi)) = r, \quad (24)$$

where $b(x; \xi)$ coincides with the principal symbol of B for $|\xi| \geq 1$.

In this case we say that

$$\begin{cases} D\varphi = \chi & \text{in } M \\ BT\varphi = f & \text{at } \partial M \end{cases} \quad (25)$$

is an *elliptic boundary problem*, and denote by D_B the closure of D acting on the sections $\varphi \in C^\infty(M, E)$ satisfying $B(T\varphi) = 0$.

An elliptic boundary problem as (25) has a solution $\varphi \in H^1(M, E)$ for any (χ, f) in a subspace of $L^2(M, E) \times H^{1/2}(\partial M, G)$ of finite codimension. Moreover, this solution is unique up to a finite dimensional kernel [16]. In other words, the operator

$$(D, BT) : H^1(M, E) \rightarrow L^2(M, E) \times H^{1/2}(\partial M, G) \quad (26)$$

is Fredholm.

For $\nu = 2$, Definition 1 not always applies. For instance, for the two dimensional chiral Euclidean Dirac operator

$$D = 2i \frac{\partial}{\partial z^*}, \quad (27)$$

acting on sections with positive chirality and taking values in the subspace of sections with negative one, it is easy to see from (22) that

$$q(x; \xi) = H(\xi). \quad (28)$$

Then, the *rank* of $q(x; \xi)$ is not constant. In fact,

$$\text{rank } q(x; \xi) = \begin{cases} 0 & \text{if } \xi < 0 \\ 1 & \text{if } \xi > 0 \end{cases}. \quad (29)$$

However, for the (full) two dimensional Euclidean Dirac operator

$$D(A) = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} \quad (30)$$

we get from (20) that $\text{rank } q(x; \xi) = 2/2 = 1 \forall \xi \neq 0$, and so Definition 1 does apply.

Local boundary conditions

When B is a local operator, Definition 1 yields the classical local elliptic boundary conditions, also called Lopatinsky-Shapiro conditions (see for instance [18]).

For Euclidean Dirac operators on \mathbf{R}^ν , $E_{/\partial M} = \partial M \times \mathbf{C}^k$, and local boundary conditions arise when the action of B is given by the multiplication by a $\frac{k}{2} \times k$ matrix of functions defined on ∂M .

Owing to *topological obstructions*, chiral Dirac operators in even dimensions do not admit local elliptic boundary conditions (see for example [19]). For instance, in four dimensions, by choosing the γ -matrices at $x = (x_1, x_2, x_3) \in \partial M$ as

$$\gamma_4 = i \begin{pmatrix} 0 & Id_{2 \times 2} \\ -Id_{2 \times 2} & 0 \end{pmatrix} \quad \text{and} \quad \gamma_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad \text{for } j = 1, 2, 3, \quad (31)$$

the principal symbol of the Calderón projector (19) associated to the full Dirac operator turns out to be

$$q(x; \xi) = \frac{1}{2} \begin{pmatrix} Id_{2 \times 2} + \frac{\xi \cdot \sigma}{|\xi|} & 0 \\ 0 & Id_{2 \times 2} - \frac{\xi \cdot \sigma}{|\xi|} \end{pmatrix}. \quad (32)$$

Thus, from the left upper block, one gets for the chiral Dirac operator

$$q_{ch}(x; \xi) = \frac{1}{2} \begin{pmatrix} 1 + \frac{\xi_3}{|\xi|} & \frac{\xi_1 - i\xi_2}{|\xi|} \\ \frac{\xi_1 + i\xi_2}{|\xi|} & 1 - \frac{\xi_3}{|\xi|} \end{pmatrix}. \quad (33)$$

So $q_{ch}(x; \xi)$ is a hermitian idempotent 2×2 matrix with $\text{rank} = 1$. If one had a local boundary condition with principal symbol $b(x) = (\beta_1(x), \beta_2(x))$, according to Definition 1, it should be $\text{rank}(b(x) q_{ch}(x; \xi)) = 1, \forall \xi \neq 0$. However, it is easy to see that for

$$\xi_1 = \frac{-2\beta_1\beta_2}{\beta_1^2 + \beta_2^2}, \quad \xi_2 = 0 \quad \text{and} \quad \xi_3 = \frac{\beta_2^2 - \beta_1^2}{\beta_1^2 + \beta_2^2}, \quad (34)$$

$\text{rank}(b(x) q_{ch}(x; \xi)) = 0$. This is an example of the so called topological obstructions.

Nevertheless, it is easy to see that local boundary conditions can be defined for the full, either free or coupled, Euclidean Dirac operator

$$D(A) = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} \quad (35)$$

on M . For instance, we see from (22) and (24) that for $\nu = 2$, the operator B defined as

$$B \begin{pmatrix} f \\ g \end{pmatrix} = (\beta_1(x), \beta_2(x)) \begin{pmatrix} f \\ g \end{pmatrix} \quad (36)$$

yields a local elliptic boundary condition for every couple of nowhere vanishing functions $\beta_1(x)$ and $\beta_2(x)$ on ∂M .

A type of non-local boundary conditions to be consider is related to the ones defined and analyzed by M. Atiyah, V. Patodi and I. Singer in [14] for a wide class of first order Dirac-like operators, including the Euclidean chiral case. Near ∂M such operators can be written as

$$\varrho (\partial_t + \mathcal{A}), \quad (37)$$

where $\rho : E \rightarrow F$ is an isometric bundle isomorphism, and

$$\mathcal{A} : L^2(\partial M, E_{/\partial M}) \rightarrow L^2(\partial M, E_{/\partial M}) \quad (38)$$

is self adjoint. The operator P_{APS} defining the boundary condition is the orthogonal projection onto the closed subspace of $L^2(\partial M, E_{/\partial M})$ spanned by the eigenfunctions of \mathcal{A} associated to non negative eigenvalues,

$$P_{APS} = \sum_{\lambda \geq 0} \phi_\lambda(\phi_\lambda, \cdot), \quad \text{where } \mathcal{A}\phi_\lambda = \lambda\phi_\lambda. \quad (39)$$

The projector P_{APS} is a zero-th order pseudo differential operator and its principal symbol coincides with the one of the corresponding Calderón projector [20].

The problem (25)

$$\begin{cases} D\varphi = \chi & \text{in } M \\ P_{APS}T\varphi = f & \text{at } \partial M \end{cases} \quad (40)$$

with $B = P_{APS}$ has a solution $\varphi \in H^1(M, E)$ for any (χ, f) with χ in a finite codimensional subspace of $L^2(M, E)$ and f in the intersection of $H^{1/2}(\partial M, E_{/\partial M})$ with the image of P_{APS} . The solution is unique up to a finite dimensional kernel. Note that, since the codimension of $P_{APS} [L^2(\partial M, E_{/\partial M})]$ is not finite, the operator

$$(D, P_{APST}) : H^1(M, E) \rightarrow L^2(M, E) \times H^{1/2}(\partial M, E_{/\partial M}) \quad (41)$$

is not Fredholm.

It is to be stressed that, even though P_{APS} has the same principal symbol as Q , their actions are, roughly speaking, *opposite*. In fact, the Calderón projector is related to the problem of the *inner extension* of section over the boundary to global solutions on the manifold. On the other hand, the action of P_{APS} is related to the *outer extension problem*, in the sense that the solutions of $D_{P_{APS}}$ admit a square-integrable prolongation on the non-compact elongation obtained from M by attaching a semi-infinite cylinder $(-\infty, 0] \times \partial M$ to the boundary.

Definition 1 for elliptic boundary conditions does not encompass Atiyah, Patodi and Singer (APS) conditions since P_{APS} takes values in $L^2(\partial M, E_{/\partial M})$ instead of $L^2(\partial M, G)$, with G an r dimensional vector bundle ($r = \text{rank } q(x; \xi)$), as required in that definition. However, it is possible to define elliptic boundary problems according to Definition 1 by using conditions *à la* APS. For instance, the following self-adjoint boundary problem for the two-dimensional full Euclidean Dirac operator is elliptic:

$$\begin{aligned} \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} &= \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} & \text{in } M, \\ (P_{APS}, \varrho(I - P_{APS}) \varrho^*) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} &= h & \text{at } \partial M, \end{aligned} \quad (42)$$

In fact, as mentioned above, the principal symbol of P_{APS} is equal to the principal symbol of the Calderón projector associated to D . So, from (28) we get

$$\sigma_0(P_{APS})(x, \xi) = H(\xi). \quad (43)$$

By taking adjoints we obtain

$$\sigma_0(\varrho(I - P_{APS})\varrho^*) = H(-\xi). \quad (44)$$

Then, the principal symbol of $B = (P_{APS}, \sigma(I - P_{APS})\sigma^*)$ is

$$b(x; \xi) = (H(\xi), H(-\xi)) \quad (45)$$

and satisfies

$$\text{rank}(b(x; \xi) q(x; \xi)) = \text{rank}(q(x; \xi)) \quad \forall \xi \neq 0. \quad (46)$$

Functional determinants

For the case of *local boundary conditions* (as in the boundaryless case), the estimates of Seeley [13] allow one to express the complex powers of D_B, D_B^z , as an integral operator with continuous kernel $J_z(x, t; y, s)$ (and, consequently, of trace class) for $\text{Re}(z) < -\nu$.

As a function of z ,

$$\zeta_{(D_B)}(-z) \equiv \text{Tr}(D_B^z) \quad (47)$$

can be extended to a meromorphic function in the whole complex plane \mathbf{C} [21], with only simple poles at $z = j - \nu$, $j = 0, 1, 2, \dots$ and vanishing residues for $z = 0, 1, 2, \dots$

So, in this case, a regularized determinant of D_B can then be defined as

$$\text{Det}(D_B) = \exp\left[-\frac{d}{dz} \text{Tr}(D_B^z)\right]_{z=0}. \quad (48)$$

This determinant can also be expressed in terms of the Green's function of the elliptic boundary value problem, as in [11,12].

But, as far as we know, the construction of complex powers for elliptic boundary problems with *global* boundary conditions is still under study [22]. So, for the global case, one can not use the previous definition.

In the following we present the complete evaluation of the determinant of the Dirac operator on a disk, in the presence of an axially symmetric Abelian flux and under spectral boundary conditions, in terms of the corresponding Green's function.

DIRAC OPERATOR ON A DISK WITH GLOBAL BOUNDARY CONDITIONS

We will evaluate the determinant of the operator $D = i\partial + \mathcal{A}$ acting on functions defined on a two dimensional disk of radius R , under APS boundary conditions.

We consider an Abelian gauge field in the Lorentz gauge, $A_\mu = \epsilon_{\mu\nu} \partial_\nu \phi$ ($\epsilon_{01} = -\epsilon_{10} = 1$), with ϕ a smooth bounded function $\phi = \phi(r)$; then

$$A_r = 0, \quad A_\theta(r) = -\partial_r \phi(r) = -\phi'(r). \quad (49)$$

The flux through the disk is

$$\kappa = \frac{\Phi}{2\pi} = \frac{1}{2\pi} \oint_{r=R} A_\theta R d\theta = -R\phi'(R). \quad (50)$$

With the conventions for the γ -matrices stated above, the full Dirac operator can be written as

$$D = e^{-\gamma_5 \phi(r)} i \partial e^{-\gamma_5 \phi(r)} = \begin{pmatrix} 0 & \varrho^{-1}(\partial_r + \mathcal{A}) \\ \varrho(-\partial_r + \mathcal{A}) & 0 \end{pmatrix}, \quad (51)$$

where

$$\varrho = -i e^{i\theta}, \quad (52)$$

and

$$\mathcal{A}(r) = -\frac{i}{r} \partial_\theta + \partial_r \phi(r). \quad (53)$$

At the boundary, the eigenvectors and eigenvalues of the self adjoint operator $\mathcal{A}(R)$ are given by

$$\mathcal{A}(R) e^{in\theta} = a_n e^{in\theta}, \quad \text{with } a_n = \frac{1}{R}(n - \kappa), \quad n \in \mathbf{Z}. \quad (54)$$

We take the radial variable to be conveniently adimensionalized through multiplication by a fixed constant with dimensions of mass.

Let k be the integer such that $k < \kappa \leq k + 1$. We will consider the action of the differential operator D on the space of functions satisfying homogeneous global boundary conditions characterized by

$$(\mathcal{P}_\geq \varrho(1 - \mathcal{P}_\geq) \varrho^*) \begin{pmatrix} \varphi(R, \theta) \\ \chi(R, \theta) \end{pmatrix} = 0, \quad (55)$$

where

$$\mathcal{P}_\geq = \frac{1}{2\pi} \sum_{n \geq k+1} e^{in\theta} (e^{in\theta}, \cdot), \quad (56)$$

$$\varrho(1 - \mathcal{P}_\geq) \varrho^* = \varrho \mathcal{P}_< \varrho^* = \frac{1}{2\pi} \sum_{n \leq k+1} e^{in\theta} (e^{in\theta}, \cdot) = \mathcal{P}_<. \quad (57)$$

Notice that the operator so defined, which we call $(D)_\kappa$, turns out to be self adjoint.

Our aim is to compute the quotient of the determinants of the operators $(D)_\kappa$ and $(i \not{\partial})_{\kappa=0}$. Since the global boundary condition in Eq. (55) is not a continuous function of the flux Φ , we will proceed in two steps:

$$(D)_\kappa \rightarrow (i \not{\partial})_\kappa \rightarrow (i \not{\partial})_{\kappa=0}. \quad (58)$$

In the first step, where there is no change of boundary conditions, we can grow the gauge field by varying α from 0 to 1 in

$$D_\alpha = i \not{\partial} + \alpha \not{A} = e^{-\alpha\gamma_5 \phi(r)} i \not{\partial} e^{-\alpha\gamma_5 \phi(r)}, \quad (59)$$

thus going smoothly from the free to the full Dirac operator. The explicit knowledge of the Green's function will allow us to perform the calculation of this step, where we will use a gauge invariant point splitting regularization of the α -derivative of the determinant. The second step will be achieved by using a ζ -function regularization, after explicitly computing the spectra.

There is an additional complication, since these global boundary conditions give rise to the presence of $|k + 1|$ linearly independent zero modes. For $k > 0$, these normalized eigenvectors are given by

$$\frac{e^{\alpha\phi(r)}}{\sqrt{2\pi} q_n(R; \alpha)} \begin{pmatrix} X^n \\ 0 \end{pmatrix}, \quad \text{with } 0 \leq n \leq k, \quad (60)$$

where $X = x_0 + ix_1 = r e^{i\theta}$, and the normalization factors are

$$q_n(u; \alpha) = \int_0^u e^{2\alpha\phi(r)} r^{2n+1} dr. \quad (61)$$

For $k < 1$ we get similar expressions, with the opposite chirality. Notice that, for $k = -1$ (in particular, when $\Phi = 0$), there is no zero mode.

For simplicity, in the following we will consider only the case $k \geq -1$. The kernel of the orthogonal projector on $\text{Ker}(D_\alpha)_\kappa$, P_α is given by

$$P_\alpha(z, w) = \sum_{n=0}^k \frac{e^{\alpha[\phi(z)+\phi(w)]}}{2\pi q_n(R; \alpha)} \begin{pmatrix} (ZW^*)^n & 0 \\ 0 & 0 \end{pmatrix}. \quad (62)$$

Now, since $(D_\alpha + P_\alpha)_\kappa$ is invertible, we can define

$$Det'(D_\alpha)_\kappa \equiv Det(D_\alpha + P_\alpha)_\kappa, \quad (63)$$

and write

$$\frac{Det'(D)_\kappa}{Det(i\partial + P_0)_\kappa} = \frac{Det(D + P_1)_\kappa}{Det(i\partial + P_0)_\kappa} \frac{Det(i\partial + P_0)_\kappa}{Det(i\partial + P_0)_\kappa}. \quad (64)$$

We can compute the first factor in the r.h.s. by taking the derivative

$$\frac{\partial}{\partial \alpha} [\ln Det(D_\alpha + P_\alpha)_\kappa] = Tr[(\mathcal{A} + \partial_\alpha P_\alpha)G_\alpha], \quad (65)$$

where $G_\alpha(x, y)$ is the Green's function of the problem

$$\begin{aligned} (D_\alpha + P_\alpha) G_\alpha(x, y) &= \delta(x, y), \\ (\mathcal{P}_\geq \quad \mathcal{P}_\leq) G_\alpha(x, y)|_{r=R} &= 0. \end{aligned} \quad (66)$$

Since $(D_\alpha)_\kappa$ is self-adjoint, $G_\alpha(x, y)$ has the structure

$$G(x, y)_\alpha = (1 - P_\alpha) \mathcal{G}_\alpha(x, y) (1 - P_\alpha) + P_\alpha, \quad (67)$$

where $\mathcal{G}_\alpha(x, y)$ is the kernel of the right-inverse of D_α on the orthogonal complement of $\text{Ker}(D_\alpha)_\kappa$ [23],

$$\mathcal{G}_\alpha(x, y) = \frac{1}{2\pi i} \times \begin{pmatrix} 0 & \frac{e^{\alpha[\phi(x)-\phi(y)]}}{X-Y} \left(\frac{X}{Y}\right)^{k+1} \\ \frac{e^{-\alpha[\phi(x)-\phi(y)]}}{X^*-Y^*} \left(\frac{Y^*}{X^*}\right)^{k+1} & 0 \end{pmatrix}, \quad (68)$$

which, replaced in (67), allows to get $G(x, y)_\alpha$.

Being P_α an orthogonal projector,

$$(P_\alpha)^2 = P_\alpha, \quad \frac{\partial P_\alpha}{\partial \alpha} (1 - P_\alpha) = P_\alpha \frac{\partial P_\alpha}{\partial \alpha}, \quad (69)$$

from (67) we get $Tr[(\partial_\alpha P_\alpha)G_\alpha] = 0$. So, (65) reduces to the evaluation of $Tr[\mathcal{A}G_\alpha]$.

As usual, the kernel of the operator inside the trace is singular at the diagonal, so we must introduce a regularization. We will employ a point-splitting one where, following Schwinger [24], we will introduce a phase factor in order to preserve gauge invariance. We thus get,

$$\begin{aligned} Tr[\mathcal{A}G_\alpha] &= \\ \text{sym. lim. } \epsilon \rightarrow 0 &\int_{r < R} d^2x \text{ tr} \left[\mathcal{A}(x) G_\alpha(x, x + \epsilon) e^{i\alpha \epsilon \cdot A(x)} \right], \end{aligned} \quad (70)$$

where by symmetric limit we mean half the sum of the lateral limits $\epsilon \rightarrow 0^\pm$.

Performing the integral in α from 0 to 1 we get [23]

$$\begin{aligned} \ln \left[\frac{Det(D + P_1)_\kappa}{Det(i\partial + P_0)_\kappa} \right] &= -\frac{1}{2\pi} \int_{r < R} d^2x \phi'^2 \\ -2(k+1)\phi(R) &+ \sum_{n=0}^k \ln \left[2(n+1) \frac{q_n(R; 1)}{R^{2(n+1)}} \right]. \end{aligned} \quad (71)$$

Notice that when there are no zero modes ($k + 1 = 0$) only the first term in the r.h.s. survives.

In the following, we will obtain the second quotient of determinants in Eq. (64) by computing explicitly the spectra of the free Dirac operators and using a ζ -function regularization.

The eigenfunctions of $(i\partial + P_0)_\kappa$ are of the form

$$\psi_n(r, \theta) = \begin{pmatrix} \varphi_n(r, \theta) \\ \chi_n(r, \theta) \end{pmatrix} = \begin{pmatrix} J_n(|\lambda|r) e^{in\theta} \\ -i \frac{|\lambda|}{\lambda} J_{n+1}(|\lambda|r) e^{i(n+1)\theta} \end{pmatrix}, \quad (72)$$

and satisfy the boundary condition

$$\mathcal{P}_\geq \varphi_n(R, \theta) = \frac{1}{2\pi} \sum_{n \geq k+1} e^{in\theta} (e^{in\theta}, \varphi_n(R, \theta)) = 0, \quad (73)$$

$$\mathcal{P}_\leq \chi_n(R, \theta) = \frac{1}{2\pi} \sum_{n \leq k+1} e^{in\theta} (e^{in\theta}, \chi_n(R, \theta)) = 0. \quad (74)$$

For $n \geq k + 1$ the corresponding eigenvalues are $\lambda = \pm j_{n,l}/R$ ($j_{n,l}$ is the l -th zero of $J_n(z)$). Analogously, for $n \leq k$, $\lambda = \pm j_{n+1,l}/R$. Notice that $j_{-n,l} = j_{n,l}$, and that, for $n = k + 1$ the eigenvalues appear twice, once for an eigenfunction with vanishing upper component at the boundary, and once for another one with vanishing lower component.

For $\Re(s)$ large enough, we can construct the ζ -function of $(i\partial + P_0)_\kappa$ as [23]

$$\zeta_{(i\partial + P_0)_\kappa}(s) = |k + 1| + (1 + e^{-i\pi s}) \times \left\{ \sum_{n=-\infty}^{\infty} \sum_{l=1}^{\infty} \left(\frac{j_{n,l}}{R} \right)^{-s} + \sum_{l=1}^{\infty} \left(\frac{j_{|k+1|,l}}{R} \right)^{-s} \right\}. \quad (75)$$

The first term, $|k + 1|$, is just the multiplicity of the 0-eigenvalue of $(i\partial)_\kappa$. It is also interesting to note that the double sum in the r.h.s. (which is independent of k) corresponds to the ζ -function of the Laplacian on a disk with Dirichlet (local) boundary conditions, thus being analytic at $s = 0$ [21].

It is easy to verify that the analytic extension of the second sum,

$$f_\nu(s) \equiv \sum_{l=1}^{\infty} (j_{\nu,l})^{-s}. \quad (76)$$

is regular at $s = 0$. Then $\zeta_{(i\partial + P_0)_\kappa}(s)$ is regular at the origin. This is interesting since, as far as we know, the regularity of the ζ -function at the origin for non local boundary conditions has not been established in general [22].

In the framework of this regularization, we thus get

$$\begin{aligned} \ln \left[\frac{\text{Det}(i\partial + P_0)_\kappa}{\text{Det}(i\partial)_{\kappa=0}} \right] &\equiv -\frac{d}{ds} [\zeta_{(i\partial + P_0)_\kappa}(s) - \zeta_{(i\partial)_{\kappa=0}}(s)]_{s=0} = \\ &-2 \left[f'_{|k+1|}(0) - f'_0(0) + (\ln R - \frac{i\pi}{2}) [f_{|k+1|}(0) - f_0(0)] \right]. \end{aligned} \quad (77)$$

Taking into account the asymptotic expansion for the zeros of Bessel functions [25], we obtain

$$f_\nu(0) = -\frac{\nu}{2} - \frac{1}{4}, \quad (78)$$

and

$$f'_\nu(0) = -\frac{1}{2} \ln 2 + \left(\frac{2\nu - 1}{4} \right) (\ln \pi - \gamma) - \sum_{l=1}^{\infty} \ln \left[\frac{j_{\nu,l}}{l\pi} e^{-\left(\frac{2\nu-1}{4l}\right)} \right], \quad (79)$$

where γ is Euler's constant.

Finally, taking into account that we have used a gauge invariant procedure, we can write

$$\begin{aligned} \ln \left[\frac{\text{Det}(D + P_1)_\kappa}{\text{Det}(i \not{\partial})_{\kappa=0}} \right] &= -\frac{1}{2\pi} \int_{r < R} d^2x A_\mu (\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}) A_\nu \\ &\quad - 2(k+1) \phi(R) + \sum_{n=0}^k \ln \left[2(n+1) \frac{q_n(R; 1)}{R^{2(n+1)}} \right] \\ &\quad - |k+1| \left[\frac{i\pi}{2} - \gamma - \ln\left(\frac{R}{\pi}\right) \right] + 2 \sum_{l=1}^{\infty} \ln \left[\frac{j_{|k+1|,l}}{j_{0,l}} e^{-\left(\frac{|k+1|}{2} l\right)} \right]. \end{aligned} \quad (80)$$

The first term is the integral on the disk of the same expression appearing in the well-known result for the boundaryless case [26].

Connection with the index theorem

The variation of the determinant under global axial transformations (ϵ constant),

$$\begin{aligned} e^{-\gamma_5 \epsilon} (D + P_1)_\kappa e^{-\gamma_5 \epsilon} &= (D + e^{-\gamma_5 \epsilon} P_1 e^{-\gamma_5 \epsilon})_\kappa, \\ e^{-\gamma_5 \epsilon} (i \not{\partial})_{\kappa=0} e^{-\gamma_5 \epsilon} &= (i \not{\partial})_{\kappa=0}, \end{aligned} \quad (81)$$

is related to the *index* of the Dirac operator:

$$\frac{\partial}{\partial \epsilon} \ln \left[\frac{\text{Det}(e^{-\gamma_5 \epsilon} (D + P_1)_\kappa e^{-\gamma_5 \epsilon})}{\text{Det}(e^{-\gamma_5 \epsilon} (i \not{\partial})_{\kappa=0} e^{-\gamma_5 \epsilon})} \right] = -2Tr[\gamma_5 P_1] = -2(N_+ - N_-), \quad (82)$$

where $N_{+(-)}$ is the number of positive(negative) chirality zero modes.

It can be verified that our strategy leads to the correct result for this index. By following the same procedure that lead to Eq. (80), we can compute the quotient of determinants in the l.h.s of (82). In fact, taking into account that the inverse of the transformed operator is given by

$$G_\alpha^{(\epsilon)}(x, y) = (1 - P_\alpha) \mathcal{G}_\alpha(x, y) (1 - P_\alpha) + e^{\gamma_5 \epsilon} P_\alpha e^{\gamma_5 \epsilon}, \quad (83)$$

the only difference appears in the first term of the r.h.s. of (75), where a factor $e^{\pm 2\epsilon s}$ arises. Thus, after performing the ϵ -derivative

$$N_+ - N_- = k + 1, \quad (84)$$

which agrees with our previous result for the number of zero modes.

The Atiyah-Patodi-Singer theorem relates the $\text{index}(D)_\kappa$ with the spectral asymmetry of the self-adjoint operator

$$\mathcal{A} = \mathcal{A}(R) = -\frac{i}{R} \partial_\theta + \partial_r \phi(R). \quad (85)$$

From the eigenvalues of \mathcal{A} , $a_n = \frac{1}{R}(n - \kappa)$, one defines the η -function through the series

$$\eta_{(\mathcal{A})}(s) = R^s \sum_{n \neq \kappa} \text{sig}(n - \kappa) |n - \kappa|^{-s}, \quad (86)$$

convergent for $\Re(s) > 1$. The analytic extension of $\eta_{(\mathcal{A})}(s)$ to $s = 0$ is given by [23]

$$\eta_{(\mathcal{A})}(0) = 2(\kappa - k) - 1 - h(\mathcal{A}), \quad (87)$$

where $h(\mathcal{A}) = \dim \text{Ker}(\mathcal{A})$.

Following the construction of APS in [14], and taking into account that $\varrho = \varrho(\theta) = -i e^{i\theta}$ in the present case, we get

$$\text{index } D = \kappa + \frac{[1 - h(\mathcal{A}) - \eta_{(\mathcal{A})}(0)]}{2} = k + 1, \quad (88)$$

in agreement with (84). The first term in the intermediate expression is the well known contribution from the bulk [26]. The second one is the boundary contribution of APS, shifted by 1/2. This correction, due to the presence of the factor ϱ in (51), has already been obtained in [27] with slightly different spectral boundary conditions.

CONCLUSIONS

We have achieved the complete evaluation of the determinant of the Dirac operator on a disk, in the presence of an axially symmetric flux, under global boundary conditions of the type introduced by Atiyah, Patodi and Singer.

To this end, we have proceeded in two steps: In the first place, we have grown the gauge field while keeping the boundary condition fixed. This calculation was possible thanks to the exact knowledge of the zero modes and the Green's function (in the complement of the null space.) Here, a gauge invariant point splitting regularization was employed.

In the second step, we have explicitly obtained the eigenvalues of $(i \not{\partial} + P_0)_\kappa$. We have shown that the corresponding ζ -function is regular at the origin and we have evaluated the quotient of the free Dirac operators for two different global boundary conditions.

We have verified that our complete result is in agreement with the APS index theorem.

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