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Supersymmetric Solutions in Three-Dimensional Heterotic String Theory

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ABSTRACT

We consider the low-energy effective field theory of heterotic string theory compactified on a seven-torus, and we construct electrically charged as well as more general solitonic solutions. These solutions preserve 1/2, 1/4 and 1/8 of $N = 8, D = 3$ supersymmetry and have Killing spinors which exist due to cancellation of holonomies. The associated space–time line elements do not exhibit the conical structure that often arises in $2 + 1$ dimensional gravity theories.

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1 Introduction

The study of three-dimensional gravity theories is interesting in several respects. For instance, general relativity in three space–time dimensions has been a useful laboratory for studying conceptual issues in classical and quantum gravity (see [1, 2] for a review on work on $2+1$ dimensional gravity). More recently, the study of duality symmetries of compactified string theories down to three dimensions has provided some information about the large internal symmetries of this sector [3, 4, 5]. These symmetries are of interest, as they can yield non-perturbative information about the full string theory.

Another interesting aspect that has been recently pointed out by Witten [6, 7] is that the vanishing of the cosmological constant and the absence of a massless dilaton in four space–time dimensions could be explained by duality between a supersymmetric string vacuum in three dimensions and a non-supersymmetric string vacuum in four dimensions. The observation that in $2+1$ dimensions the usual connection between supersymmetry of the vacuum and the bose–fermi degeneracy of the excited states does not hold [6, 7], has been subsequently explored in certain three-dimensional models [8, 9]. Other models that have been studied are supersymmetric spacetimes in $2+1$ anti-de Sitter supergravity [10], and some new $2+1$ dimensional Poincaré supergravity theories with central charges and Killing spinors [11]. All these considerations add renewed interest to the study of three-dimensional supergravity theories.

In this paper, we will consider the low-energy effective theory of heterotic string theory compactified on a seven-torus [3], and we will construct various static soliton solutions. Rather than using the criteria of the saturation of the Bogomol'nyi bound to characterise these solutions, we will use the criteria of unbroken supersymmetry [3]. The construction of these supersymmetric solutions will thus be achieved by solving the associated Killing spinor equations. The associated space–time metric does not approach flat space–time at infinity, as is the case in four dimensions, and this renders the existence of covariantly constant spinors uncertain at first sight, due to the phase acquired by a spinor when parallel transported around a closed curve at infinity. We show, however, that it is possible to construct such Killing spinors due to the cancellation of the holonomies. The existence of non-trivial supercovariantly constant Killing spinors in asymptotically conical spacetimes [12] due to the cancellation of phases has already been noticed in various other three-dimensional models [10, 8, 11, 13, 14, 15, 16, 9].

This paper is organised as follows. In section 2 we review some properties of the low-energy effective action of heterotic string theory compactified on a seven dimensional torus [3]. In section 3 we present the Killing spinor equations associated to the three-dimensional heterotic low-energy effective Lagrangian. Consistency with the Clifford algebra in ten dimensions forces us to introduce a chirality operator in three dimensions [17] (see appendix). In order to be able to do so, we promote the three-dimensional Killing spinors to four-component spinors (no two-dimensional representation for the three-dimensional Dirac matrices exists admitting a gamma matrix anticommuting with all of them).

In section 4 we present static soliton solutions, which we obtain by solving the Killing spinor equations along the lines of [18]. We find that the space–time line element differs from the line element associated with conical geometries [12]. We proceed in several steps. First, we construct electrically charged solutions. We take the associated gauge fields to be the ones arising from the compactification of the heterotic string from ten dimensions down to three. We further restrict the internal metric G_{mn} to be diagonal. This restriction has the consequence that the electrically charged solution can, at most, carry two electric charges associated with two different $U(1)$ factors. In subsection 4.1 we construct electrically charged solutions carrying both charges, and we show that they preserve 1/2 of $N = 8, D = 3$ supersymmetry. The associated internal metric G_{mn} is constant, whereas the internal antisymmetric tensor field B_{mn} is zero. Next, since the low-energy effective theory is invariant under $O(8, 24)$ transformations of the background fields, we apply a particular $O(8, 24)$ transformation on the background fields of the electrically charged solution, and we obtain two types of solitonic solutions which also preserve 1/2 of $N = 8, D = 3$ supersymmetry. In particular, the type of solitonic solutions given in subsection 4.2.1 has an off-diagonal non-constant internal metric G_{mn} as well as a non-vanishing internal antisymmetric tensor field B_{mn}. In addition, the associated gauge field strengths vanish. Then, we proceed to construct solitonic solutions preserving 1/4 of $N = 8, D = 3$ supersymmetry, by combining features of the electrically charged solutions and of the solitonic solutions of subsection 4.2.1. That is, they have non-vanishing gauge field strengths as well as a non-diagonal non-constant internal metric and a non-vanishing internal antisymmetric tensor field. Finally, this procedure can be generalised to yield solitonic solutions preserving 1/8 of $N = 8, D = 3$ supersymmetry. This is achieved by increasing the number of non-vanishing entries (blocks) in the B_{mn}-field.

In section 5 we repeat the analysis given in section 4, starting from electrically charged solutions carrying only one electric charge. These electrically charged solutions have a non-constant internal metric G_{mn}, as opposed to the ones discussed in section 4. We proceed to construct solitonic solutions preserving 1/2, 1/4 and 1/8 of $N = 8, D = 3$ supersymmetry along the line of section 4. Here we find in all cases that the internal metric G_{mn} is non-constant, but diagonal.

The space–time curvature of each of the solutions constructed in sections 4 and 5 vanishes at spatial infinity, but the associated space–time metric does not asymptotically approach either a flat metric or an anti-de Sitter metric. Thus, these solutions do not describe black hole solutions in the usual sense [19, 20]. Our supersymmetric solutions do not appear to interpolate spatially between two vacuum-type supersymmetric configurations, as is the case for the extreme Reissner–Nordström metric in four dimensions, for example. This latter solution interpolates between flat space–time at spatial infinity and a Bertotti–Robinson metric near the horizon [21]. We nevertheless refer to our supersymmetric solutions as solitonic solutions.

In [3], Sen constructed a particular three-dimensional solution by first considering the fundamental string solution of the four dimensional theory [22] and then winding the direction along which the string extends once in the third direction. In section 6, we construct the associated Killing spinor in three dimensions, as an application of our formalism.

All solutions discussed in sections 4, 5 and 6 have $H_{\mu\nu\rho} = 0$. In section 7, we consider solutions to the Killing spinor equations with $H_{\mu\nu\rho} \neq 0$, which preserve 1/2 of $N = 8, D = 3$ supersymmetry. We show that all such solutions, with the exception of one, do not solve the equations of motion. This should be compared with the common expectation [23, 13, 14] that (under some suitable general assumptions) every solution to the Killing spinor equations also solves the equations of motion.

Finally, in section 8, we present our conclusions. Our conventions are summarised in the appendix.

2 The three-dimensional effective action

The effective low-energy field theory of the ten-dimensional heterotic string compactified on a seven-dimensional torus is obtained from reducing the ten-dimensional $N = 1$ supergravity theory coupled to $U(1)^{16}$ super Yang-Mills multiplets (at a generic point in the moduli space) [24, 25, 3]. The massless ten-dimensional bosonic fields are the metric $G^{(10)}_{MN}$, the anti-symmetric tensor field $B^{(10)}_{MN}$, the $U(1)$ gauge fields $A^{(10)I}_M$ and the scalar dilaton $\Phi^{(10)}$ with $(0 \leq M, N \leq 9, \quad 1 \leq I \leq 16)$. The field strengths are $F^{(10)I}_{MN} = \partial_M A^{(10)I}_N - \partial_N A^{(10)I}_M$ and $H^{(10)}_{MNP} = (\partial_M B^{(10)}_{NP} - \frac{1}{2} A^{(10)I}_M F^{(10)I}_{NP})+$ cyclic permutations of M, N, P.

The bosonic part of the ten dimensional action is

$$\mathcal{S} \;\propto\; \int d^{10}x\sqrt{-G^{(10)}}e^{-\Phi^{(10)}}[\mathcal{R}^{(10)} + G^{(10)MN}\partial_M\Phi^{(10)}\partial_N\Phi^{(10)} - \frac{1}{12}H^{(10)}_{MNP}H^{(10)MNP} - \frac{1}{4}F^{(10)I}_{MN}F^{(10)IMN}]. \tag{2.1}$$

The reduction to three dimensions [26, 25, 3] introduces the graviton $g_{\mu\nu}$, the dilaton $\phi \equiv \Phi^{(10)} - \ln\sqrt{\det G_{mn}}$, with G_{mn} the internal 7D metric, 30 $U(1)$ gauge fields $A^{(a)}_\mu \equiv (A^{(1)m}_\mu,\, A^{(2)}_{\mu m},\, A^{(3)I}_\mu)$ $(a = 1,\dots,30,\ m = 1,\dots,7,\ I = 1,\dots,16)$, where $A^{(1)m}_\mu$ are the 7 Kaluza–Klein gauge fields coming from the reduction of $G^{(10)}_{MN}$, $A^{(2)}_{\mu m} \equiv B_{\mu m} + B_{mn}A^{(1)n}_\mu + \frac{1}{2}a^I_m A^{(3)I}_\mu$ are the 7 gauge fields coming from the reduction of $B^{(10)}_{MN}$ and $A^{(3)I}_\mu \equiv A^I_\mu - a^I_m A^{(1)m}_\mu$ are the 16 gauge fields from $A^{(10)I}_M$.

The field strengths $F^{(a)}_{\mu\nu}$ are given by $F^{(a)}_{\mu\nu} = \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu$. Finally, $B^{(10)}_{MN}$ induces the two form field $B_{\mu\nu}$ with field strength $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2}A^{(a)}_\mu L_{ab}F^{(b)}_{\nu\rho}+$ cyclic permutations.

The 161 scalars G_{mn}, a^I_m and B_{mn} can be arranged into a 30×30 matrix M (we use here the conventions of [25])

$$M = \begin{pmatrix} G^{-1} & -G^{-1}C & -G^{-1}a^T \\ -C^TG^{-1} & G + C^TG^{-1}C + a^Ta & C^TG^{-1}a^T + a^T \\ -aG^{-1} & aG^{-1}C + a & I_{16} + aG^{-1}a^T \end{pmatrix}, \tag{2.2}$$

where $G \equiv [G_{mn}],\ C \equiv [\frac{1}{2}a^I_m a^I_n + B_{mn}]$ and $a \equiv [a^I_m]$.

We have $MLM^T = L, \quad M^T = M, \quad L^{-1} = L$, where

$$L = \begin{pmatrix} 0 & I_7 & 0 \\ I_7 & 0 & 0 \\ 0 & 0 & I_{16} \end{pmatrix}. \tag{2.3}$$

We use the following ansatz for the Kaluza-Klein 10D vielbein E^A_M and inverse vielbein E^M_A, in the string frame

$$E^A_M = \begin{pmatrix} e^\phi e^\alpha_\mu & A^{(1)m}_\mu e^a_m \\ 0 & e^a_m \end{pmatrix}, \qquad E^M_A = \begin{pmatrix} e^{-\phi}e^\mu_\alpha & -e^{-\phi}e^\mu_\alpha A^{(1)m}_\mu \\ 0 & e^m_a \end{pmatrix}, \tag{2.4}$$

where e^a_m is the internal and e^α_μ the space–time vielbein in the Einstein frame (the relation between string metric $G_{\mu\nu}$ and Einstein metric $g_{\mu\nu}$ in three dimensions is $G_{\mu\nu} = e^{2\phi}g_{\mu\nu}$).

The three-dimensional action in the Einstein frame is then [25, 3],

$$\mathcal{S} \;=\; \frac{1}{4}\int d^3x\sqrt{-g}\{\mathcal{R} - g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{12}e^{-4\phi}g^{\mu\mu'}g^{\nu\nu'}g^{\rho\rho'}H_{\mu\nu\rho}H_{\mu'\nu'\rho'} - \frac{1}{4}e^{-2\phi}g^{\mu\mu'}g^{\nu\nu'}F^{(a)}_{\mu\nu}(LML)_{ab}F^{(b)}_{\mu'\nu'} + \frac{1}{8}g^{\mu\nu}\mathrm{Tr}\,(\partial_\mu ML\partial_\nu ML)\}, \tag{2.5}$$

where $a = 1, \ldots, 30$.

This action is invariant under the $O(7, 23)$ transformations

$$M \to \tilde{\Omega} M \tilde{\Omega}^T, \quad A_\mu^{(a)} \to \tilde{\Omega}_{ab} A_\mu^{(b)}, \quad g_{\mu\nu} \to g_{\mu\nu}, \quad B_{\mu\nu} \to B_{\mu\nu}, \quad \phi \to \phi, \quad \tilde{\Omega}^T L \tilde{\Omega} = L, \tag{2.6}$$

where $\tilde{\Omega}$ is a 30×30 $O(7, 23)$ matrix.

The equations of motion for $A_\mu^{(a)}$, ϕ, $H^{\mu\nu\rho}$ and $g^{\mu\nu}$ are, respectively,

$$\partial_\mu (e^{-2\phi} \sqrt{-g} (LML)_{ab} F^{(b)\mu\nu}) + \frac{1}{2} e^{-4\phi} \sqrt{-g}\, L_{ab}\, F_{\mu\rho}^{(b)} H^{\nu\mu\rho} = 0 \,, \tag{2.7}$$

$$D_\mu D^\mu \phi + \frac{1}{4} e^{-2\phi} F_{\mu\nu}^{(a)} (LML)_{ab} F^{\mu\nu(b)} + \frac{1}{6} e^{-4\phi} H^{\mu\nu\rho} H_{\mu\nu\rho} = 0 \,, \tag{2.8}$$

$$\partial_\mu (\sqrt{-g} e^{-4\phi} H^{\mu\nu\rho}) = 0 \,, \tag{2.9}$$

$$\begin{aligned} \mathcal{R}_{\mu\nu} = {} & \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} e^{-2\phi} F_{\mu\rho}^{(a)} (LML)_{ab} F_\nu^{\rho(b)} - \frac{1}{8} \mathrm{Tr}\, (\partial_\mu ML \partial_\nu ML) \\ & - \frac{1}{4} e^{-2\phi} g_{\mu\nu} F_{\rho\tau}^{(a)} (LML)_{ab} F^{\rho\tau(b)} + \frac{1}{4} e^{-4\phi} H_\mu^{\tau\sigma} H_{\nu\tau\sigma} - \frac{1}{6} g_{\mu\nu} e^{-4\phi} H^{\tau\sigma\rho} H_{\tau\sigma\rho} \,. \end{aligned} \tag{2.10}$$

We note that after dimensional reduction on a seven torus, the only massless bosonic fields remaining are the spin two (non-propagating) graviton $g_{\mu\nu}$ and a set of scalar fields, since in three dimensions vector fields are dual to scalar fields. In three dimensions the field $B_{\mu\nu}$ has no physical degrees of freedom. We will therefore consider backgrounds where either $H_{\mu\nu\rho} = 0$, or $H_{\mu\nu\rho} = \sqrt{-g} \epsilon_{\mu\nu\rho} \Lambda e^{4\phi}$.

Let us now consider the case where $H_{\mu\nu\rho} = 0$. From the equations of motion for the gauge fields $A_\mu^{(a)}$ (2.7) one can define a set of scalar fields Ψ^a, $a = 1, \ldots, 30$, through [3]

$$\begin{aligned} \sqrt{-g} e^{-2\phi} g^{\mu\mu'} g^{\nu\nu'} (ML)_{ab} F_{\mu'\nu'}^{(b)} &= \epsilon^{\mu\nu\rho} \partial_\rho \Psi^a, \\ F^{(a)\mu\nu} &= \frac{1}{\sqrt{-g}} e^{2\phi} (ML)_{ab} \epsilon^{\mu\nu\rho} \partial_\rho \Psi^b \,. \end{aligned} \tag{2.11}$$

Then, from the Bianchi identity $\epsilon^{\mu\nu\rho} \partial_\mu F_{\nu\rho}^{(a)} = 0$,

$$D^\mu (e^{2\phi} (ML)_{ab} \partial_\mu \Psi^b) = 0. \tag{2.12}$$

Following [3], the charge quantum numbers of elementary string excitations are characterized by a 30 dimensional vector $\vec{\alpha} \in \Lambda_{30}$. The asymptotic value of the field strength $F_{\mu\nu}^{(a)}$ associated with such an elementary particle can be calculated to be [3]

$$\sqrt{-g} F^{(a)tr} \simeq -\frac{1}{2\pi} e^{2\phi} M_{ab} \alpha^b. \tag{2.13}$$

The asymptotic form of Ψ^a is then

$$\Psi^a \simeq -\frac{\theta}{2\pi} L_{ab} \alpha^b + \text{constant}. \tag{2.14}$$

Arranging now the Ψ's into a 30 dimensional column vector, one can define a new 32×32 matrix $\mathcal{M}$ [3]

$$\mathcal{M} = \begin{pmatrix} M + e^{2\phi}\Psi\Psi^T & -e^{2\phi}\Psi & ML\Psi + \frac{1}{2}e^{2\phi}\Psi(\Psi^T L\Psi) \\ -e^{2\phi}\Psi^T & e^{2\phi} & -\frac{1}{2}e^{2\phi}\Psi^T L\Psi \\ \Psi^T LM + \frac{1}{2}e^{2\phi}\Psi^T(\Psi^T L\Psi) & -\frac{1}{2}e^{2\phi}\Psi^T L\Psi & e^{-2\phi} + \Psi^T LML\Psi + \frac{1}{4}e^{2\phi}(\Psi^T L\Psi)^2 \end{pmatrix}, \tag{2.15}$$

where $\mathcal{M}^T = \mathcal{M}, \quad \mathcal{M}^T\mathcal{L}\mathcal{M} = \mathcal{L},$ and $\mathcal{L}$ is a 32×32 matrix

$$\mathcal{L} = \begin{pmatrix} L & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} . \tag{2.16}$$

Then the action in the Einstein frame can be written as [3]

$$\mathcal{S} = \frac{1}{4}\int d^3x\sqrt{-g}\left[\mathcal{R} + \frac{1}{8}g^{\mu\nu}\mathrm{Tr}\,(\partial_\mu\mathcal{M}\mathcal{L}\partial_\nu\mathcal{M}\mathcal{L})\right], \tag{2.17}$$

and is invariant under the $O(8,24)$ transformation

$$\mathcal{M} \to \Omega\mathcal{M}\Omega^T, \qquad g_{\mu\nu} \to g_{\mu\nu}, \tag{2.18}$$

with the 32×32 matrix Ω satisfying $\Omega^T\mathcal{L}\Omega = \mathcal{L}$. The low energy effective three dimensional field theory becomes then invariant under $O(8,24)$ transformations.

As explained in [3], this $O(8,24)$ symmetry may be understood as a combination of the $O(7,23)$ symmetry (2.6) and the $\mathrm{SL}(2,\mathbb{R})$ symmetry of the four dimensional effective action. The three dimensional theory may be regarded as arising from compactification of the four dimensional theory on a circle, i.e. consider the four dimensional theory to be obtained by compactifying the directions 4-9. The three-dimensional theory is then obtained by compactifying the direction 3 on a circle. Then the $\mathrm{SL}(2,\mathbb{R})$ transformation of the four dimensional axion-dilaton complex scalar field $\lambda \to (a\lambda + b)/(c\lambda + d)$ [27] generated by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$, corresponds to the following transformation on the three dimensional fields $\mathcal{M} \to \Omega\mathcal{M}\Omega^T$ [3] :

$$\Omega = \begin{pmatrix} a & 0 & 0 & 0 & 0 & b & 0 \\ 0 & I_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & 0 & 0 & -c \\ 0 & 0 & 0 & I_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{16} & 0 & 0 \\ c & 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & -b & 0 & 0 & 0 & a \end{pmatrix}, \quad ad - bc = 1 \ , \tag{2.19}$$

with Ω being a $O(8,24)$ transformation. The full $O(8,24)$ group of transformations is then generated from the $O(7,23)$ transformations (2.6) and the $\mathrm{SL}(2,\mathbb{R})$ transformation written above. In fact, a $O(8,24;\mathbb{Z})$ subgroup of this group is a symmetry of the full string theory [3].

3 The Killing spinor equations

In ten dimensions, the supersymmetry transformation rules for the gaugini χ^I, dilatino λ and gravitino ψ_M are, in the string frame, given by [28, 29, 30, 31, 32]

$$
\begin{aligned}
\delta\chi^I &= \frac{1}{2}F^I_{MN}\Gamma^{MN}\varepsilon \ , \\
\delta\lambda &= -\frac{1}{2}\Gamma^M\partial_M\Phi\varepsilon + \frac{1}{12}H_{MNP}\Gamma^{MNP}\varepsilon \ , \\
\delta\psi_M &= \partial_M\varepsilon + \frac{1}{4}(\omega_{MAB} - \frac{1}{2}H_{MAB})\Gamma^{AB}\varepsilon \ .
\end{aligned} \tag{3.1}
$$

These equations become, when reduced to three dimensions in the Einstein frame,

$$
\begin{aligned}
\delta\chi^I &= \frac{1}{2}e^{-2\phi}(F^{(3)I}_{\mu\nu} + F^{(1)m}_{\mu\nu}a^I_m)\gamma^{\mu\nu}\varepsilon + e^{-\phi}\partial_\mu a^I_m\gamma^\mu\gamma^4\otimes\Sigma^m\varepsilon \ , \\
\delta\lambda &= -\frac{1}{2}e^{-\phi}\partial_\mu\{\phi + \ln\det e^a_m\}\gamma^\mu\otimes\mathbf{I}_8\,\varepsilon + \frac{1}{12}e^{-3\phi}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\varepsilon \\
&\quad + \frac{1}{4}e^{-2\phi}[-C_{mn}F^{(1)n}_{\mu\nu} + F^{(2)}_{\mu\nu m} - a^I_mF^{(3)I}_{\mu\nu}]\gamma^{\mu\nu}\gamma^4\otimes\Sigma^m\varepsilon \\
&\quad + \frac{1}{4}e^{-\phi}[\partial_\mu B_{mn} + \frac{1}{2}(a^I_m\partial_\mu a^I_n - a^I_n\partial_\mu a^I_m)]\gamma^\mu\otimes\Sigma^{mn}\varepsilon \ , \\
\delta\psi_\mu &= \partial_\mu\varepsilon + \frac{1}{4}\omega_{\mu\alpha\beta}\gamma^{\alpha\beta}\varepsilon + \frac{1}{4}(e_{\mu\alpha}e^\nu_\beta - e_{\mu\beta}e^\nu_\alpha)\partial_\nu\phi\gamma^{\alpha\beta}\varepsilon + \frac{1}{8}(e^n_a\partial_\mu e_{nb} - e^n_b\partial_\mu e_{na})\mathbf{I}_4\otimes\Sigma^{ab}\varepsilon \\
&\quad -\frac{1}{8}e^{-2\phi}H_{\mu\nu\delta}\gamma^{\nu\delta}\varepsilon - \frac{1}{4}e^{-\phi}[e^m_aF^{(2)}_{\mu\nu(m)} - e_{ma}F^{(1)m}_{\mu\nu}]\gamma^\nu\gamma^4\otimes\Sigma^a\varepsilon - \frac{1}{8}[\partial_\mu B_{mn} + \frac{1}{2}(a^I_m\partial_\mu a^I_n \\
&\quad - a^I_n\partial_\mu a^I_m)\,]\,\mathbf{I}_4\otimes\Sigma^{mn}\varepsilon - \frac{1}{4}e^{-\phi}[-C_{mn}F^{(1)n}_{\mu\nu} - a^I_mF^{(3)I}_{\mu\nu}\,]\,\gamma^\nu\gamma^4\otimes\Sigma^m\varepsilon \ , \\
\delta\psi_d &= -\frac{1}{4}e^{-\phi}(e^m_d\partial_\mu e_{ma} + e^m_a\partial_\mu e_{md})\gamma^\mu\gamma^4\otimes\Sigma^a\varepsilon - \frac{1}{8}e^{-2\phi}e^m_d[-C_{mn}F^{(1)n}_{\mu\nu} - a^I_mF^{(3)I}_{\mu\nu}]\gamma^{\mu\nu}\varepsilon \\
&\quad + \frac{1}{4}e^{-\phi}e^m_de^n_a(\partial_\mu B_{mn} + \frac{1}{2}(a^I_m\partial_\mu a^I_n - a^I_n\partial_\mu a^I_m))\gamma^\mu\gamma^4\otimes\Sigma^a\varepsilon \\
&\quad - \frac{1}{8}e^{-2\phi}[\,e_{md}F^{(1)m}_{\mu\nu} + e^m_dF^{(2)}_{\mu\nu m}\,]\gamma^{\mu\nu}\varepsilon \ ,
\end{aligned} \tag{3.2}
$$

where $\delta\psi_d \equiv e^m_d\delta\psi_m$ denotes the variation of the internal gravitini, and where we have suppressed the label i indicating the supersymmetries ($i = 1,\ldots,8$) as well as the index A for the space–time dimensionality of the spinors ($A = 1,\ldots,4$) (see appendix).

We would now like to construct static solutions to the Killing spinor equations by taking the supersymmetry variations of the fermionic fields to zero. This will insure that the bosonic configuration so obtained will be supersymmetric.

We will take the space–time metric to be diagonal. In all cases, with the exception of the one discussed in section 6, the space–time metric will be given by

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + R^2(r)d\theta^2 \quad , \tag{3.3}$$

for which

$$\begin{aligned}
&\omega_{t\alpha\beta}\,\gamma^{\alpha\beta} = -2\sqrt{V}\,\partial_r\sqrt{V}\,\gamma^{01}, \quad \omega_{r\alpha\beta}\gamma^{\alpha\beta} = 0, \quad \omega_{\theta\alpha\beta}\,\gamma^{\alpha\beta} = -2\sqrt{V}\,\partial_r R\,\gamma^{12} \ , \\
&(e_{\mu\alpha}\, e^r_\beta - e_{\mu\beta}\, e^r_\alpha)\partial_r\phi\gamma^{\alpha\beta} \quad = \quad -2V\,\partial_r\phi\,\gamma^{01}, \qquad \text{for} \quad \mu = t, \\
&\qquad\qquad = \quad 0, \qquad \text{for} \quad \mu = r, \\
&\qquad\qquad = \quad -2R\sqrt{V}\,\partial_r\phi\,\gamma^{12} \quad \text{for} \quad \mu = \theta \ .
\end{aligned} \tag{3.4}$$

Then, the Ricci tensor has the following non-zero components

$$\begin{aligned}
\mathcal{R}_{tt} &= \frac{V}{2}(V'' + V'\frac{R'}{R}) \ , \\
\mathcal{R}_{rr} &= -\frac{V''}{2V} - \frac{V'R'}{2VR} - \frac{R''}{R} \ , \\
\mathcal{R}_{\theta\theta} &= -VRR'' - V'R'R \ ,
\end{aligned} \tag{3.5}$$

and the curvature scalar is given by

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu} = -V'' - \frac{2V'R'}{R} - \frac{2VR''}{R} \ , \tag{3.6}$$

where $V' = \partial_r V, \ \ R' = \partial_r R.$

In all cases, we will make the following ansatz for the Killing spinors

$$\varepsilon = \epsilon \otimes \chi \quad , \tag{3.7}$$

where $\epsilon^T = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$ is a SO(1,2) spinor and χ is a SO(7) spinor of the internal space. In all cases, with the exception of the ones discussed in sections 6 and 7, we will be able to solve the Killing spinor equations by imposing the following two conditions on ϵ:

$$\gamma^1\epsilon = ip\,\gamma^2\,\epsilon \ , \tag{3.8}$$

$$\gamma^1\epsilon = \tilde{p}\,J\gamma^2\gamma^4\,\epsilon \ , \tag{3.9}$$

where $p = \pm$, $\tilde{p} = \pm$.

It follows that

$$\epsilon = \tilde{\epsilon}(r, \theta) \begin{pmatrix} ip \\ 1 \\ \tilde{p} \\ -ip\tilde{p} \end{pmatrix} \quad , \tag{3.10}$$

and, hence, ϵ contains only two real independent degrees of freedom. χ, on the other hand, contains eight real degrees of freedom; thus there are a priori a total of 16 real degrees of freedom. These will be further reduced by conditions on χ specific to each case considered. Up to three such independent conditions ($m = 1, 2, 3$) can be imposed on χ, thereby allowing for the construction of solutions preserving $1/2^m$ of the $N = 8, D = 3$ supersymmetry.

In all cases where $H_{\mu\nu\rho} = 0$, we find that

$$\tilde{\epsilon} = e^{\frac{\phi}{2}}\, e^{iY(r,\,\theta)} \quad , \tag{3.11}$$

up to a multiplicative constant.

4 Supersymmetric solutions with $\vec{\alpha}^2 \neq 0$

In this section, we will consider a particular class of solutions to the Killing spinor equations, namely solutions for which $\vec{\alpha}^2 = \alpha^T L \alpha \neq 0$. We will construct solutions which preserve $1/2^m$ of $N = 8, D = 3$ supersymmetry, where $m = 1, 2, 3$. The solutions are obtained with $H_{\mu\nu\rho} = 0$ and $a_m^I = 0$.

We will find that the space–time metric (3.3) is given in terms of

$$V = 1, \qquad\qquad R = a\, r^{1-\frac{\gamma}{2}} \quad , \tag{4.1}$$

and that the dilaton is given by

$$e^{2\phi} = r^{-\gamma} \quad , \tag{4.2}$$

where

$$\gamma = \frac{2}{n+1} \quad , \quad a = \frac{\sqrt{|\alpha_i\, \alpha_{i+7}|}}{\gamma\, \pi} \quad . \tag{4.3}$$

By the coordinate transformation $r = (\frac{\gamma}{2})^{\frac{2}{\gamma}}\, (a \ln \tilde{r})^{\frac{2}{\gamma}}$, $1 \leq \tilde{r} \leq \infty$, the associated space–time metric can be put into the form

$$ds^2 = -dt^2 + a^{\frac{4}{\gamma}}\, (\frac{\gamma}{2})^{\frac{2(2-\gamma)}{\gamma}}\, \frac{(\ln \tilde{r})^{\frac{2(2-\gamma)}{\gamma}}}{\tilde{r}^2}\, (d\tilde{r}^2 + \tilde{r}^2 d\theta^2). \tag{4.4}$$

This differs from the line element associated with conical geometries [12].

The curvature scalar, $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$, is computed to be

$$\mathcal{R} = \gamma(1-\frac{\gamma}{2})\,\frac{1}{r^2} = \frac{2n}{(n+1)^2}\,\frac{1}{r^2} \ . \tag{4.5}$$

4.1 Electrically charged solutions

We will first consider the case where the internal vielbein e_m^a is diagonal and given by $e_m^a = \delta_a^m e_m(r)$. We will also take $\phi = \phi(r)$ and $B_{mn} = 0$.

The Killing equations (3.2) reduce to

$$\begin{aligned}
\delta\chi^I &= \frac{1}{2}e^{-2\phi}F^{(3)I}_{\mu\nu}\gamma^{\mu\nu}\varepsilon \ , \\
\delta\lambda &= -\frac{1}{2}e^{-\phi}\partial_\mu\{\phi + \ln\det e_m^a\}\gamma^\mu\varepsilon + \frac{1}{4}e^{-2\phi}e_d^m F^{(2)}_{\mu\nu m}\gamma^{\mu\nu}\gamma^4\otimes\Sigma^d\varepsilon \ , \\
\delta\psi_\mu &= \partial_\mu\varepsilon + \frac{1}{4}\omega_{\mu\alpha\beta}\gamma^{\alpha\beta}\varepsilon + \frac{1}{4}(e_{\mu\alpha}e_\beta^r - e_{\mu\beta}e_\alpha^r)\partial_r\phi\gamma^{\alpha\beta}\varepsilon \\
&\quad -\frac{1}{4}e^{-\phi}[e_a^m F^{(2)}_{\mu\nu(m)} - e_{ma}F^{(1)m}_{\mu\nu}]\gamma^\nu\gamma^4\otimes\Sigma^a\varepsilon \ , \\
\delta\psi_d &= -\frac{1}{2}e^{-\phi}e_d^m\partial_\mu e_{md}\gamma^\mu\gamma^4\otimes\Sigma^d\varepsilon - \frac{1}{8}e^{-2\phi}[e_d^m F^{(2)}_{\mu\nu m} + e_{md}F^{(1)m}_{\mu\nu}]\gamma^{\mu\nu}\varepsilon \ .
\end{aligned} \tag{4.6}$$

Using (3.3), as well as (3.4), we get

$$\begin{aligned}
\delta\chi^I &= \frac{1}{2}e^{-2\phi}F^{(3)I}_{\mu\nu}\gamma^{\mu\nu}\varepsilon \ , \\
\delta\lambda &= -\frac{1}{2}e^{-\phi}\partial_r\{\phi + \ln\det e_m^a\}\gamma^r\,\varepsilon + \frac{1}{2}e^{-2\phi}e_a^m F^{(2)}_{trm}\gamma^{tr}\gamma^4\otimes\Sigma^a\varepsilon \ , \\
\delta\psi_t &= -\frac{1}{2}[\sqrt{V}\partial_r\sqrt{V} + V\partial_r\phi]\gamma^{tr}\varepsilon - \frac{1}{4}e^{-\phi}[e_a^m F^{(2)}_{trm} - e_{ma}F^{(1)m}_{tr}]\gamma^r\gamma^4\otimes\Sigma^a\varepsilon \ , \\
\delta\psi_r &= \partial_r\varepsilon - \frac{1}{4}e^{-\phi}[e_a^m F^{(2)}_{rtm} - e_{ma}F^{(1)m}_{rt}]\gamma^t\gamma^4\otimes\Sigma^a\varepsilon \ , \\
\delta\psi_\theta &= \partial_\theta\varepsilon - \frac{1}{2}[\sqrt{V}\partial_r R + R\sqrt{V}\partial_r\phi]\gamma^{12}\varepsilon \ , \\
\delta\psi_d &= -\frac{1}{4}e^{-2\phi}[e_d^m F^{(2)}_{trm} + e_{md}F^{(1)m}_{tr}]\gamma^{tr}\varepsilon - \frac{1}{2}e^{-\phi}\partial_r\ln e_m^d\gamma^r\gamma^4\otimes\Sigma^d\varepsilon, \qquad d = 1,\dots 7.
\end{aligned} \tag{4.7}$$

In the following, we will set $F^{(3)I}_{\mu\nu} = 0, \quad I = 1,\dots,16$.

We take the Killing spinor ε to be given by (3.7).

Let us now determine how many electric charges can be non-zero. Let us assume that $\Sigma^a\chi = \eta\,\chi$. Since $(\Sigma^a)^2 = 1, \quad \eta = \pm 1$. Suppose now that we also have $\Sigma^b\chi = \eta\,\chi$ with $a \neq b$. Then, $\Sigma^a\Sigma^b\,\chi = \chi$. Since however $(\Sigma^a\Sigma^b)^2 = -1$ (for $a \neq b$), we must have

$\Sigma^a\Sigma^b\,\chi = \pm iM\,\chi$, where $M^2 = \mathbf{I}$. Therefore the above assumption $a \neq b$ is not valid. So, out of the 14 remaining electric charges, only two are non-zero, one of them arising from the Kaluza-Klein sector and the other from the two-form gauge fields.

For concreteness, we choose $a = 2$, and hence, the two non-vanishing charges are α_2 and α_9 (see equation 2.13). Note that $\vec{\alpha}^2 \neq 0$.

We now set $\Sigma^2\chi = \chi$. Then

$$\begin{aligned}
\delta\lambda &= -\frac{1}{2}e^{-\phi}\partial_r\{\phi + \ln\det e^a_m\}\sqrt{V}\gamma^1\varepsilon - \frac{1}{2}e^{-2\phi}\sqrt{G^{22}}\,F^{(2)}_{tr2}\,J\,\gamma^2\gamma^4\varepsilon \ , \\
\delta\psi_t &= \frac{1}{2}[\sqrt{V}\partial_r\sqrt{V} + V\partial_r\phi]\,J\,\gamma^2\varepsilon - \frac{1}{4}e^{-\phi}\sqrt{V}[\sqrt{G^{22}}\,F^{(2)}_{tr2} - \sqrt{G_{22}}\,F^{(1)2}_{tr}]\,\gamma^1\gamma^4\varepsilon \ , \\
\delta\psi_r &= \partial_r\varepsilon + \frac{1}{4}e^{-\phi}\sqrt{V^{-1}}[\sqrt{G^{22}}\,F^{(2)}_{tr2} - \sqrt{G_{22}}\,F^{(1)2}_{tr}]\,\gamma^0\gamma^4\varepsilon \ , \\
\delta\psi_\theta &= \partial_\theta\varepsilon - \frac{1}{2}[\sqrt{V}\partial_r R + R\sqrt{V}\partial_r\phi]\,J\,\gamma^0\varepsilon \ , \qquad\qquad (4.8)\\
\delta\psi_2 &= \frac{1}{4}e^{-2\phi}[\sqrt{G^{22}}\,F^{(2)}_{tr2} + \sqrt{G_{22}}\,F^{(1)2}_{tr}]\,J\gamma^2\varepsilon - \frac{1}{2}e^{-\phi}\sqrt{V}\partial_r\ln\sqrt{G_{22}}\,\gamma^1\gamma^4\varepsilon\,, \\
\delta\psi_d &= -\frac{1}{2}e^{-\phi}\sqrt{V}\partial_r\ln\sqrt{G_{dd}}\,\gamma^1\gamma^4\varepsilon \ .
\end{aligned}$$

In order for the equations to be compatible, we will impose conditions (3.8) and (3.9) on the four-dimensional spinor ε.

Setting the variations of the supersymmetry equations to zero, we have

$$\begin{aligned}
\partial_r\{\phi + \ln\det e^a_m\}\sqrt{V} &= -\tilde{p}\,e^{-\phi}\,\sqrt{G^{22}}\,F^{(2)}_{tr2} \ , && (4.9)\\
\sqrt{V}\,[\frac{1}{2}\partial_r\ln V + \partial_r\phi] &= \frac{\tilde{p}}{2}\,e^{-\phi}\,[\sqrt{G_{22}}\,F^{(1)2}_{tr} - \sqrt{G^{22}}\,F^{(2)}_{tr2}] \ , && (4.10)\\
\sqrt{V}\,\partial_r\ln\sqrt{G_{22}} &= -\frac{\tilde{p}}{2}\,e^{-\phi}[\sqrt{G^{22}}\,F^{(2)}_{tr2} + \sqrt{G_{22}}\,F^{(1)2}_{tr}] \ , && (4.11)\\
\partial_r\,\tilde{\epsilon} &= \frac{\tilde{p}}{4}\frac{e^{-\phi}}{\sqrt{V}}\,[\sqrt{G_{22}}\,F^{(1)2}_{tr} - \sqrt{G^{22}}F^{(2)}_{tr2}]\,\tilde{\epsilon} \ , && (4.12)\\
\partial_\theta\,\tilde{\epsilon} &= -\frac{ip}{2}\,[\sqrt{V}\partial_r R + R\sqrt{V}\partial_r\phi]\,\tilde{\epsilon} \ , && (4.13)\\
G_{dd} &= \text{constant}, \qquad d \neq 2 \ . && (4.14)
\end{aligned}$$

From (4.10), (4.12) and (4.13), one has

$$\partial_r\,\tilde{\epsilon} - \frac{1}{2}\,[\frac{1}{2}\partial_r\ln V + \partial_r\phi]\,\tilde{\epsilon} = 0 \ , \qquad (4.15)$$

$$\partial_\theta\,\tilde{\epsilon} + ip\,\frac{1}{2}\,[\sqrt{V}\partial_r R + R\sqrt{V}\partial_r\phi]\,\tilde{\epsilon} = 0 \ . \qquad (4.16)$$

In order for these equations to be compatible with respect to the mixed derivative $\partial^2_{r\theta}$, we need to impose $\frac{\partial}{\partial r}[\sqrt{V}\partial_r R + R\sqrt{V}\partial_r \phi] = 0$. In the following, we will set $[\sqrt{V}\partial_r R + R\sqrt{V}\partial_r \phi] = 0$, and hence $\partial_\theta \tilde{\varepsilon} = 0$. Then it follows that

$$\partial_r \phi = -\frac{R'}{R} \qquad \longrightarrow \qquad R = a e^{-\phi}. \tag{4.17}$$

We note here, however, that if the spinor were to have a phase of the form $e^{i\eta\theta}$, η would have to be $(2n+1)/2$ (with n integer), such that $e^{i\eta(\theta=2\pi)} = -e^{i\eta(\theta=0)}$ [33, 10], so we would need to have $[\sqrt{V}\partial_r R + R\sqrt{V}\partial_r \phi] = -(2n+1)p$.

We take the two electric fields to be given by

$$\begin{aligned} F^{(1)m}_{tr} &= \frac{1}{2\pi}\frac{e^{2\phi}}{R} G^{22}\, \alpha_2 \ , \qquad\qquad m = 2 \ , \\ F^{(2)}_{trm} &= \frac{1}{2\pi}\frac{e^{2\phi}}{R} G_{22}\, \alpha_9 \ , \end{aligned} \tag{4.18}$$

which is consistent with the asymptotic behavior given in (2.13).

We now look for solutions with the internal metric G_{22} constant, noting here that a more general internal metric could be generated by $O(7,23)$ rotations from this one. Then equation (4.11) can be solved by

$$\sqrt{G^{22}}\, F^{(2)}_{tr2} = -\sqrt{G_{22}}\, F^{(1)2}_{tr} \qquad \longrightarrow \qquad G_{22} = -\frac{\alpha_2}{\alpha_9} \ . \tag{4.19}$$

It follows that α_2 and α_9 have opposite signs. We will in the following denote the signs of α_2, α_9 by η_{α_2}, η_{α_9}.

It further follows by inspection of (4.10), (4.11) and (4.9) that $\partial_r\sqrt{V} = 0$, thus V is a constant which we set equal to 1.

One can now solve straightforwardly for ϕ from (4.9) and (4.10). By doing so, we find that $\tilde{p} = -\eta_{\alpha_2}$ as well as

$$e^{2\phi} = \frac{c^2}{|r-r_0|}, \qquad R = \frac{a}{c}\sqrt{|r-r_0|}, \qquad \frac{a}{c^2} = \frac{1}{\pi}\sqrt{|\alpha_2\alpha_9|} \ , \tag{4.20}$$

where r_0 and c are integration constants which will be set to zero and one, respectively, from now on.

Note that the coupling constant $g^2 = e^{2\phi} \longrightarrow 0$ as $r \to \infty$.

The space–time metric is then of the form

$$ds^2 = -dt^2 + dr^2 + a^2\, r\, d\theta^2 \tag{4.21}$$

or, equivalently,

$$ds^2 = -dt^2 + \frac{a^4}{4}\left(\frac{\ln \tilde{r}}{\tilde{r}}\right)^2 (d\tilde{r}^2 + \tilde{r}^2 d\theta^2)\,, \tag{4.22}$$

where $r = \frac{a^2}{4}(\ln \tilde{r})^2$.

The behavior of the spinor $\tilde{\epsilon}$ can also be determined. We have from (4.15) and (4.16)

$$\begin{aligned} \partial_r \, \tilde{\epsilon} - \frac{1}{2}\partial_r \phi \, \tilde{\epsilon} &= 0 \ , \\ \partial_\theta \, \tilde{\epsilon} &= 0 \ , \end{aligned} \tag{4.23}$$

which can easily be solved by

$$\tilde{\epsilon} \, = \, e^{\frac{\phi}{2}} \, = \, r^{-1/4} \ , \tag{4.24}$$

up to a multiplicative constant. The existence of such a Killing spinor is made possible due to the cancellation of holonomies, that is due to a cancellation between the spin connection and a term involving the dilaton (see equation (4.16)).

This solution preserves 1/2 of the $N = 8, D = 3$ supersymmetry.

Computing the curvature $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$, we have $\mathcal{R} = \frac{1}{2r^2}$ which blows up at $r = 0$ but goes to zero at $r \to \infty$.

Let us now consider the equations of motion (2.10):

$$\begin{aligned} \mathcal{R}_{tt} &= \frac{1}{2}e^{-2\phi}F^{(a)}_{tr}F^{r(a)}_{t} - \frac{1}{4}e^{-2\phi}g_{tt}F^{(a)}_{\alpha\beta}F^{\alpha\beta(a)} = 0 \ , \\ \mathcal{R}_{rr} &= \partial_r\phi\partial_r\phi + \frac{1}{2}e^{-2\phi}F^{(a)}_{rt}(LML)_{aa}F^{t(a)}_{r} - \frac{1}{4}e^{-2\phi}g_{11}F^{(a)}_{\alpha\beta}(LML)_{aa}F^{\alpha\beta(a)} \\ &- \frac{1}{8}\mathrm{Tr}\ (\partial_r ML\partial_r ML) \ = \ (\partial_r\phi)^2 \ , \\ \mathcal{R}_{\theta\theta} &= -\frac{1}{2}e^{-2\phi}R^2F^{(a)}_{tr}(LML)_{aa}F^{tr(a)} \\ &= \frac{1}{8\pi^2}e^{2\phi}[\sqrt{G^{22}}\ (\alpha_2)^2 + \sqrt{G_{22}}\ (\alpha_9)^2] \ . \end{aligned} \tag{4.25}$$

Using now (3.5), it can be checked that our solution (4.20) solves the equations of motion.

4.2 Soliton solutions preserving $N = 4$ supersymmetry

4.2.1 Case $\alpha_2 \neq 0, \alpha_9 \neq 0$

Here, we will discuss the soliton solution which is obtained by dualizing the charged solution discussed in subsection 4.1. That is, we will utilize the $O(8, 24)$ transformation Ω given in (2.19) to generate the dual background $\mathcal{M} \to \tilde{\mathcal{M}} = \Omega\mathcal{M}\Omega^T$. We will, for simplicity, set the transformation parameter d to $d = 0$ in the following, so that $bc = -1$.

Recall that the bosonic background fields of the charged solution discussed in subsection 4.1 are given by ϕ, $(G_{mn}) = \text{diagonal}\,(G_{11}, G_{22}, \ldots, G_{77})$, $G_{22} = |\alpha_2/\alpha_9|, B_{mn} = 0, a^I_m =$

0 and $\Psi^T = (0, \Psi_2, 0, \ldots, 0, \Psi_9, 0, \ldots, 0) = (0, -\frac{\theta}{2\pi}\,\alpha_9\,, 0, \ldots, 0, -\frac{\theta}{2\pi}\,\alpha_2\,, 0, \ldots, 0)$. The dual background fields are then given by

$$\tilde{G}^{-1} = \begin{pmatrix} \tilde{G}^{11} & \tilde{G}^{12} & 0 & \cdots & 0 \\ \tilde{G}^{21} & \tilde{G}^{22} & 0 & \cdots & 0 \\ 0 & 0 & \tilde{G}^{33} & 0\cdots & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & 0 & \tilde{G}^{77} \end{pmatrix} = \begin{pmatrix} b^2 e^{2\phi} & -be^{2\phi}\Psi_2 & 0 & \cdots & 0 \\ -be^{2\phi}\Psi_2 & G^{22} + e^{2\phi}\Psi_2^2 & 0 & \cdots & 0 \\ 0 & 0 & G^{33} & 0\cdots & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & 0 & G^{77} \end{pmatrix},$$

$$\tilde{B} = (\tilde{B}_{mn}) = \begin{pmatrix} 0 & \tilde{B}_{12} & 0 & \cdots & 0 \\ \tilde{B}_{21} & 0 & & & \\ 0 & & 0 & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & & 0 \end{pmatrix} = \begin{pmatrix} 0 & -c\Psi_9 & 0 & \cdots & 0 \\ c\Psi_9 & 0 & & & \\ 0 & & 0 & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & & 0 \end{pmatrix} \tag{4.26}$$

as well as

$$e^{2\tilde{\phi}} = c^2 G^{11} \quad , \quad \tilde{\Psi} = \begin{pmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \\ \vdots \\ \tilde{\Psi}_9 \\ \vdots \end{pmatrix} = \begin{pmatrix} -a/c \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} \quad , \quad \tilde{a}^I_m = 0 \quad . \tag{4.27}$$

Note that the associated gauge field strengths $F^{(a)}_{\mu\nu}$ are all zero for this solitonic solution. The internal inverse vielbein $\tilde{e}_a^m$ associated to (4.26) is given by

$$\tilde{e}_a^m = \begin{pmatrix} be^{\phi} & -e^{\phi}\Psi_2 & 0 & \cdots & 0 \\ 0 & \sqrt{G^{22}} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{G^{33}} & 0\cdots & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & 0 & \sqrt{G^{77}} \end{pmatrix} . \tag{4.28}$$

Note that the space–time metric is duality invariant and hence given as before (see (4.21)).

Next, we would like to determine the Killing spinor associated with the soliton background (4.26). The Killing spinor equations (3.2) now take the form

$$\delta\chi^I = 0 \quad ,$$

$$\delta\lambda = -\frac{1}{2}e^{-\tilde{\phi}}\partial_\mu \log\det \tilde{e}_m^a \gamma^\mu\varepsilon + \frac{1}{4}e^{-\tilde{\phi}}\partial_\mu \tilde{B}_{mn}\gamma^\mu \otimes \Sigma^{mn}\varepsilon \quad , \tag{4.29}$$

$$\delta\psi_\mu = \partial_\mu\varepsilon + \frac{1}{4}\omega_{\mu\alpha\beta}\gamma^{\alpha\beta}\varepsilon + \frac{1}{8}(\tilde{e}_a^n\partial_\mu\tilde{e}_{nb} - \tilde{e}_b^n\partial_\mu\tilde{e}_{na})\Sigma^{ab}\varepsilon - \frac{1}{8}\partial_\mu\tilde{B}_{mn}\Sigma^{mn}\varepsilon \quad , \tag{4.30}$$

$$\delta\psi_d = -\frac{1}{4}e^{-\tilde{\phi}}(\tilde{e}_d^m\partial_\mu\tilde{e}_{ma}+\tilde{e}_a^m\partial_\mu\tilde{e}_{md})\gamma^\mu\gamma^4\otimes\Sigma^a\varepsilon + \frac{1}{4}e^{-\tilde{\phi}}\tilde{e}_d^m\tilde{e}_a^n\partial_\mu\tilde{B}_{mn}\gamma^\mu\gamma^4\otimes\Sigma^a\varepsilon. \tag{4.31}$$

The Killing spinor $\varepsilon = \epsilon\otimes\chi$ will be taken to satisfy (3.8) and (3.9). For the solitonic background under consideration, the vanishing of the Killing spinor equation (4.29) then yields

$$-\partial_r \log\det\tilde{e}_m^a\gamma^1\varepsilon + b\frac{\sqrt{G^{22}}}{R}e^\phi\partial_\theta\tilde{B}_{12}\gamma^2\Sigma^{12}\varepsilon = 0 \quad , \tag{4.32}$$

which can be solved by demanding that the Killing spinor $\varepsilon = \epsilon\otimes\chi$ should also satisfy

$$\Sigma^{12}\chi = q\,i\,\chi \quad , \quad q = \pm \quad . \tag{4.33}$$

Then, equation (4.32) turns into

$$-p\,\partial_r \log\det\tilde{e}_m^a + b\,q\,\frac{\sqrt{G^{22}}}{R}e^\phi\partial_\theta\tilde{B}_{12} = 0 \quad , \tag{4.34}$$

which is indeed satisfied, provided one takes $q = -p\,\eta_{\alpha_2}$, where η_{α_2} denotes the sign of the charge α_2, $\eta_{\alpha_2} = \operatorname{sign}\alpha_2$.

Next, consider solving the Killing spinor equations (4.30). We will again make the ansatz that the Killing spinor is static, that is $\varepsilon = \varepsilon(r,\theta)$. Then, the equation $\delta\psi_t = 0$ is automatically satisfied. The condition $\delta\psi_r = 0$, on the other hand, yields

$$\partial_r\varepsilon = 0 \quad . \tag{4.35}$$

Finally, the condition $\delta\psi_\theta = 0$ results in

$$\partial_\theta\varepsilon - \frac{1}{2}\partial_r R\gamma^{12}\varepsilon + \frac{1}{4}\sqrt{G_{22}}e^\phi\partial_\theta\Psi_2\Sigma^{12}\varepsilon - \frac{1}{4}\sqrt{G^{22}}e^\phi\partial_\theta\Psi_9\Sigma^{12}\varepsilon = 0 \quad . \tag{4.36}$$

Inserting the conditions (3.8) and (4.33) into (4.36) yields that

$$\partial_\theta\varepsilon = 0 \quad . \tag{4.37}$$

Thus, it follows that the Killing spinor ε is constant.

Finally, it can be checked that the Killing spinor equations (4.31) for $\delta\psi_1$ and $\delta\psi_2$ are also satisfied.

The solitonic background under consideration preserves 1/2 of $N = 8$ supersymmetry.

4.2.2 Case $\alpha_1 \neq 0, \alpha_8 \neq 0$

Next, we will discuss a different soliton solution, which will be obtained by dualizing a charged solution with bosonic background fields ϕ, $(G_{mn}) = \text{diagonal}\,(G_{11}, \ldots, G_{77})$, $G_{11} = |\alpha_1/\alpha_8|$, $B_{mn} = 0$, $a^I_m = 0$ and $\Psi^T = (\Psi_1, 0, \ldots, 0, \Psi_8, 0, \ldots, 0) = (-\frac{\theta}{2\pi}\,\alpha_8\,, 0, \ldots, 0, -\frac{\theta}{2\pi}\,\alpha_1\,, 0, \ldots, 0)$. This charged solution is similar to the one discussed in subsection 4.1.

The dual background fields can be read off from $\tilde{\mathcal{M}} = \Omega \mathcal{M} \Omega^T$, where Ω is again given by (2.19). We will, for simplicity, set the transformation parameter d to $d = 0$ in the following, so that $bc = -1$. For this choice, the dual background fields are given by $\tilde{B} = (\tilde{B}_{mn}) = 0$, $\tilde{a}^I_m = 0$,

$$\tilde{G} = \begin{pmatrix} \tilde{G}_{11} & 0 & 0 & \cdots & 0 \\ 0 & \tilde{G}_{22} & 0 & \cdots & 0 \\ 0 & 0 & \tilde{G}_{33} & 0 \cdots & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & 0 & \tilde{G}_{77} \end{pmatrix} = \begin{pmatrix} c^2\left(e^{-2\phi} + G_{11}\Psi_1^2\right) & 0 & 0 & \cdots & 0 \\ 0 & G_{22} & 0 & \cdots & 0 \\ 0 & 0 & G_{33} & 0 \cdots & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & 0 & G_{77} \end{pmatrix} \tag{4.38}$$

as well as

$$e^{2\tilde{\phi}} = c^2\left(G^{11} + e^{2\phi}\Psi_1^2\right) \quad , \quad \tilde{\Psi} = \begin{pmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \\ \vdots \\ \tilde{\Psi}_7 \\ \tilde{\Psi}_8 \\ \tilde{\Psi}_9 \\ \vdots \end{pmatrix} = \begin{pmatrix} -\left(\frac{a}{c} + \frac{e^{2\phi}\Psi_1}{c^2(G^{11} + e^{2\phi}\Psi_1^2)}\right) \\ 0 \\ \vdots \\ 0 \\ \Psi_8 \\ 0 \\ \vdots \end{pmatrix} . \tag{4.39}$$

Note that $\tilde{\phi}$ now depends on both r and θ.

Next, we would like to determine the Killing spinor associated with the soliton background (4.38). The Killing spinor equations (3.2) now take the form

$$\begin{aligned} \delta\chi^I &= 0 \quad , \\ \delta\lambda &= -\frac{1}{2}e^{-\tilde{\phi}}\partial_\mu\{\tilde{\phi} + \ln\det\tilde{e}^a_m\}\gamma^\mu\varepsilon + \frac{1}{4}e^{-2\tilde{\phi}}\tilde{e}^m_d\tilde{F}^{(2)}_{\mu\nu m}\gamma^{\mu\nu}\gamma^4 \otimes \Sigma^d\varepsilon \quad , \end{aligned} \tag{4.40}$$

$$\begin{aligned} \delta\psi_\mu &= \partial_\mu\varepsilon + \frac{1}{4}\omega_{\mu\alpha\beta}\gamma^{\alpha\beta}\varepsilon + \frac{1}{2}e_{\mu\alpha}e^\nu_\beta\partial_\nu\tilde{\phi}\gamma^{\alpha\beta}\varepsilon \\ &- \frac{1}{4}e^{-\tilde{\phi}}[\tilde{e}^m_a\tilde{F}^{(2)}_{\mu\nu(m)} - \tilde{e}_{ma}\tilde{F}^{(1)m}_{\mu\nu}]\gamma^\nu\gamma^4 \otimes \Sigma^a\varepsilon \quad , \end{aligned} \tag{4.41}$$

$$\delta\psi_d = -\frac{1}{2}e^{-\tilde{\phi}}\tilde{e}_d^m\partial_\mu\tilde{e}_{md}\gamma^\mu\gamma^4\otimes\Sigma^d\varepsilon - \frac{1}{8}e^{-2\tilde{\phi}}[\tilde{e}_d^m\tilde{F}^{(2)}_{\mu\nu m} + \tilde{e}_{md}\tilde{F}^{(1)m}_{\mu\nu}]\gamma^{\mu\nu}\varepsilon \quad . \tag{4.42}$$

Note that in (4.42) there is no summation over d.

We will again take the Killing spinor $\varepsilon = \epsilon\otimes\chi$ to satisfy (3.8) and (3.9). Hence, ϵ is given by (3.10).

Using that

$$e^{-2\tilde{\phi}}\tilde{F}^{(1)}_{tr1} = -\frac{\tilde{G}^{11}}{R}\partial_\theta\tilde{\Psi}_8 \quad , \quad e^{-2\tilde{\phi}}\tilde{F}^{(2)}_{tr1} = -\frac{\tilde{G}_{11}}{R}\partial_\theta\tilde{\Psi}_1 \quad , \quad e^{-2\tilde{\phi}}\tilde{F}^{(2)}_{t\theta 1} = -\tilde{G}_{11}R\partial_r\tilde{\Psi}_1 \; , \tag{4.43}$$

it can be checked that the Killing spinor equation $\delta\lambda = 0$ is satisfied provided that

$$\Sigma^1\chi = \chi \quad , \quad \tilde{p} = \eta_{\alpha_8} \quad , \tag{4.44}$$

where $\eta_{\alpha_8} = \alpha_8/|\alpha_8|$ denotes the sign of the charge α_8. Similarly, it can be checked that the Killing spinor equation $\delta\psi_1 = 0$ (eq. (4.42)) is satisfied.

Next, consider solving the Killing spinor equations (4.41). We will again make the ansatz that the Killing spinor is static, that is $\varepsilon = \varepsilon(r,\theta)$. Then, the equation $\delta\psi_t = 0$ is satisfied. The condition $\delta\psi_r = 0$, on the other hand, results in

$$\partial_r \log\tilde{\epsilon} = i\frac{p}{2a}\frac{1}{\sqrt{r}}\frac{e^{2\phi}\Psi_1\partial_\theta\Psi_1}{G^{11}+e^{2\phi}\Psi_1^2} + \frac{\eta_{\alpha_8}\sqrt{G_{11}}}{2a}\frac{1}{r^2}\frac{\Psi_1^2\partial_\theta\Psi_1}{G^{11}+e^{2\phi}\Psi_1^2} \quad , \tag{4.45}$$

whereas the condition $\delta\psi_\theta = 0$ results in

$$\partial_\theta\log\tilde{\epsilon} = -i\frac{p}{4}\frac{a\,G^{11}}{\sqrt{r}}\frac{1}{G^{11}+e^{2\phi}\Psi_1^2} - \frac{\eta_{\alpha_8}a\sqrt{G^{11}}}{4}\frac{e^{2\Phi}\Psi_1}{G^{11}+e^{2\phi}\Psi_1^2} \quad . \tag{4.46}$$

Clearly, the solution to both (4.45) and (4.46) will be of the form $\log\tilde{\epsilon} = X + iY$ with real X and Y, namely

$$\tilde{\epsilon} = e^{\tilde{\phi}/2}e^{iY} \quad , \quad Y = -\frac{\eta_{\alpha_8}p}{2}\,\arctan\left(\frac{\eta_{\alpha_8}a}{2}\frac{\theta}{\sqrt{r}}\right) , \tag{4.47}$$

up to a multiplicative constant. Comparison with (4.24) shows that, whereas the form of X was to be expected on the grounds of the replacement $\phi\to\tilde{\phi}$ under duality, the duality transformation $\mathcal{M}\to\tilde{\mathcal{M}} = \Omega\mathcal{M}\Omega^T$ actually also produces a complicated phase Y.

Note that when $r\to\infty$, the Killing spinor approaches a constant value given by

$$\tilde{\epsilon}\to(c^2\,G^{11})^{\frac{1}{4}}\,e^{\pm i(\frac{\eta}{2}\pi)}. \tag{4.48}$$

This solitonic solution preserves 1/2 of $N = 8$ supersymmetry.

4.3 Soliton solutions preserving $N = 2$ supersymmetry

In this subsection, we will consider soliton solutions preserving 1/4 of $N = 8, D = 3$ supersymmetry.

A particular class of such solutions can be obtained by combining certain features of the electrically charged solutions, discussed in subsection 4.1, and of the solitonic solution discussed in subsection 4.2.1. Namely, we will make the following ansatz for the background fields G^{-1} and B,

$$\begin{pmatrix} G^{11} & G^{12} & 0 & 0 & \cdots & 0 \\ G^{21} & G^{22} & 0 & 0 & \cdots & 0 \\ 0 & 0 & G^{33} & 0 & \cdots & \\ 0 & 0 & 0 & G^{44} & 0 & \vdots \\ \vdots & & & & \ddots & \\ 0 & & \cdots & & 0 & G^{77} \end{pmatrix} = \begin{pmatrix} f^2(r) & -f^2(r)\Upsilon_2 & 0 & 0 & \cdots & 0 \\ -f^2(r)\Upsilon_2 & |\frac{\alpha_9}{\alpha_2}| + f^2(r)\Upsilon_2^2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & G^{33} & 0 & \cdots & \\ 0 & 0 & 0 & G^{44} & 0 & \vdots \\ \vdots & & & & \ddots & \\ 0 & & \cdots & & 0 & G^{77} \end{pmatrix},$$

$$B = (B_{mn}) = \begin{pmatrix} 0 & B_{12} & 0 & \cdots & 0 \\ B_{21} & 0 & & & \\ 0 & & 0 & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Upsilon_9 & 0 & \cdots & 0 \\ -\Upsilon_9 & 0 & & & \\ 0 & & 0 & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & & 0 \end{pmatrix}, \tag{4.49}$$

where

$$f(r) = D\, r^{-\frac{\gamma}{2}} \quad , \quad \Upsilon_2 = -\frac{\theta}{2\pi}\alpha_9 \quad , \quad \Upsilon_9 = -\frac{\theta}{2\pi}\alpha_2 \quad , \quad G^{44} = |\frac{\alpha_{11}}{\alpha_4}|. \tag{4.50}$$

We will also take

$$e^{2\phi} \;=\; r^{-\beta} \quad , \quad \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \vdots \\ \Psi_{10} \\ \Psi_{11} \\ \Psi_{12} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\theta}{2\pi}\alpha_{11} \\ 0 \\ \vdots \\ 0 \\ -\frac{\theta}{2\pi}\alpha_4 \\ 0 \\ \vdots \end{pmatrix} \quad , \quad a_m^I = 0 \quad . \tag{4.51}$$

For the space–time metric we will make the ansatz

$$ds^2 = -dt^2 + dr^2 + R^2(r)\,d\theta^2 \quad , \quad R(r) = ar^\rho \quad . \tag{4.52}$$

The constants D, β, γ and ρ will be fixed below.

The internal inverse vielbein e_a^m associated to (4.49) is given by

$$e_a^m = \begin{pmatrix} f(r) & -f(r)\Upsilon_2 & 0 & \cdots & & 0 \\ 0 & \sqrt{|\frac{\alpha_9}{\alpha_2}|} & 0 & & \cdots & 0 \\ 0 & 0 & \sqrt{G^{33}} & 0 & \cdots & \\ 0 & 0 & 0 & \sqrt{|\frac{\alpha_{11}}{\alpha_4}|} & 0 & \vdots \\ \vdots & & & & \ddots & \\ 0 & & \cdots & & 0 & \sqrt{G^{77}} \end{pmatrix} \quad . \tag{4.53}$$

Next, we would like to determine the Killing spinor associated with the soliton background (4.49). The Killing spinor equations (3.2) now take the form

$$\begin{aligned} \delta\chi^I &= 0 \quad , \\ \delta\lambda &= -\frac{1}{2}e^{-\phi}\partial_\mu\{\phi + \ln\det e_m^a\}\gamma^\mu\,\varepsilon + \frac{1}{4}e^{-2\phi}F^{(2)}_{\mu\nu m}\gamma^{\mu\nu}\gamma^4 \otimes \Sigma^m\varepsilon \\ &\quad + \frac{1}{4}e^{-\phi}\partial_\mu B_{mn}\gamma^\mu \otimes \Sigma^{mn}\varepsilon \quad , \qquad (4.54) \\ \delta\psi_\mu &= \partial_\mu\varepsilon + \frac{1}{4}\omega_{\mu\alpha\beta}\gamma^{\alpha\beta}\varepsilon + \frac{1}{4}(e_{\mu\alpha}e^\nu_\beta - e_{\mu\beta}e^\nu_\alpha)\partial_\nu\phi\gamma^{\alpha\beta}\varepsilon + \frac{1}{8}(e^n_a\partial_\mu e_{nb} - e^n_b\partial_\mu e_{na})\Sigma^{ab}\varepsilon \\ &\quad -\frac{1}{4}e^{-\phi}[e^m_a F^{(2)}_{\mu\nu(m)} - e_{ma}F^{(1)m}_{\mu\nu}]\gamma^\nu\gamma^4 \otimes \Sigma^a\varepsilon - \frac{1}{8}\partial_\mu B_{mn}\Sigma^{mn}\varepsilon \quad , \qquad (4.55) \\ \delta\psi_d &= -\frac{1}{4}e^{-\phi}(e^m_d\partial_\mu e_{ma} + e^m_a\partial_\mu e_{md})\gamma^\mu\gamma^4 \otimes \Sigma^a\varepsilon + \frac{1}{4}e^{-\phi}e^m_d e^n_a\partial_\mu B_{mn}\gamma^\mu\gamma^4 \otimes \Sigma^a\varepsilon \\ &\quad -\frac{1}{8}e^{-2\phi}[e_{md}F^{(1)m}_{\mu\nu} + e^m_d F^{(2)}_{\mu\nu m}]\gamma^{\mu\nu}\varepsilon \quad . \qquad (4.56) \end{aligned}$$

As before, the Killing spinor $\varepsilon = \epsilon \otimes \chi$ will be taken to satisfy (3.8) and (3.9) and, hence, also (3.10). The Killing spinor equation $\delta\lambda = 0$ can be solved by demanding that

$$\Sigma^{12}\chi = q\,i\,\chi \quad , \quad \Sigma^4\chi = \chi \quad , \quad q = \pm \quad . \tag{4.57}$$

Note that the condition (4.57) reduces the degrees of freedom of ε to 4 real degrees of freedom, and thus the solitonic background under consideration preserves 1/4 of $N = 8$ supersymmetry. Then, the Killing spinor equation $\delta\lambda = 0$ is solved provided that

$$\beta = \gamma \quad , \quad \rho = 1 - \frac{\gamma}{2} \quad , \quad D = \gamma\frac{\pi a}{\sqrt{|\alpha_2\alpha_9|}} \quad , \quad \pi a = \frac{1}{\beta}\sqrt{|\alpha_4\alpha_{11}|} \tag{4.58}$$

as well as $q = -p\eta_{\alpha_2}$ and $\tilde{p} = \eta_{\alpha_{11}}$, where η_{α_2} and $\eta_{\alpha_{11}}$ denote the signs of the charges α_2 and α_{11}, respectively ($\eta_{\alpha_2} = -\eta_{\alpha_9}, \eta_{\alpha_4} = -\eta_{\alpha_{11}}$).

Next, consider solving the Killing spinor equations (4.55). We will again make the ansatz that the Killing spinor is static. Then, the equation $\delta\psi_t = 0$ is automatically satisfied. The condition $\delta\psi_r = 0$, on the other hand, yields

$$\partial_r \log \tilde{\epsilon} = \frac{1}{2}\partial_r \phi \quad . \tag{4.59}$$

Finally, the condition $\delta\psi_\theta = 0$ can be solved by setting

$$\beta = 2(1-\gamma) \quad . \tag{4.60}$$

Then $\delta\psi_\theta = \partial_\theta \varepsilon = 0$ and, hence,

$$\tilde{\epsilon} = e^{\frac{\phi}{2}} \quad , \tag{4.61}$$

up to a multiplicative constant. Comparison of (4.58) and (4.60), on the other hand, yields that $\beta = \gamma = \rho = 2/3$. Thus, it follows that

$$\begin{aligned} f^2(r) &= D^2 r^{-\frac{2}{3}} \quad , \quad R(r) = a r^{\frac{2}{3}} \quad , \quad e^{2\phi} = r^{-\frac{2}{3}} \quad , \\ D &= \frac{\sqrt{|\alpha_4 \alpha_{11}|}}{\sqrt{|\alpha_2 \alpha_9|}} \quad , \quad a = \frac{3}{2\pi}\sqrt{|\alpha_4\alpha_{11}|} \quad . \end{aligned} \tag{4.62}$$

Then, finally, it can be checked that the Killing spinor equations $\delta\psi_1 = 0$, $\delta\psi_2 = 0$ and $\delta\psi_4 = 0$ (eqs. (4.56)) are also satisfied.

Other solitonic solutions preserving $1/4$ of $N = 8$ supersymmetry can be obtained by applying the $O(8,24)$ duality transformation

$$\Omega = \begin{pmatrix} I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & I_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & -c \\ 0 & 0 & 0 & 0 & 0 & I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{16} & 0 & 0 \\ 0 & c & 0 & 0 & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & -b & 0 & 0 & 0 & a \end{pmatrix} \quad , \quad ad - bc = 1 \tag{4.63}$$

to (4.49). We will, for concreteness, set $a = d = 0, b = -c = 1$ in the following. The resulting dual background fields $\tilde{G}^{-1}$ and $\tilde{B}$ are then given by

$$\begin{pmatrix} \tilde{G}^{11} & \tilde{G}^{12} & 0 & 0 & \cdots & 0 \\ \tilde{G}^{21} & \tilde{G}^{22} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \tilde{G}^{33} & \tilde{G}^{34} & \cdots & \\ 0 & 0 & \tilde{G}^{43} & \tilde{G}^{44} & 0 & \vdots \\ \vdots & & & & \ddots & \\ 0 & & \cdots & & 0 & \tilde{G}^{77} \end{pmatrix} = \begin{pmatrix} G^{11} & G^{12} & 0 & 0 & \cdots & \\ G^{21} & G^{22} & 0 & 0 & \cdots & 0 \\ 0 & 0 & e^{2\phi} & -e^{2\phi}\Psi_4 & \cdots & \\ 0 & 0 & -e^{2\phi}\Psi_4 & G^{44}+e^{2\phi}\Psi_4^2 & 0 & \vdots \\ \vdots & & & & \ddots & \\ 0 & & \cdots & & 0 & G^{77} \end{pmatrix} ,$$

$$\tilde{B} = \begin{pmatrix} 0 & \tilde{B}_{12} & 0 & & \cdots & 0 \\ \tilde{B}_{21} & 0 & & & & \\ 0 & & 0 & \tilde{B}_{34} & 0 & \\ 0 & & \tilde{B}_{43} & 0 & & \vdots \\ \vdots & & 0 & & \ddots & \\ 0 & & \cdots & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Upsilon_9 & 0 & & \cdots & 0 \\ -\Upsilon_9 & 0 & & & & \\ 0 & & 0 & \Psi_{11} & 0 & \\ 0 & & -\Psi_{11} & 0 & & \vdots \\ \vdots & & 0 & & \ddots & \\ 0 & & \cdots & & & 0 \end{pmatrix} ,$$

as well as

$$e^{2\tilde{\phi}} = G^{33} \quad , \quad \tilde{\Psi} = 0 \quad , \quad \tilde{a}^I_m = 0 \quad . \tag{4.64}$$

Note that the dual dilaton field $\tilde{\phi}$ is constant.

It can be checked that the associated Killing spinor equations are satisfied by a constant Killing spinor $\varepsilon = \epsilon \otimes \chi$ provided that

$$\Sigma^{12}\chi = -i\, p\, \eta_{\alpha_2}\chi \quad , \quad \Sigma^{34}\chi = -i\, p\, \eta_{\alpha_4}\chi \quad , \tag{4.65}$$

where, again, η_{α_2} and η_{α_4} denote the sign of the charges α_2 and α_4, respectively.

4.4 Soliton solutions preserving $N = 1$ supersymmetry

It is now straightforward to construct solutions which preserve 1/8 of $D = 3, N = 8$ supersymmetry.

One class of solitonic solutions preserving 1/8 of $D = 3, N = 8$ supersymmetry is given as follows. The background fields are given by

$$e_a^{\ m} = \begin{pmatrix} f_2(r) & -f_2(r)\Upsilon_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{|\frac{\alpha_9}{\alpha_2}|} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_4(r) & -f_4(r)\Upsilon_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{|\frac{\alpha_{11}}{\alpha_4}|} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{G^{55}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{G^{66}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{G^{77}} \end{pmatrix} , \tag{4.66}$$

$$B = \begin{pmatrix} 0 & B_{12} & 0 & 0 & 0 & 0 & 0 \\ B_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{34} & 0 & 0 & 0 \\ 0 & 0 & B_{43} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Upsilon_9 & 0 & 0 & 0 & 0 & 0 \\ -\Upsilon_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Upsilon_{11} & 0 & 0 & 0 \\ 0 & 0 & -\Upsilon_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

as well as

$$e^{2\phi} = r^{-\gamma} \quad , \quad \Psi = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_6 \\ \Psi_7 \\ \Psi_8 \\ \vdots \\ \Psi_{13} \\ \Psi_{14} \\ \Psi_{15} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\frac{\theta}{2\pi}\alpha_{14} \\ 0 \\ \vdots \\ 0 \\ -\frac{\theta}{2\pi}\alpha_7 \\ 0 \\ \vdots \end{pmatrix} \quad , \quad a^I_m = 0 \quad . \tag{4.67}$$

Here

$$\Upsilon_2 = -\frac{\theta}{2\pi}\alpha_9 \quad , \quad \Upsilon_4 = -\frac{\theta}{2\pi}\alpha_{11} \quad , \quad \Upsilon_9 = -\frac{\theta}{2\pi}\alpha_2 \quad , \quad \Upsilon_{11} = -\frac{\theta}{2\pi}\alpha_4 \quad ,$$

$$G_{77} = |\frac{\alpha_7}{\alpha_{14}}| \tag{4.68}$$

as well as

$$f_2 = D_2\, r^{-\frac{\gamma}{2}} \quad , \quad D_2 = \frac{\sqrt{|\alpha_7\alpha_{14}|}}{\sqrt{|\alpha_2\alpha_9|}} \quad ,$$

$$f_4 = D_4\, r^{-\frac{\gamma}{2}} \quad , \quad D_4 = \frac{\sqrt{|\alpha_7\alpha_{14}|}}{\sqrt{|\alpha_4\alpha_{11}|}} \quad . \tag{4.69}$$

The space–time metric is given by

$$ds^2 = -dt^2 + dr^2 + R^2 d\theta^2 \quad , \quad R = ar^{1-\frac{\gamma}{2}} \quad , \tag{4.70}$$

where

$$a = \frac{1}{\gamma\,\pi}\sqrt{|\alpha_7\alpha_{14}|} \quad , \quad \gamma = \frac{1}{2} \quad . \tag{4.71}$$

The associated Killing spinor $\varepsilon = \epsilon \otimes \chi$ satisfies (3.8) and (3.9) with

$$\tilde{\epsilon} = e^{\frac{\phi}{2}} \quad , \tag{4.72}$$

as well as

$$\Sigma^{12}\chi = iq\chi \quad , \quad \Sigma^{34}\chi = i\tilde{q}\chi \quad , \quad \Sigma^{7}\chi = \chi \quad , \quad q = \pm \quad , \quad \tilde{q} = \pm \,, \tag{4.73}$$

where $\tilde{p} = \eta_{\alpha_{14}}, q = -p\eta_{\alpha_2}$ and $\tilde{q} = -p\eta_{\alpha_4}$ $(\eta_{\alpha_2} = -\eta_{\alpha_9}, \eta_{\alpha_4} = -\eta_{\alpha_{11}}, \eta_{\alpha_7} = -\eta_{\alpha_{14}})$.

Another class of solitonic solutions preserving 1/8 of $D = 3, N = 8$ supersymmetry is given as follows. The background fields are given by

$$e_a^{\ m} = \begin{pmatrix} f_2(r) & -f_2(r)\Upsilon_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{|\frac{\alpha_9}{\alpha_2}|} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f_4(r) & -f_4(r)\Upsilon_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{|\frac{\alpha_{11}}{\alpha_4}|} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_6(r) & -f_6(r)\Upsilon_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{|\frac{\alpha_{13}}{\alpha_6}|} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{G^{77}} \end{pmatrix} ,$$

$$B = \begin{pmatrix} 0 & B_{12} & 0 & 0 & 0 & 0 & 0 \\ B_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{34} & 0 & 0 & 0 \\ 0 & 0 & B_{43} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{56} & 0 \\ 0 & 0 & 0 & 0 & B_{65} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Upsilon_9 & 0 & 0 & 0 & 0 & 0 \\ -\Upsilon_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Upsilon_{11} & 0 & 0 & 0 \\ 0 & 0 & -\Upsilon_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Upsilon_{13} & 0 \\ 0 & 0 & 0 & 0 & -\Upsilon_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.74}$$

as well as

$$e^{2\phi} = r^{-\gamma} \quad , \quad \Psi = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_6 \\ \Psi_7 \\ \Psi_8 \\ \vdots \\ \Psi_{13} \\ \Psi_{14} \\ \Psi_{15} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\frac{\theta}{2\pi}\alpha_{14} \\ 0 \\ \vdots \\ 0 \\ -\frac{\theta}{2\pi}\alpha_7 \\ 0 \\ \vdots \end{pmatrix} \quad , \quad a_m^I = 0 \quad . \tag{4.75}$$

Here

$$\begin{aligned} \Upsilon_2 &= -\frac{\theta}{2\pi}\alpha_9 \quad , \quad \Upsilon_4 = -\frac{\theta}{2\pi}\alpha_{11} \quad , \quad \Upsilon_6 = -\frac{\theta}{2\pi}\alpha_{13} \quad , \\ \Upsilon_9 &= -\frac{\theta}{2\pi}\alpha_2 \quad , \quad \Upsilon_{11} = -\frac{\theta}{2\pi}\alpha_4 \quad , \quad \Upsilon_{13} = -\frac{\theta}{2\pi}\alpha_6 \quad , \\ G_{77} &= |\frac{\alpha_7}{\alpha_{14}}| \end{aligned} \tag{4.76}$$

as well as

$$
\begin{aligned}
f_2 &= D_2\, r^{-\frac{\gamma}{2}} \quad , \quad D_2 = \frac{\sqrt{|\alpha_7\alpha_{14}|}}{\sqrt{|\alpha_2\alpha_9|}} \quad , \\
f_4 &= D_4\, r^{-\frac{\gamma}{2}} \quad , \quad D_4 = \frac{\sqrt{|\alpha_7\alpha_{14}|}}{\sqrt{|\alpha_4\alpha_{11}|}} \quad , \\
f_6 &= D_6\, r^{-\frac{\gamma}{2}} \quad , \quad D_6 = \frac{\sqrt{|\alpha_7\alpha_{14}|}}{\sqrt{|\alpha_6\alpha_{13}|}} \quad .
\end{aligned}
\tag{4.77}
$$

The space–time metric is given by

$$
ds^2 = -dt^2 + dr^2 + R^2 d\theta^2 \quad , \quad R = a r^{1-\frac{\gamma}{2}} \quad , \tag{4.78}
$$

where

$$
a = \frac{1}{\gamma\,\pi}\sqrt{|\alpha_7\alpha_{14}|} \quad , \quad \gamma = \frac{2}{5} \quad . \tag{4.79}
$$

The associated Killing spinor $\varepsilon = \epsilon \otimes \chi$ satisfies (3.8) and (3.9) with

$$
\tilde{\epsilon} = e^{\frac{\phi}{2}} \quad , \tag{4.80}
$$

as well as

$$
\Sigma^{12}\chi = iq\chi \ , \ \Sigma^{34}\chi = i\tilde{q}\chi \ , \ \Sigma^{56}\chi = i\hat{q}\chi \ , \ \Sigma^{7}\chi = \chi \ , \ q = \pm \ , \ \tilde{q} = \pm \ , \ \hat{q} = \pm, \tag{4.81}
$$

where $\tilde{p} = \eta_{\alpha_{14}}, q = -p\eta_{\alpha_2}, \tilde{q} = -p\eta_{\alpha_4}$ and $\hat{q} = -p\eta_{\alpha_6}$ ($\eta_{\alpha_2} = -\eta_{\alpha_9}, \eta_{\alpha_4} = -\eta_{\alpha_{11}}, \eta_{\alpha_6} = -\eta_{\alpha_{13}}, \eta_{\alpha_7} = -\eta_{\alpha_{14}}$).

5 Supersymmetric solutions with $\vec{\alpha}^2 = 0$

In this section, we will consider a particular class of solutions to the Killing spinor equations, namely solutions for which $\vec{\alpha}^2 = \alpha^T L \alpha = 0$. We will construct solutions which preserve $1/2^m$ of $N = 8, D = 3$ supersymmetry, where $m = 1, 2, 3$. The solutions are obtained with $H_{\mu\nu\rho} = 0$ and $a^I_m = 0$.

We will find that the space–time metric (3.3) is given in terms of

$$
V = 1, \qquad\qquad R = a\, r^{1-\gamma} \quad , \tag{5.1}
$$

and that the dilaton is given by

$$
e^{2\phi} = r^{-\gamma} \quad , \tag{5.2}
$$

where

$$\gamma = \frac{2}{n+2} \quad , \quad a = \frac{|\alpha_i|}{2\pi\,\gamma} \quad . \tag{5.3}$$

By the coordinate transformation $r = (\gamma)^{\frac{1}{\gamma}} \, (a \ln \tilde{r})^{\frac{1}{\gamma}}$, $1 \leq \tilde{r} \leq \infty$, the associated space–time metric can be put into the form

$$ds^2 = -dt^2 + a^{\frac{2}{\gamma}} \, (\gamma)^{\frac{2(1-\gamma)}{\gamma}} \, \frac{(\ln \tilde{r})^{\frac{2(1-\gamma)}{\gamma}}}{\tilde{r}^2} \, (d\tilde{r}^2 + \tilde{r}^2 d\theta^2). \tag{5.4}$$

The curvature scalar, $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$, is computed to be

$$\mathcal{R} \; = \; 2\gamma(1-\gamma) \, \frac{1}{r^2} \; = \; \frac{4n}{(n+2)^2} \, \frac{1}{r^2}. \tag{5.5}$$

5.1 Electrically charged solutions

We will be solving the same Killing spinor equations subject to the same assumptions as in subsection 4.1, where in addition we take $\alpha_9 = 0$.

Looking back at (4.9), with $\alpha_9 = 0$, i.e. $F^{(2)}_{\mu\nu 2} = 0$, we have $\partial_r \phi = -\partial_r \ln \sqrt{G_{22}}$. From (4.16) we still have $\partial_r \phi = -\partial_r \ln R$.

These relations can be satisfied with the following ansatz

$$\sqrt{G_{22}} = de^{-\phi}, \qquad\qquad R = a\, e^{-\phi}, \tag{5.6}$$

where d is an integration constant that is set to one in the following.

We can now solve equations (4.10), (4.11) and find, (with again $V = 1$),

$$\begin{aligned} \sqrt{G_{22}} &= r^{\frac{1}{3}}, \\ R &= \frac{a}{c}\, r^{\frac{1}{3}}, \qquad\qquad \frac{a}{c^3} = -\frac{3\,\tilde{p}\,\alpha_2}{4\pi} \quad , \\ e^{2\phi} &= c^2 r^{-\frac{2}{3}}, \end{aligned} \tag{5.7}$$

where $\tilde{p}\alpha_2 = -|\alpha_2|$ and where the integration constant c will be set to one.

The space–time metric is now of the form

$$ds^2 = -dt^2 + dr^2 + a^2\, r^{\frac{2}{3}}\, d\theta^2. \tag{5.8}$$

or, equivalently,

$$ds^2 = -dt^2 + \frac{2a^3}{3} \left(\frac{\ln \tilde{r}}{\tilde{r}^2} \right) (d\tilde{r}^2 + \tilde{r}^2 d\theta^2) \, , \tag{5.9}$$

where $r = (\frac{2a}{3})^{3/2} (\ln \tilde{r})^{3/2}$.

The dependence of the spinor in terms of ϕ is the same as before

$$\partial_r \tilde{\epsilon} - \frac{1}{2}\partial_r \phi \, \tilde{\epsilon} = 0 \ , \qquad \partial_\theta \tilde{\epsilon} = 0 \ , \tag{5.10}$$

or

$$\tilde{\epsilon} = e^{\frac{\phi}{2}} = r^{-\frac{1}{6}} \ . \tag{5.11}$$

These electric solutions preserve again 1/2 of the $N = 8$ supersymmetry.

5.2 Soliton solutions preserving $N = 4$ supersymmetry

Now we discuss the soliton solution which is obtained by dualizing the charged solution discussed in subsection 5.1, with one electric charge only ($\alpha_9 = 0$). The bosonic background fields of the charged solution are given by $\phi, (G_{mn}) = \text{diagonal}\,(G_{11}, G_{22}, \ldots, G_{77})\,, G_{22} = r^{\frac{2}{3}}, B_{mn} = 0, a^I_m = 0$ and $\Psi^T = (0, 0, 0, \ldots, 0, \Psi_9, 0, \ldots, 0) = (0,\, 0\,, 0, \ldots, 0, -\frac{\theta}{2\pi}\,\alpha_2\,, 0, \ldots, 0)$. We will utilize the $O(8, 24)$ transformation Ω given in (2.19) to generate the dual background $\mathcal{M} \to \Omega\mathcal{M}\Omega^T$. We will for simplicity set the transformation parameter d to $d = 0$ in the following so that $bc = -1$. The dual background fields are then given by

$$\tilde{G}^{-1} = \begin{pmatrix} \tilde{G}^{11} & 0 & 0 & \cdots & 0 \\ 0 & \tilde{G}^{22} & 0 & \cdots & 0 \\ 0 & 0 & \tilde{G}^{33} & 0\cdots & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & 0 & \tilde{G}^{77} \end{pmatrix} = \begin{pmatrix} b^2 e^{2\phi} & 0 & 0 & \cdots & 0 \\ 0 & G^{22} & 0 & \cdots & 0 \\ 0 & 0 & G^{33} & 0\cdots & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & 0 & G^{77} \end{pmatrix} ,$$

$$\tilde{B} = (\tilde{B}_{mn}) = \begin{pmatrix} 0 & \tilde{B}_{12} & 0 & \cdots & 0 \\ \tilde{B}_{21} & 0 & & & \\ 0 & & 0 & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & & 0 \end{pmatrix} = \begin{pmatrix} 0 & -c\Psi_9 & 0 & \cdots & 0 \\ c\Psi_9 & 0 & & & \\ 0 & & 0 & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & & 0 \end{pmatrix} \tag{5.12}$$

as well as

$$e^{2\tilde{\phi}} = c^2 G^{11} \quad , \quad \tilde{\Psi} = \begin{pmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \\ \vdots \\ \tilde{\Psi}_9 \\ \vdots \end{pmatrix} = \begin{pmatrix} -a/c \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} \quad , \quad \tilde{a}^I_m = 0 \ . \tag{5.13}$$

The associated gauge field strengths $F^{(a)}_{\mu\nu}$ are again all zero for this solitonic solution. The internal inverse vielbein $\tilde{e}_a^{\ m}$ associated to (5.12) is given by

$$\tilde{e}_a^{\ m} = \begin{pmatrix} be^{\phi} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{G^{22}} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{G^{33}} & 0\cdots & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & 0 & \sqrt{G^{77}} \end{pmatrix} . \tag{5.14}$$

Note that the space–time metric is duality invariant and hence given as in (5.8).

The Killing spinor equations in the new background (5.12) are of the same form as in equations (4.29), (4.30) and (4.31). It is easy to check that the Killing spinor will be of the same form as before with the same conditions (3.8), (3.9) and (4.33) to be satisfied.

For the solitonic background under consideration, the Killing spinor equation (4.29) then yields

$$-p\,\partial_r \log\det \tilde{e}_m^{\ a} + b\,q\,\frac{\sqrt{G^{22}}}{R} e^{\phi}\partial_\theta \tilde{B}_{12} = 0 \quad , \tag{5.15}$$

which is satisfied, provided one takes $q = p\tilde{p}$.

Next, consider solving the Killing spinor equations (4.30). The Killing spinor being static, the equation $\delta\psi_t = 0$ is again automatically satisfied. The condition $\delta\psi_r = 0$ yields again

$$\partial_r \varepsilon = 0 \quad . \tag{5.16}$$

Finally, the condition $\delta\psi_\theta = 0$ results in

$$\partial_\theta \varepsilon - \frac{1}{2}\partial_r R\gamma^{12}\varepsilon - \frac{1}{4}\sqrt{G^{22}} e^{\phi}\partial_\theta \Psi_9 \Sigma^{12}\varepsilon = 0 \quad , \tag{5.17}$$

from which it follows again, if $p\tilde{p} = q$, that

$$\partial_\theta \varepsilon = 0 \quad . \tag{5.18}$$

Hence the Killing spinor ε is constant.

Finally, it can be checked that the Killing spinor equations (4.31) for $\delta\psi_1$ and $\delta\psi_2$ are automatically satisfied.

The solitonic background preserves 1/2 of $N = 8$ supersymmetry.

5.3 Soliton solutions preserving $N = 2$ supersymmetry

We take the following ansatz for the background fields G^{-1} and B

$$\begin{pmatrix} G^{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & G^{22} & 0 & 0 & \cdots & 0 \\ 0 & 0 & G^{33} & 0 & \cdots & \\ 0 & 0 & 0 & G^{44} & 0 & \vdots \\ \vdots & & & & \ddots & \\ 0 & & \cdots & & 0 & G^{77} \end{pmatrix} = \begin{pmatrix} f_1^2(r) & 0 & 0 & 0 & \cdots & 0 \\ 0 & f_2^2(r) & 0 & 0 & \cdots & 0 \\ 0 & 0 & G^{33} & 0 & \cdots & \\ 0 & 0 & 0 & f_4^2(r) & 0 & \vdots \\ \vdots & & & & \ddots & \\ 0 & & \cdots & & 0 & G^{77} \end{pmatrix},$$

$$B = (B_{mn}) = \begin{pmatrix} 0 & B_{12} & 0 & \cdots & 0 \\ B_{21} & 0 & & & \\ 0 & & 0 & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Upsilon_9 & 0 & \cdots & 0 \\ -\Upsilon_9 & 0 & & & \\ 0 & & 0 & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & & 0 \end{pmatrix}, \qquad (5.19)$$

where

$$f_i^2(r) = \frac{D_i^2}{r^\gamma} \quad , \quad D_4 = 1, \quad \Upsilon_9 = -\frac{\theta}{2\pi}\alpha_2 \quad . \qquad (5.20)$$

We will also take

$$e^{2\phi} = r^{-\gamma} \quad , \quad \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \vdots \\ \Psi_{10} \\ \Psi_{11} \\ \Psi_{12} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{\theta}{2\pi}\alpha_4 \\ 0 \\ \vdots \end{pmatrix} \quad , \quad a_m^I = 0 \quad . \qquad (5.21)$$

For the space–time metric we will make the ansatz

$$ds^2 = -dt^2 + dr^2 + R^2(r)\, d\theta^2 \quad , \qquad R(r) = a\, r^{1-\gamma} \quad . \qquad (5.22)$$

The constants D_i and γ will be fixed by the Killing spinor equations.

The Killing spinor equations (3.2) now take the form

$$\begin{aligned} \delta\chi^I &= 0 \quad , \\ \delta\lambda &= -\frac{1}{2}e^{-\phi}\partial_\mu\{\phi + \ln\det e_m^a\}\gamma^\mu\,\varepsilon + \frac{1}{4}e^{-\phi}\partial_\mu B_{mn}\gamma^\mu \otimes \Sigma^{mn}\varepsilon \quad , \end{aligned} \qquad (5.23)$$

$$\delta\psi_\mu = \partial_\mu\varepsilon + \frac{1}{4}\omega_{\mu\alpha\beta}\gamma^{\alpha\beta}\varepsilon + \frac{1}{4}(e_{\mu\alpha}e^\nu_\beta - e_{\mu\beta}e^\nu_\alpha)\partial_\nu\phi\gamma^{\alpha\beta}\varepsilon$$

$$+\frac{1}{4}e^{-\phi}\, e_{ma}F^{(1)m}_{\mu\nu}\,\gamma^\nu\gamma^4\otimes\Sigma^a\varepsilon - \frac{1}{8}\partial_\mu B_{mn}\Sigma^{mn}\varepsilon \quad , \tag{5.24}$$

$$\delta\psi_d = -\frac{1}{4}e^{-\phi}(e^m_d\partial_\mu e_{ma} + e^m_a\partial_\mu e_{md})\gamma^\mu\gamma^4\otimes\Sigma^a\varepsilon + \frac{1}{4}e^{-\phi}e^m_d e^n_a\partial_\mu B_{mn}\gamma^\mu\gamma^4\otimes\Sigma^a\varepsilon$$

$$-\frac{1}{8}e^{-2\phi}\, e_{md}F^{(1)m}_{\mu\nu}\,\gamma^{\mu\nu}\varepsilon \quad . \tag{5.25}$$

The Killing spinor $\varepsilon = \epsilon\otimes\chi$ will be taken to satisfy (3.8) and (3.9) as well as $\Sigma^{12}\,\chi = iq\,\chi$ and $\Sigma^4\,\chi = \chi$.

The condition $\delta\lambda = 0$ (5.23) then implies

$$D_1 D_2 = -\frac{pq\gamma 2\pi a}{\alpha_2} \quad , \tag{5.26}$$

while $\delta\psi_\theta = 0$ in (5.24) gives the condition

$$pa(1-\frac{3}{2}\gamma) + \frac{q\alpha_2}{4\pi}D_1 D_2 = 0 \quad . \tag{5.27}$$

These last two equations yield $\gamma = \frac{1}{2}$, whereas $\delta\psi_4 = 0$ (5.25) gives the relation

$$a = -\frac{\alpha_4}{2\pi\gamma\tilde{p}} \quad . \tag{5.28}$$

Since this last quantity is positive, this implies $\tilde{p} = -\eta_{\alpha_4}$, where η_{α_4} denotes the sign of α_4.

The above shows that we can then take

$$D_1 = D_2 = \sqrt{-\frac{pq\gamma 2\pi a}{\alpha_2}} = \sqrt{\frac{\gamma 2\pi a}{|\alpha_2|}} \quad , \tag{5.29}$$

with $-pq = \eta_{\alpha_2}$.

This solution preserves 1/4 of $N = 8$ supersymmetry.

5.4 Soliton solutions preserving $N = 1$ supersymmetry

It is now straightforward to construct solutions which preserve 1/8 of $D = 3, N = 8$ supersymmetry.

One class of solitonic solutions preserving 1/8 of $D = 3, N = 8$ supersymmetry is given as follows. The background fields G^{-1} and B are given by

$$\begin{pmatrix} G^{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G^{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G^{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G^{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G^{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & G^{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G^{77} \end{pmatrix} = \begin{pmatrix} f_1^2(r) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_2^2(r) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G^{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_4^2(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_5^2(r) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f_6^2(r) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G^{77} \end{pmatrix},$$

$$\begin{pmatrix} 0 & B_{12} & 0 & 0 & 0 & 0 & 0 \\ B_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{56} & 0 \\ 0 & 0 & 0 & 0 & B_{65} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Upsilon_9 & 0 & 0 & 0 & 0 & 0 \\ -\Upsilon_9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Upsilon_{13} & 0 \\ 0 & 0 & 0 & 0 & -\Upsilon_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$f_i^2(r) = \frac{D_i^2}{r^\gamma} \quad , \quad D_4 = 1, \;\; \Upsilon_9 = -\frac{\theta}{2\pi}\alpha_2 \quad , \quad \Upsilon_{13} = -\frac{\theta}{2\pi}\alpha_6 \; . \tag{5.30}$$

Again, we have

$$e^{2\phi} \;\; = \;\; r^{-\gamma} \qquad , \qquad \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \\ \Psi_5 \\ \vdots \\ \Psi_{10} \\ \Psi_{11} \\ \Psi_{12} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{\theta}{2\pi}\alpha_4 \\ 0 \\ \vdots \end{pmatrix} \qquad , \qquad a_m^I = 0 \quad . \tag{5.31}$$

As before, we will make the following ansatz for the space–time metric

$$ds^2 = -dt^2 + dr^2 + R^2(r)\, d\theta^2 \quad , \quad R(r) = a\, r^{1-\gamma} \quad . \tag{5.32}$$

Now, the Killing spinor equations give the following constraints

$$\begin{aligned} -2\gamma p - \frac{\alpha_2 q}{2\pi a} D_1 D_2 - \frac{\alpha_6 \tilde{q}}{2\pi a} D_5 D_6 = 0, \\ pa(1 - \frac{3}{2}\gamma) + \frac{q\alpha_2}{4\pi} D_1 D_2 + \frac{\tilde{q}\alpha_6}{4\pi} D_5 D_6 = 0, \end{aligned} \tag{5.33}$$

as well as

$$D_1 D_2 = -pq\,\frac{2\pi a\gamma}{\alpha_2}, \qquad D_5 D_6 = -p\tilde{q}\,\frac{2\pi a\gamma}{\alpha_6} \qquad \text{and} \qquad a = -\frac{\alpha_4 \tilde{p}}{2\pi\gamma} \quad . \tag{5.34}$$

Here we have used the conditions

$$\begin{aligned} \Sigma^4\chi \;&=\; \chi \;, \\ \Sigma^{12}\chi \;&=\; iq\chi, \qquad\qquad q = -p\eta_{\alpha_2} \;, \\ \Sigma^{56}\chi \;&=\; i\tilde{q}\chi, \qquad\qquad \tilde{q} = -p\,\eta_{\alpha_6} \;, \end{aligned} \tag{5.35}$$

which shows that the background preserves 1/8 of the $N = 8$ supersymmetry.

Note that equation (5.33) yields that γ here is $\frac{2}{5}$.

Looking back at B_{mn} we notice now that we can add one more block with $B_{37} = -B_{73} = \Upsilon_{14} = -\frac{\theta}{2\pi}\,\alpha_7$. The Killing equations will imply a further condition on χ,

$$\Sigma^{37}\chi = i\hat{q}\,\chi, \qquad\qquad \hat{q} = -p\,\eta_{\alpha_7}\ . \tag{5.36}$$

Furthermore $\gamma = 1/3$ and $D_3 D_7 = -p\hat{q}\,\frac{2\pi a\gamma}{\alpha_7}$. We note that the condition (5.36) does not break any additional supersymmetry, and hence this solution also preserves 1/8 of the $N = 8$ supersymmetry.

6 The compactified cosmic string solution

In [3], Sen constructed a particular three-dimensional solution by considering the fundamental string solution of the four-dimensional theory [22] and by winding the direction along which the string extends once in the third direction. As shown in [3], this solution is related to the cosmic string solution of [34] by a O(8,24;$\mathbb{Z}$) transformation. Other three-dimensional solutions with internal winding can be obtained from the four-dimensional string solutions given in [35].

The fundamental string solution in four space–time dimensions is known to have partial space–time supersymmetry [36]. Here, we will presently construct the Killing spinor associated with the particular three-dimensional solution mentioned above.

The field configuration representing a fundamental string solution winding once in the third direction is described below, following [3].

The three-dimensional space–time metric is now of the form

$$\begin{aligned} ds^2 &= -dt^2 + \lambda_2\,(dr^2 + r^2\,d\theta^2) \\ &= -dt^2 + \lambda_2\,dzd\bar{z}\ , \end{aligned} \tag{6.1}$$

where $z = re^{i\theta}$ is the complex coordinate labelling the two-dimensional space. The scalar fields are $e^{-2\phi} = \lambda_2$, $\Psi \equiv \Psi^1 = -\lambda_1$ and $G_{11} = e^{2\phi}$. From (2.11), the only non-vanishing field strength is

$$F^{(2)}_{\mu\nu 1} = e^{4\phi}\epsilon^{\alpha\beta\gamma}\,e_{\mu\alpha}\,e_{\nu\beta}\,e^{\rho}_{\gamma}\partial_\rho\Psi\ . \tag{6.2}$$

It will turn out to be convenient to combine λ_1 and λ_2 into a complex scalar field $S = \lambda_1 + i\lambda_2$. Here, as opposed to the previous cases, we take Ψ and ϕ to depend on both r and θ.

With these assignments the Killing spinor equations take the form (where now $\Sigma^1\chi=\chi$),

$$
\begin{aligned}
\delta\lambda &= -(\partial_r\phi\,\gamma^1+\frac{1}{r}\partial_\theta\phi\,\gamma^2)\,\varepsilon+\frac{1}{2}e^{2\phi}(\partial_r\Psi\,\gamma^1+\frac{1}{r}\partial_\theta\Psi\gamma^2)\,J\,\gamma^4\,\varepsilon\ ,\\
\delta\psi_t &= -\frac{1}{2}e^{\phi}(-\partial_r\phi\,\gamma^2+\frac{1}{r}\partial_\theta\phi\,\gamma^1)\,J\,\varepsilon+\frac{1}{4}e^{3\phi}(-\partial_r\Psi\,\gamma^2+\frac{1}{r}\partial_\theta\Psi\,\gamma^1)\gamma^4\varepsilon\ ,\\
\delta\psi_r &= \partial_r\,\varepsilon-\frac{1}{4}\frac{e^{2\phi}}{r}\partial_\theta\Psi\,\gamma^0\gamma^4\varepsilon\ ,\\
\delta\psi_\theta &= \partial_\theta\,\varepsilon-\frac{1}{2}\gamma^{12}\,\varepsilon+\frac{r}{4}\,e^{2\phi}\partial_r\Psi\,\gamma^0\gamma^4\varepsilon\ ,\\
\delta\psi_1 &= -\frac{1}{2}(\partial_r\phi\,\gamma^1+\frac{1}{r}\partial_\theta\phi\,\gamma^2)\,\gamma^4\,\varepsilon-\frac{1}{4}e^{2\phi}(\partial_r\Psi\,\gamma^1+\frac{1}{r}\partial_\theta\Psi\,\gamma^2)\,J\,\varepsilon\ .
\end{aligned}
\tag{6.3}
$$

Compatibility of the form of ϵ for $\delta\lambda,\delta\psi_t$ and $\delta\psi_1$ can be achieved by imposing $J\gamma^4\,\epsilon=i\alpha\,\epsilon$, which yields

$$
\begin{pmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\end{pmatrix}=\begin{pmatrix}\epsilon_1\\ \epsilon_2\\ -i\alpha\epsilon_1\\ -i\alpha\epsilon_2\end{pmatrix}, \tag{6.4}
$$

where $\alpha=\pm$.

Equations $\delta\lambda,\delta\psi_t$ and $\delta\psi_1$ are actually equivalent to each other and can be written in the form

$$
\delta\lambda\propto\partial_\mu S\,\gamma^\mu\,\epsilon \tag{6.5}
$$

with $S=-\Psi+ie^{-2\phi}$, provided that $\alpha=+1$.

The equation $\delta\lambda=0$ can be solved by assuming that S is a holomorphic function of z, therefore $\partial_{\bar z}S=0$. Note that a holomorphic function $S(z)$ solves the equation of motion (2.8).

Then, $\delta\lambda\propto\partial_z S\,\gamma^z\,\epsilon=0$, which can be solved by setting $\gamma^z\,\epsilon=0$.

By using that $\gamma^z\propto\gamma^1(1+i)-\gamma^2(1-i)$, one finds

$$
\begin{pmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\end{pmatrix}=\tilde\epsilon\begin{pmatrix}1\\ i\\ -i\\ 1\end{pmatrix}. \tag{6.6}
$$

The remaining equations to be solved are then $\delta\psi_r=0$ and $\delta\psi_\theta=0$. We find

$$
\begin{aligned}
&\partial_r\,\tilde\epsilon+\frac{1}{4}\frac{e^{2\phi}}{r}\partial_\theta\Psi\,\tilde\epsilon=0\ ,\\
&\partial_\theta\,\tilde\epsilon-\frac{i}{2}\tilde\epsilon-\frac{r}{4}e^{2\phi}\partial_r\Psi\,\tilde\epsilon=0.
\end{aligned}
\tag{6.7}
$$

The holomorphicity of S implies the relations

$$\begin{aligned} \partial_r \Psi &= \frac{2}{r} e^{-2\phi}\, \partial_\theta \phi \ , \\ \frac{1}{r}\, \partial_\theta \Psi &= -2\partial_r \phi\, e^{-2\phi} \ , \end{aligned} \tag{6.8}$$

which we use to solve for the spinors completely:

$$\tilde{\epsilon} = e^{\frac{\phi}{2}}\, e^{\frac{i}{2}\theta}. \tag{6.9}$$

The above solution breaks 1/2 of $N = 8, D = 3$ supersymmetry.

7 String solutions with $H_{\mu\nu\rho} \neq 0$

Here we show how the Killing spinor equations (3.2) determine a three-dimensional solution with a non-vanishing $H_{\mu\nu\rho}$ and a non-constant dilaton (other solutions with non-zero $H_{\mu\nu\rho}$ have been considered in [37, 38]).

We consider a solution without gauge fields and with constant internal metric G_{mn}. We will take $H_{\mu\nu\rho} = \sqrt{-g}\epsilon_{\mu\nu\rho}\Lambda e^{p\phi}$, where p is an integer, in the Einstein frame. We will take the space–time metric to be of the form (3.3), and we will also take $\phi = \phi(r)$.

The Killing spinor equations reduce to

$$\begin{aligned} \delta\chi^I &= 0 \ , \\ \delta\lambda &= -\frac{1}{2} e^{-\phi} \partial_\mu (\phi + \ln\det e^a_m) \gamma^\mu\, \varepsilon + \frac{1}{12} e^{-3\phi} H_{\mu\nu\rho}\, \gamma^{\mu\nu\rho}\, \varepsilon \ , \\ \delta\psi_\mu &= \partial_\mu \varepsilon + \frac{1}{4}\omega_{\mu\alpha\beta}\gamma^{\alpha\beta}\varepsilon + \frac{1}{4}(e_{\mu\alpha}e^\nu_\beta - e_{\mu\beta}e^\nu_\alpha)\partial_\nu\phi\gamma^{\alpha\beta}\varepsilon - \frac{1}{8}e^{-2\phi} H_{\mu\nu\delta}\gamma^{\nu\delta}\varepsilon \ , \\ \delta\psi_d &= -\frac{1}{4} e^{-\phi}(e^m_d \partial_\mu e_{ma} + e^m_a \partial_\mu e_{md})\gamma^\mu \gamma^4 \otimes \Sigma^a \varepsilon \ . \end{aligned} \tag{7.1}$$

By using identity (A.11) of the appendix, these equations become

$$\begin{aligned} \delta\chi^I &= 0 \ , \\ \delta\lambda &= -\frac{1}{2} e^{-\phi} \partial_r (\phi + \ln\det e^a_m) \sqrt{V} \gamma^1\, \varepsilon + \frac{1}{2} e^{(p-3)\phi} \Lambda\, J\, \varepsilon \ , \\ \delta\psi_t &= \frac{1}{2}[\sqrt{V}\partial_r \sqrt{V} + V\partial_r\phi]\, J\, \gamma^2\, \varepsilon + \frac{1}{4} e^{(p-2)\phi}\sqrt{V}\Lambda\, J\gamma^0\, \varepsilon \ , \\ \delta\psi_r &= \partial_r\varepsilon - \frac{1}{4} e^{(p-2)\phi} \frac{\Lambda}{\sqrt{V}}\, J\gamma^1\, \varepsilon \ , \\ \delta\psi_\theta &= \partial_\theta\varepsilon - \frac{1}{2}[\sqrt{V}\partial_r R + R\sqrt{V}\partial_r\phi]\, J\, \gamma^0\, \varepsilon - \frac{1}{4} e^{(p-2)\phi} R\Lambda\, J\gamma^2 \varepsilon \ . \end{aligned} \tag{7.2}$$

Compatibility of the spinor $\varepsilon = \epsilon \otimes \chi$ within these equations is obtained by demanding

$$\begin{aligned} \gamma^1\,\epsilon &= \alpha\,J\,\epsilon\ , \qquad \alpha = \pm\,, \\ \gamma^2\,\epsilon &= -\alpha\,\gamma^0\,\epsilon\ . \end{aligned} \tag{7.3}$$

Then, one has

$$\begin{aligned} \delta\lambda &= -\frac{1}{2}\alpha e^{-\phi}\partial_r(\phi + \ln\det e^a_m)\sqrt{V}J\,\varepsilon + \frac{1}{2}e^{(p-3)\phi}\Lambda\,J\,\varepsilon\ , \\ \delta\psi_t &= -\frac{1}{2}\alpha[\sqrt{V}\partial_r\sqrt{V} + V\partial_r\phi]\,J\,\gamma^0\,\varepsilon + \frac{1}{4}e^{(p-2)\phi}\sqrt{V}\Lambda\,J\gamma^0\,\varepsilon\ , \\ \delta\psi_r &= \partial_r\varepsilon - \frac{1}{4}\alpha e^{(p-2)\phi}\frac{\Lambda}{\sqrt{V}}\,\varepsilon\ , \\ \delta\psi_\theta &= \partial_\theta\varepsilon - \frac{1}{2}[\sqrt{V}\partial_r R + R\sqrt{V}\partial_r\phi]\,J\,\gamma^0\,\varepsilon + \frac{\alpha}{4}e^{(p-2)\phi}R\Lambda\,J\gamma^0\varepsilon\ . \end{aligned} \tag{7.4}$$

For $\alpha = +1$ the spinor ϵ is of the form

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix} = \begin{pmatrix} \epsilon_1 \\ 0 \\ \epsilon_3 \\ 0 \end{pmatrix} = \tilde{\epsilon}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{7.5}$$

(where we have imposed $\epsilon_1 = \epsilon_3$ in order to reduce the degrees of freedom contained in ϵ to the degrees of freedom contained in a two component spinor, see appendix), whereas for $\alpha = -1$ it is

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \epsilon_2 \\ 0 \\ \epsilon_4 \end{pmatrix} = \tilde{\epsilon}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \tag{7.6}$$

Demanding the vanishing of the Killing spinor equations (7.4) and imposing the condition $\partial_\theta\,\epsilon = 0$, as well as $\alpha\Lambda = |\Lambda|$, leaves us with the following constraints

$$\begin{aligned} \sqrt{V}\partial_r\phi &= e^{(p-2)\phi}|\Lambda|\ , \\ \sqrt{V}\partial_r\sqrt{V} + V\partial_r\phi &= \frac{1}{2}\sqrt{V}e^{(p-2)\phi}|\Lambda|\ , \\ \partial_r\,\tilde{\epsilon} &= \frac{1}{4}\frac{|\Lambda|}{\sqrt{V}}e^{(p-2)\phi}\tilde{\epsilon}\ , \\ \sqrt{V}\partial_r R + R\sqrt{V}\partial_r\phi &= \frac{1}{2}e^{(p-2)\phi}|\Lambda|\ . \end{aligned} \tag{7.7}$$

These equations can be solved to give

$$\begin{aligned} V &= a^2 \, e^{-\phi} \ , \\ R &= b^2 \, e^{-\phi/2} \ , \\ \tilde{\epsilon} &= e^{\frac{\phi}{4}} \epsilon_0, \end{aligned} \tag{7.8}$$

where a, b, ϵ_0 are integration constants.

For $p \neq \frac{3}{2}$, the dilaton is given by

$$e^{(\frac{3}{2}-p)\phi} = |\frac{\Lambda}{a}(\frac{3}{2} - p)(r - r_0)|, \tag{7.9}$$

whereas for $p = \frac{3}{2}$,

$$\partial_r \phi = \frac{|\Lambda|}{a} \quad \longrightarrow \quad \phi = \frac{|\Lambda|}{a}(r - r_0). \tag{7.10}$$

Note that ϵ is real and hence the background preserves 1/2 of $N = 8, D = 3$ supersymmetry.

We notice here that whatever the value of p, there is a solution to the Killing spinor equations. On the other hand, the equations of motion are satisfied provided $H^{\mu\nu\rho} = \frac{\epsilon^{\mu\nu\rho}}{\sqrt{-g}} \, \Lambda \, e^{4\phi}$, that is $p = 4$. Thus, contrary to common experience [23, 13, 14], not every solution to the Killing spinor equations solves the equations of motion.

For $p = 4$, the curvature scalar is computed to be $\mathcal{R} = \frac{5}{2}\Lambda^2 e^{4\phi}$, which is always positive.

8 Conclusions

We have considered in the present work the low-energy effective Lagrangian of heterotic string theory compactified on a seven-torus, and we have constructed a variety of electrically charged and solitonic backgrounds preserving $1/2^m$ of $N = 8, D = 3$ supersymmetry ($m = 1, 2, 3$). The construction of the solutions is done by using the criteria of unbroken supersymmetries and solving for the associated Killing spinor equations. The space–time line elements of the solutions constructed here have the form (4.4), which differ from the usual line element associated with conical geometries [12]. These line elements do not seem to correspond to small deformations of flat space–time. Thus they seem to contain some interesting structure which deserves further study.

Further solitonic solutions with diagonal space–time line elements can be obtained by applying more general $O(8, 24)$ transformations to the electrically charged solutions of section 4 and 5. It would also be interesting to consider a non-diagonal ansatz for the space–time metric.

We have also found a solution to the Killing spinor equations with $H_{\mu\nu\rho} \neq 0$ which preserves 1/2 of the $N = 8$ supersymmetry. Furthermore, we have shown that the compactified cosmic string solution constructed by Sen [3] satisfies our Killing spinor equations.

Most of the solutions presented here are charged with the associated gauge field strengths given everywhere by (2.13). We note, however, that one should generally expect these solitonic solutions to get modified by quantum corrections, at least in the strong coupling regime [3]. Recall the fundamental string solution discussed by Sen [3] that we have considered in section 6. There we showed that a holomorphic solution $S(z)$ satisfies our Killing spinor equations. As Sen points out, the holomorphic function $S(z) = \lambda_1 + i\lambda_2$ has the behavior $\lambda_2 = e^{-2\phi} \sim -\ln r$ for $r \longrightarrow 0$, whereas at $r \longrightarrow \infty$, this behavior needs to be modified in order to make sense. This is achieved by replacing S by the $SL(2,\mathbb{Z})$ invariant function $j(S)$, such that $j(S(z)) \simeq 1/z$ for $r \longrightarrow \infty$. Therefore the solution $S(z)$ should be considered only as an approximate solution that gets modified as the theory enters the strong coupling regime. In analogy with the above, we would, for example, expect our electrically charged solutions in subsections 4.1 and 5.1 to get modified at short distances, where the coupling becomes strong. Similar phenomena have been shown to occur in $N = 4, D = 3$ supersymmetric gauge theories [39], where the classical moduli space receives perturbative as well as non-perturbative quantum corrections. An extension of these ideas to string theory remains to be explored.

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A Appendix

In ten dimensions, the Γ^A matrices satisfy

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB} \quad , \tag{A.1}$$

where A, B are D=10 tangent space indices and where $\eta_{AB} = (-,+,\ldots,+)$. The decomposition of the gamma matrices that is appropriate to the 10=3+7 split is

$$\Gamma^A \equiv (\Gamma^\alpha, \Gamma^a) = (\gamma^\alpha \otimes \mathbf{I}_8, \gamma^4 \otimes \Sigma^a) \quad , \tag{A.2}$$

where

$$\{\gamma^\alpha, \gamma^\beta\} \quad = \quad 2\eta^{\alpha\beta}, \qquad \eta^{\alpha\beta} = (-,+,+) \quad ,$$

$$
\begin{aligned}
\{\Sigma^a, \Sigma^b\} &= 2\eta^{ab}, \qquad \eta^{ab} = (+,+,\ldots,+) \ , \\
\{\gamma^4, \gamma^\alpha\} &= 0, \qquad (\gamma^4)^2 = \mathbf{I}_4 \ .
\end{aligned}
\tag{A.3}
$$

γ^α and Σ^a are the 3D space–time and 7D internal Dirac matrices respectively, and γ^4 plays the role of a chirality operator [17] by enhancing the three-dimensional space–time spinor to a four component spinor (instead of the usual two component spinor). Note that in order to have $\{\Gamma^a, \Gamma^\alpha\} = 0$ we are forced to introduce the chirality operator γ^4. We decompose the 10D spinor into the form $\varepsilon^{A,i} = \epsilon^A \otimes \chi^i$, where ϵ^A is a four component spinor of SO(1,2) ($A = 1, \ldots, 4$) and χ^i is a SO(7) spinor, with $i = 1, \ldots, 8$ indicating the $N = 8$ supersymmetries.

We have

$$
\begin{aligned}
&\Gamma^{\alpha\beta} = \gamma^{\alpha\beta} \otimes \mathbf{I}_8, \qquad \Gamma^{\alpha a} = \gamma^\alpha \gamma^4 \otimes \Sigma^a, \qquad \Gamma^{ab} = \mathbf{I}_4 \otimes \Sigma^{ab} \ , \\
&\Gamma^{\alpha ab} = \gamma^\alpha \otimes \Sigma^{ab}, \qquad \Gamma^{\alpha\beta a} = \gamma^{\alpha\beta}\gamma^4 \otimes \Sigma^a,
\end{aligned}
\tag{A.4}
$$

with $\Gamma^{AB\ldots C} = \Gamma^{[a}\Gamma^b \ldots \Gamma^{C]}$, thus $\Gamma^{AB} = \frac{1}{2}(\Gamma^A\Gamma^B - \Gamma^B\Gamma^A)$. The 10D chirality operator Γ^{11} is given by $\Gamma^{11} = \gamma^0\gamma^1\gamma^2\gamma^4 \otimes i\mathbf{I}_8$.

As a representation of the γ matrices, we take

$$
\gamma^0 = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} \mathbf{I}_2 & 0 \\ 0 & -\mathbf{I}_2 \end{pmatrix},
$$

with

$$
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.5}
$$

Then

$$
\gamma^{01} = -J\gamma^2, \qquad \gamma^{02} = J\gamma^1, \qquad \gamma^{12} = J\gamma^0 \ , \tag{A.6}
$$

where

$$
J = \begin{pmatrix} 0 & \mathbf{I}_2 \\ \mathbf{I}_2 & 0 \end{pmatrix}. \tag{A.7}
$$

We have $\gamma^\mu = e^\mu_\alpha \gamma^\alpha$, $\Sigma^m = e^m_a \Sigma^a$, where e^μ_α and e^m_a are the space–time and internal inverse vielbeins respectively.

The conventions for the Christoffel symbols and the Ricci tensor are the following:

$$
\begin{aligned}
&\Gamma^\rho_{\mu\nu} = \frac{1}{2}\sum_\sigma g^{\rho\sigma}\{\frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma}\} \ , \\
&\mathcal{R}_{\mu\rho} = \sum_\nu \mathcal{R}_{\mu\nu\rho}{}^\nu = \sum_\nu \frac{\partial}{\partial x^\nu}\Gamma^\nu_{\mu\rho} - \frac{\partial}{\partial x^\mu}(\sum_\nu \Gamma^\nu_{\nu\rho}) + \sum_{\alpha,\nu}(\Gamma^\alpha_{\mu\rho}\Gamma^\nu_{\alpha\nu} - \Gamma^\alpha_{\nu\rho}\Gamma^\nu_{\alpha\mu}) \ , \\
&\sum_\nu \Gamma^\nu_{\nu\mu} = \frac{1}{2}\sum_{\nu,\alpha} g^{\nu\alpha}\frac{\partial g_{\nu\alpha}}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}\ln\sqrt{|g|} \ .
\end{aligned}
\tag{A.8}
$$

Our convention for the spin connection is

$$\omega_{MAB} = \frac{1}{2}E^N_A(\partial_M E_{NB}-\partial_N E_{MB})+\frac{1}{2}E^N_B(\partial_N E_{MA}-\partial_M E_{NA})-\frac{1}{2}E^P_A E^Q_B(\partial_P E^C_Q-\partial_Q E^C_P)E_{MC} \tag{A.9}$$

and

$$G_{MN} = E^A_M\,\eta_{AB}\,E^B_N, \qquad\qquad E_{MB} = E^C_M\,\eta_{CB}. \tag{A.10}$$

Other useful identities are

$$\begin{aligned}
\epsilon_{\mu\nu\rho}\,\gamma^{\mu\nu\rho} &= \frac{1}{\det e}\,\epsilon_{\alpha\beta\eta}\,\gamma^{\alpha\beta\eta} \\
&= \frac{6}{\det e}\,\epsilon_{012}\,\gamma^{012} \qquad\qquad \text{with } \epsilon^{012} = -\epsilon_{012} = 1\ , \\
\epsilon_{\mu\nu\rho}\,\gamma^{\nu\rho} &= \frac{1}{\det e}\,e^\alpha_\mu\,\epsilon_{\alpha\beta\eta}\,\gamma^{\beta\eta}\ .
\end{aligned} \tag{A.11}$$

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