Bosonization in $d > 2$ dimensions *

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Abstract

I discuss in this talk a bosonization approach recently developed in refs.[1]-[7]. It leads to the (exact) bosonization rule for fermion currents in $d \geq 2$ dimensions and also provides a systematic way of constructing the bosonic action in different regimes.

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1 Introduction

I describe in this talk recent work on bosonization developed in colaboration with my friends **Nino Bralić** from Universidad Católica de Santiago de Chile, **César Fosco** from Centro Atómico de Bariloche, **Eduardo Fradkin** from the University of Illinois at Urbana Champaign, **Jean Claude Le Guillou** from LAPP-Annecy and ENS de Lyon, **Enrique Moreno** from City College and Baruch College of the CUNY, and **Virginia Manías** and **Carlos Núñez** from the University of La Plata. More details can be found in references [1]-[7].

Bosonization is a mapping of quantum field theories of fermion fields onto equivalent theories of boson fields. Well-established in $1+1$ dimensions, bosonization constitutes there one of the main tools available for the study of the non-perturbative behavior of both quantum field theories [8] and of condensed matter systems [9]. In dimensions other than $1+1$, much less is known. In this talk I will precisely discuss the issue of $d > 2$ bosonization within the path-integral approach making special emphasis in the $d = 3$ case, of particular interest in condensed matter problems. The bosonization approach that I will present corresponds to the path-integral framework and basically establishes an identity between the generating functional of the fermionic theory and the generating functional of the equivalent bosonic theory. From this relation, the recipe for bosonization of fermion currents is derived and the current commutator algebra is presented. To complete the bosonization program one should also calculate the energy-momentum tensor algebra. This is not done in this work.

Related and unrelated approaches to $d > 2$ bosonization can be found in [10]-[23].

2 The method

The method is straightforward. We shall basically consider the case of free fermions but we shall also discuss an interacting (Thirring) model.

One starts from the fermion Lagrangian for N massive free fermions in d dimensions,

$$L = \bar{\psi}(i\partial\!\!\!/ + m)\psi \tag{1}$$

The corresponding generating functional reads

$$Z_{fer}[s] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp[-\int_M d^dx \bar{\psi}(i\partial\!\!\!/ + s\!\!\!/ + m)\psi] \tag{2}$$

where s_μ is the source for fermion currents, $s_\mu = s_\mu^a t^a$ with t^a the generators of a group G and M a d-dimensional manifold.

Our derivation heavily relies on the invariance of the fermion measure under local gauge transformations $h(x) \in \hat{G}$ with $\hat{G}$ the group of continuous maps $M \to G$. This ensures that

$$Z[s^h] = Z[s] \tag{3}$$

with

$$s_\mu^h = h^{-1}s_\mu h + h^{-1}\partial_\mu h \tag{4}$$

Evidently, fermions can be integated in (2),

$$Z_{fer}[s] = \det(i\partial\!\!\!/ + s\!\!\!/ + m) \tag{5}$$

and the determinant in the r.h.s. of eq.(5) will be used to introduce an auxiliary field b_μ taking values in the Lie algebra of G through the trivial formula

$$Z_{fer}[s] = \int \mathcal{D}b_\mu \, \delta^{(d)}(b_\mu - s_\mu) \, \det(i\partial\!\!\!/ + b\!\!\!/ + m) \tag{6}$$

Now, it will be convenient to replace the delta function in (6) as follows

$$\delta^{(d)}(b_\mu - s_\mu) \to \Delta[b] \, \delta^{(n)} \left(\varepsilon_{\mu_1\mu_2\ldots\mu_d} (f_{\mu_1\mu_2}[b] - f_{\mu_1\mu_2}[s]) \right) \tag{7}$$

Here we have used that the equation

$$f_{\mu\nu}[b] = f_{\mu\nu}[s] \tag{8}$$

has for $s_\mu \neq 0$ the unique solution

$$b_\mu = s_\mu \tag{9}$$

and $\Delta[b]$ is a Faddeev-Popov-like jacobian,

$$\Delta[b] = | \det(2\varepsilon_{\mu_1\mu_2\ldots\mu_d} D_{\mu_1}[b])| \tag{10}$$

with $D_\mu[b]$ the covariant derivative,

$$D_\mu[b] = \partial_\mu + [b_\mu, \;] \tag{11}$$

We do not consider for the moment Gribov like problems that could arise for certain manifolds and groups. Concerning the delta function in the r.h.s. of eq.(7), n depends on the space-time dimensions according to $n = d(d-1)/2$ since one needs for enforcing eq.(8) as many δ-functions as independent components the curvature has.

It is at this point that the bosonic field whose dynamics will be equivalent to that of the original Fermi field comes into play. We introduce it as a Lagrange multiplier $A_{\mu_3\ldots\mu_d}$ enforcing the δ-function in the path-integral (6),

$$\begin{aligned} Z_{fer}[s] &= \int \mathcal{D}b_\mu \mathcal{D}A_{\mu_3\ldots\mu_d} \, \det(i\partial\!\!\!/ + b\!\!\!/ + m)\Delta[b] \\ &\quad \exp\left[\lambda \, tr \int d^dx A_{\mu_3\ldots\mu_d} \varepsilon_{\mu_1\mu_2\ldots\mu_d} (f_{\mu_1\mu_2}[b] - f_{\mu_1\mu_2}[s])\right] \end{aligned} \tag{12}$$

Here λ is a constant which can be adjusted so as to obtain an adequate normalization for the currents. Now, we rewrite eq.(12) in the form

$$Z_{fer}[s] = \int \mathcal{D}A_{\mu_3\ldots\mu_d}\,\exp(S_{bos}[A])\exp\left(-\lambda\, tr\int d^dx A_{\mu_3\ldots\mu_d}\varepsilon_{\mu_1\mu_2\ldots\mu_d}f_{\mu_1\mu_2}[s]\right) \tag{13}$$

where the bosonic action is defined as

$$\begin{aligned}\exp(S_{bos}[A]) &= \int \mathcal{D}b_\mu \det(i\partial\!\!\!/ + b\!\!\!/ + m)\Delta[b] \\ &\exp\left(\lambda\, tr\int d^dx A_{\mu_3\ldots\mu_d}\varepsilon_{\mu_1\mu_2\ldots\mu_d}f_{\mu_1\mu_2}[b]\right)\end{aligned} \tag{14}$$

Formulae (13)-(14) constitute our basic bosonization recipe: eq.(13) allows to compute fermion current correlation functions in terms of the bosonic field A and eq.(14) gives the bosonic action defining the dynamics of A. It can be now appreciated in what sense we consider our bosonization recipe exact: we have arrived *with no approximation* to a bosonization recipe of the form

$$\bar\psi\gamma_\mu t^a\psi \to 2\,\lambda\,\varepsilon_{\mu\mu_2\ldots\mu_d}\partial_{\mu_2}A^a_{\mu_3\ldots\mu_d} \tag{15}$$

However, except in $d=2$ dimensions where we know how to compute exactly the fermion determinant appearing in (14) and to resolve the path-integral defining S_{bos}, one should appeal to some approximation scheme to evaluate the bosonic action accompanying recipe (15). This means that only in $d=2$ dimensions the complete bosonization recipe is exact.

It should be stressed that the bosonization recipe (15) should be taken as illustrative of the bosonization since the rigorous equivalence between the fermionic and the bosonic theory is at the level of the generating functional $Z_{fer}[s]$ of Green functions. It is from $Z_{fer}[s]$ written in the form (13) that one has to compute current correlation functions in the bosonic language. Note also that in writing recipe (15) we have ignored terms which are non linear in the source. Although correlation functions of currents acquire a contribution from this terms, this contribution is irrelevant in the calculation of conmutator algebra since they have local support. This can be easily seen, for example, using the BJL method (see [6] for a discussion and [7] for the application of the BJL method within the present bosonization approach). Concerning the bosonic field $A^a_{\mu_3\ldots\mu_d}$ note that it corresponds to scalar fields in $d=2$ dimensions (see ref.[6] for details on how to make contact with the usual bosonization rules), to a vector field in $d=3$ dimensions and to an antisymmetric (Kalb-Rammond) field in $d>3$ dimensions [3],[5].

3 The abelian case in $d = 3$

The Abelian case in 3 dimensions is particularly simple. To begin with, the bosonization recipe for the fermion current reads

$$\bar{\psi}\gamma_\mu\psi \to \pm\frac{i}{8\pi}\varepsilon_{\mu\nu\alpha}\partial_\nu A_\alpha \tag{16}$$

where we have chosen λ so as to make contact with the normalization employed in ref.[7]. Concerning the bosonic action, $\Delta[b]$ is trivial so that eq.(14) simply reads

$$\exp(S_{bos}[A]) = \int \mathcal{D}b_\mu \det(i\partial\!\!\!/ + b\!\!\!/ + m) \exp\left(\mp\frac{i}{16\pi}\, tr \int d^3x A_\mu \varepsilon_{\mu\nu\alpha} f_{\nu\alpha}[b]\right) \tag{17}$$

or, calling

$$-\log\det(i\partial\!\!\!/ + b\!\!\!/ + m) = \int d^3x L[b] \tag{18}$$

we can write

$$\exp(S_{bos}[A]) = \int \mathcal{D}b_\mu \exp(-S_{eff}[b, A]) \tag{19}$$

where S_{eff} is defined as

$$S_{eff}[b, A] = \int d^3x (L[b] \pm \frac{i}{16\pi}\, tr\, A_\mu \varepsilon_{\mu\nu\alpha} f_{\nu\alpha}[b]) \tag{20}$$

(The double sign in eqs.(17)-(20) is included for convenience, see the discussion below)

Being in general $L[b]$ non-quadratic in b one cannot path-integrate in (17) so as to obtain $S_{bos}[A]$. We shall see however that there is a change of variables allowing to decouple A_μ from b_μ in $S_{eff}[b, A]$ so that one can control the A_μ dependence of $S_{bos}[A]$ without necessity of explicitly integrating over b_μ. Let us define a new variable b'_μ through the equation

$$b_\mu = (1-\theta)b'_\mu + \theta A_\mu + V_\mu[A] \tag{21}$$

where $V_\mu[A]$ is a gauge invariant function of A_μ so that b'_μ, the variable which will replace b_μ, transforms as a a gauge field. θ is an arbitrary parameter to be adjusted later. The idea is to choose V_μ so as to decouple b'_μ from A. This amounts to impose the following condition

$$\frac{\delta^2 S_{eff}}{\delta b'_\mu(x)\delta A_\nu(y)} = 0 \tag{22}$$

which in terms of V_μ reads

$$2i\lambda\varepsilon_{\rho\sigma\alpha}\partial_x^\alpha\delta(x-y)+\theta_y\int d^3z\frac{\delta^2 L[b(z)]}{\delta b_\rho(y)\delta b_\sigma(x)}$$
$$+\int d^3u\left(\int d^3z\frac{\delta^2 L[b(z)]}{\delta b_\beta(u)\delta b_\sigma(x)}\right)\frac{\delta V_\beta(u)}{\delta A_\rho(y)}=0 \tag{23}$$

To go on we need an explicit and necessarily approximate expression for $L[b]$. If we are interested in the large-distance regime of the bosonic theory we can use the result first [24]-[25] for the $d=3$ fermion determinant as an expansion in inverse powers of the fermion mass

$$L[b]=\mp\frac{i}{8\pi}\varepsilon_{\mu\nu\alpha}b_\mu\partial_\nu b_\alpha+\frac{1}{24\pi|m|}f_{\mu\nu}^2[b]+O(\frac{1}{m^2}) \tag{24}$$

The first term in (24) is the well-honored Chern-Simons action introduced in [24] as a way of generating a mass for gauge fields in three dimensions. The double sign in this term is originated in a regularisation ambiguity characteristic of odd-dimensions (see ref.[26]). The second one corresponds to the leading parity-even contribution to the fermion determinant. One easily sees that if one tries for $V_\mu[A]$ the functional form

$$V_\mu[A]=i\frac{C}{m}\varepsilon_{\mu\nu\alpha}f_{\nu\alpha}[A] \tag{25}$$

one gets, from the decoupling equation (23),

$$C=\pm 1/3 \tag{26}$$

Then, if for simplicity one chooses $\theta=-1$, the bosonic action for A_μ can be easily found to be

$$S_{bos}[A]=\pm\frac{i}{8\pi}\int d^3x\,\varepsilon_{\mu\nu\alpha}A_\mu\partial_\nu A_\alpha+\frac{1}{24\pi|m|}\int d^3x\,f_{\mu\nu}^2[A]+O(1/m^2) \tag{27}$$

One can in principle determine, following the same procedure, the following terms in the $1/m$ expansion of S_{bos} by including the corresponding terms in the fermion determinant expansion. This result extends that originally presented in ref.[1]. It shows that the bosonic counterpart of the three dimensional free fermionic theory is, to order $1/m^2$, a Maxwell-Chern-Simons theory which is equivalent, as it is well-known, in turn to a self-dual system [27]-[28].

Alternatively to the $1/m$ determinant expansion, one can consider an expansion in powers of b_μ retaining up to quadratic terms. The result can be written in the form [17]

$$L[b] = \frac{i}{2}\varepsilon_{\mu\nu\alpha} b_\mu P \partial_\nu b_\alpha + \frac{1}{4|m|} f_{\mu\nu}[b] Q f_{\mu\nu}[b] \tag{28}$$

where P and Q are functionals to be calculated within a loop expansion,

$$P \equiv P\left(\frac{\partial^2}{m^2}\right) \qquad\qquad Q \equiv Q\left(\frac{\partial^2}{m^2}\right) \tag{29}$$

Details of the calculations of P and Q and results within the loop-expansion can be found in refs. [17],[29].

In order to decouple the b_μ field one again proposes a change of variables like in (21) but now trying for V_μ the (gauge-invariant) functional form

$$V_\mu[A] = \frac{i}{m}\varepsilon_{\mu\nu\alpha} R f_{\nu\alpha}[A] = 2\frac{i}{m}\varepsilon_{\mu\nu\alpha} R \partial_\nu A_\alpha \tag{30}$$

with

$$R \equiv R\left(\frac{\partial^2}{m^2}\right) \tag{31}$$

One finds, from the decoupling conditions (23),

$$\begin{aligned} &\frac{\delta^2 S_{eff}[b', A]}{\delta A_\rho(y)\delta b'_\sigma(x)} = \\ &(1-\theta)\ \left(i\varepsilon_{\rho\sigma\alpha}\left(2\lambda + \theta P - 2\frac{\partial^2}{m^2} Q R\right)\partial_\alpha \delta(x-y) \right. \\ &\left. + \frac{2}{m}\left(\frac{1}{2}\theta Q - P R\right)\left(\partial_\rho\partial_\sigma - \delta_{\rho\sigma}\partial^2\right)\delta(x-y)\ \right) \quad = 0 \end{aligned} \tag{32}$$

θ being here a functional of ∂^2/m^2. The solution of this equation is

$$R = -\lambda \frac{Q}{\left(P^2 - \frac{\partial^2}{m^2} Q^2\right)} \qquad\qquad \theta = -2\lambda \frac{P}{\left(P^2 - \frac{\partial^2}{m^2} Q^2\right)} \tag{33}$$

With this choice, the change of variables decouples the b_μ integration so that one can finally get the bosonic action for A_μ which now reads

$$S_{bos}[A] = \int d^3x \left(-(2\lambda)^2 \frac{i}{2} \varepsilon_{\mu\nu\alpha} A_\mu \frac{P}{\left(P^2 - \frac{\partial^2}{m^2} Q^2\right)} \partial_\nu A_\alpha \right. \\ \left. +(2\lambda)^2 \frac{1}{4m} f_{\mu\nu}[A] \frac{Q}{\left(P^2 - \frac{\partial^2}{m^2} Q^2\right)} f_{\mu\nu}[A] \right) \tag{34}$$

This result coincides with that found in ref.[17], obtained by a direct functional integration on b_μ. As it was proven in this last work, it corresponds for massless fermions to the bosonization action proposed in ref.[15] since in the $m \to 0$ limit eq.(34) takes the form

$$S_{bos} = \frac{2}{\pi} \int d^3x (\frac{1}{4} F_{\mu\nu} \frac{1}{\sqrt{-\partial^2}} F_{\mu\nu} - \frac{i}{2} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda) \tag{35}$$

4 Interacting models

One can apply the bosonization approach described above to analyse interacting fermionic models. Let us consider for example the Thirring model with a current-current interaction Lagrangian of the form

$$L_{int} = -\frac{g^2}{2N} j_\mu j_\mu \tag{36}$$

where ψ^i are N two-component Dirac spinors and j^μ the $U(1)$ current,

$$j_\mu = \bar{\psi}^i \gamma^\mu \psi^i. \tag{37}$$

The coupling constant g^2 has dimensions of inverse mass. (Although non-renormalizable by power counting, four fermion interaction models in $2+1$ dimensions are known to be renormalizable in the $1/N$ expansion [30].) One can directly apply the bosonization recipe found in the precedent section to this interaction Lagrangian. Making a choice of λ so as to coincide with the normalization in [1], this meaning

$$j_\mu \to i\sqrt{\frac{N}{4\pi}} \varepsilon_{\mu\nu\alpha} \partial_\nu A_\alpha \tag{38}$$

one has

$$L_{int} \to \frac{g^2}{16\pi} F_{\mu\nu}^2 \tag{39}$$

Alternatively, one can eliminate the quartic fermionic interaction by introducing an auxiliary field a_μ via the identity

$$\exp(\int \frac{g^2}{2N} j^\mu j_\mu d^3x\,) = \int \mathcal{D}a_\mu \exp[-\int (\frac{1}{2}a^\mu a_\mu + \frac{g}{\sqrt{N}} j^\mu a_\mu) d^3x\,] \tag{40}$$

and then proceed to integrate fermions as in the free case thus obtaining a determinant in which the a_μ field can be eliminated by a shift $b_\mu \to b_\mu - a_\mu$. One confirms in this way that the bosonization recipe (39) is correct so that the three dimensional Thirring model is equivalent, in the $1/m$ approximation to a Maxwell-Chern-Simons model. To leading order in $1/m$ we can then write (after rescaling the field A_μ)

$$L_{Th} \to \frac{1}{4} F_{\mu\nu}^2 \pm i \frac{2\pi}{g^2} \epsilon^{\mu\alpha\nu} A_\mu \partial_\alpha A_\nu \tag{41}$$

We can give now a first application of the bosonization formulas and, in this way, explore their physical content. The Lagrangian in (41) has a Chern-Simons term which controls its long distance behavior. It is well known[11, 31] that the Chern-Simons gauge theory is a theory of knot invariants which realizes the representations of the Braid group. These knot invariants are given by expectation values of Wilson loops in the Chern-Simons gauge theory. In this way, it is found that the expectation values of the Wilson loop operators imply the existence of excitations with fractional statistics. Thus, it is natural to seek the fermionic analogue of the Wilson loop operator W_Γ which, in the Maxwell-Chern-Simons theory is given by

$$W_\Gamma = \langle \exp\{ i \frac{\sqrt{N}}{g} \oint_\Gamma A_\mu dx^\mu \} \rangle \tag{42}$$

where Γ is the union of a an arbitrary set of closed curves (loops) in three dimensional euclidean space. Given a closed loop (or union of closed loops) Γ, it is always possible to define a set of open surfaces Σ whose boundary is Γ, *i.e.* $\Gamma = \partial\Sigma$. Stokes' theorem implies that

$$\langle \exp\{ i \frac{\sqrt{N}}{g} \oint_\Gamma A_\mu dx^\mu \} \rangle \;\; = \;\; \langle \exp\{ i \frac{\sqrt{N}}{g} \int_\Sigma dS_\mu \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda \} \rangle$$

$$= \langle \exp\{i\frac{\sqrt{N}}{g}\int d^3x\, \epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda\, b_\lambda\}\rangle \tag{43}$$

is an identity. Here $b_\lambda(x)$ is the vector field

$$b_\lambda(x) = n_\lambda(x)\delta_\Sigma(x) \tag{44}$$

where n_λ is a field of unit vectors normal to the surface Σ and $\delta_\Sigma(x)$ is a delta function with support on Σ. Using eq.(38) we find that this expectation value becomes, in the Thirring Model, equivalent to

$$W_\Gamma = \langle \exp\{i\frac{\sqrt{N}}{g}\oint_{\partial\Sigma} dx_\mu A^\mu\}\rangle_{MCS} = \langle \exp\{\int_\Sigma dS_\mu \bar{\psi}\gamma^\mu\psi\}\rangle_{Th} \tag{45}$$

More generally we find that the Thirring operator $\mathcal{W}_\Sigma$

$$\mathcal{W}_\Sigma = \langle \exp\{q\int_\Sigma dS_\mu \bar{\psi}\gamma^\mu\psi\}\rangle_{Th} \tag{46}$$

obeys the identity

$$\langle \exp\{q\int_\Sigma dS_\mu \bar{\psi}\gamma^\mu\psi\}\rangle_{Th} = \langle \exp\{iq\frac{\sqrt{N}}{g}\oint_{\partial\Sigma} A_\mu dx^\mu\}\rangle_{MCS} \tag{47}$$

for an arbitrary fermionic charge q.

The identity (47) relates the flux of the fermionic current through an open surface Σ with the Wilson loop operator associated with the boundary Γ of the surface. The Wilson loop operator can be trivially calculated in the Maxwell-Chern-Simons theory. For very large and smooth loops the behavior of the Wilson loop operators is dominated by the Chern-Simons term of the action. The result is a topological invariant which depends only on the linking number ν_Γ of the set of curves Γ [11, 31]. By an explict calculation one finds

$$\langle \exp\{q\int_\Sigma dS_\mu \bar{\psi}\gamma^\mu\psi\}\rangle_{Th} = \exp\{\mp i\nu_\Gamma\frac{Nq^2}{8\pi}\} \tag{48}$$

This result implies that the non-local Thirring loop operator $\mathcal{W}_\Sigma$ exhibits fractional statistics with a statistical angle $\delta = Nq^2/8\pi$. The topological significance of this result bears close resemblance with the bosonization identity in $1+1$ dimensions between the circulation of the fermionic current on a closed curve and the topological charge (or instanton number) enclosed in the interior of the curve [32]. From the point of view of the Thirring model, this is a most surprising result which reveals the power of the bosonization identities. To the best of our knowledge, this is the first example of a purely fermionic operator, albeit non-local, which is directly related to a topological invariant.

5 Current algebra

We have seen in precedent sections that in 3 dimensional space-time the fermion action bosonizes, in the large m limit, to a Maxwell-Chern-Simons theory. Now, the gauge invariant algebra of such theory has been studied in refs.[24],[28]. One has for instance with our conventions,

$$[E_i(\vec{x},t), B(\vec{y},t)] = -3\,|m|\epsilon_{ij}\partial_j\delta^{(2)}(\vec{x}-\vec{y}) \tag{49}$$

If one now relates the electric field $E_i = F_{i0}$ and the magnetic field $B = \epsilon_{ij}\partial_i A_j$ to the fermionic currents through the bosonization recipe for the fermion current,

$$j_o \to \frac{1}{\sqrt{4\pi}}B \tag{50}$$

$$j_i \to \frac{1}{\sqrt{4\pi}}\epsilon_{ij}E_j \tag{51}$$

then, the resulting fermion current commutator algebra is not the one to be expected for three-dimensional free fermions. Indeed, the $d = 3$ fermion current algebra should contain an infinite Schwinger term [33]-[35] which is absent in eq.(49). The point is that calculations leading to a bosonic theory of the Maxwell-Chern-Simons type are valid only for large fermion mass while calculation of equal-time current commutators imply, as we shall see, a limiting procedure which cannot be naively followed for large masses.

Since the exact bosonic partition function is much too complicated to handle, a possible strategy is to use the quadratic (in auxiliary fields) approximation mentioned in the precedent section working with an arbitrary (not necessarily large) mass so as to obtain a bosonized version of the original fermionic model in which the equal-time limit can be safely taken. One should then compute current commutators for this bosonized theory, and test whether they coincide with those satisfied by fermionic currents in the original model. Details of this calculation can be found in [4]. I will just sketch here the principal steps leading to the consistent equal-time current commutators in the bosonic language.

As explained above, within the quadratic (in b_μ) approximation, one can write the fermionic partition function in terms of the bosonic fields A_μ in the form [17]

$$\begin{aligned} Z_{fer} &= \int DA_\mu \exp\left[-\int d^3x(\frac{1}{4}F_{\mu\nu}C_1F_{\mu\nu} - \frac{i}{2}A_\mu C_2\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda \right. \\ & \left. +is_\mu\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda)\right] \end{aligned} \tag{52}$$

with C_1 and C_2 now given through their momentum-space representation $\tilde{C}_1$ and $\tilde{C}_2$

$$\tilde{C}_1(k) = \frac{1}{4\pi}\frac{\tilde{F}(k)}{k^2\tilde{F}^2(k)+\tilde{G}^2(k)} \tag{53}$$

$$\tilde{C}_2(k) = \frac{1}{4\pi}\frac{\tilde{G}(k)}{k^2\tilde{F}^2(k)+\tilde{G}^2(k)} \tag{54}$$

and $\tilde{F}(k)$ and $\tilde{G}(k)$ given by [17]

$$\tilde{F}(k) = \frac{\mid m \mid}{4\pi k^2}\left[1 - \frac{1 - \frac{k^2}{4m^2}}{(\frac{k^2}{4m^2})^{\frac{1}{2}}} \arcsin(1 + \frac{4m^2}{k^2})^{-\frac{1}{2}}\right],$$

$$\tilde{G}(k) = \frac{q}{4\pi} + \frac{m}{2\pi \mid k \mid} \arcsin(1 + \frac{4m^2}{k^2})^{-\frac{1}{2}}$$

Being quadratic in A_μ, eq.(52) can be integrated leading to

$$Z_{bos}[s] = [detD_{\mu\nu}]^{-\frac{1}{2}} \exp\left[\frac{1}{8\pi}\int d^3xd^3y\partial_\nu s_\mu(x)\epsilon_{\mu\nu\lambda}D^{-1}_{\lambda\rho}(x,y)\partial_\sigma s_\tau(y)\epsilon_{\rho\sigma\tau}\right] \tag{55}$$

where $D^{-1}_{\mu\nu}$ is just the propagator of the bosonic action which, in the Lorentz gauge we adopt from here on, reads

$$D^{-1}_{\mu\nu}(x,y) = \int \frac{d^3k}{(2\pi)^3}\left[P(k)g_{\mu\nu} + Q(k)k_\mu k_\nu + R(k)\epsilon_{\mu\nu\alpha}k_\alpha\right] \exp ik(x-y) \tag{56}$$

with

$$P(k) = \frac{\tilde{C}_1(k)}{k^2\tilde{C}_1^2(k) + \tilde{C}_2^2(k)} = 4\pi\tilde{F}(k) \tag{57}$$

$$Q(k) = \frac{\tilde{C}_1(k)}{k^2\tilde{C}_1^2(k) + \tilde{C}_2^2(k)} \left(\frac{\tilde{C}_2(k)}{k^2\tilde{C}_1(k)}\right)^2 \tag{58}$$

$$R(k) = \frac{\tilde{C}_2(k)}{k^2(k^2\tilde{C}_1^2(k) + \tilde{C}_2^2(k))} \tag{59}$$

Let us briefly recall how one can compute current commutators within the path-integral scheme using the so-called BJL method [36]-[38]. To this end we define the correlator

$$G_{\mu\nu}(x,y) = \left.\frac{\delta^2 log Z_{fer}[s]}{\delta s_\mu(x)\delta s_\nu(y)}\right|_{s=0} \tag{60}$$

from which one can easily derive equal time current commutators using the relation

$$< [j_0(\vec{x},t), j_i(\vec{y},t)] >= \lim_{\epsilon\to 0^+}[G_{0i}(\vec{x},t+\epsilon;\vec{y},t) - G_{0i}(\vec{x},t-\epsilon;\vec{y},t)] \tag{61}$$

The current commutator evaluated using eqs.(60)-(61) corresponds to $Z_{fer}[s]$ written in terms of bosonic fields. That is, eq.(61) gives the equal-time commutator for the bosonic currents $j_\mu = (1/\sqrt{4\pi})\epsilon_{\mu\nu\alpha}\partial_\nu A_\alpha$. This result should then be compared with that arising in the original 3-dimensional fermionic model for which $j_\mu = -i\bar{\psi}\gamma_\mu\psi$ [35].

Starting from eqs.(55)-(56) and using the BJL method we get, after some calculations,

$$G_{\mu\nu}(x,y) = -\frac{1}{4\pi}\epsilon_{\mu\alpha\rho}\epsilon_{\nu\beta\sigma}\partial_\alpha\partial_\beta D^{-1}_{\rho\sigma} \tag{62}$$

or

$$G_{\mu\nu}(x,y) = \frac{1}{4\pi}\int \frac{d^3k}{(2\pi)^3}[P(k)(k^2 g_{\mu\nu} - k_\mu k_\nu) + k^2 R(k)\epsilon_{\mu\nu\alpha}k_\alpha]\exp[ik(x-y)] \tag{63}$$

With this, we can rewrite eq.(61) in the form

$$< [j_0(\vec{x},t), j_i(\vec{y},t] >= \lim_{\epsilon\to 0^+} I^\epsilon(\vec{x}-\vec{y}) \tag{64}$$

with

$$I^\epsilon(\vec{x}) = -2i\int \frac{d^3k}{(2\pi)^3}k_0 k_i \sin(k_0\epsilon)\tilde{F}(k)\exp{(i\vec{k}.\vec{x})} \tag{65}$$

where we have written $(k_\mu) = (k_o, k_i)$, $i = 1,2$. It will be convenient to define

$$k_0' = \epsilon k_0 \tag{66}$$

In terms of this new variable and using the explicit form for $\tilde{F}(k)$ given in [17],[4], with $k = (k_0^2 + \vec{k}^2)^{1/2}$, integral I^ϵ becomes

$$I^\epsilon(\vec{x}) = -\frac{1}{8\pi^2|m|}\frac{1}{\epsilon^2}\partial_i\int \frac{d^2k}{(2\pi)^2}\exp i(\vec{k}.\vec{x})\int_0^\infty dk_0' k_0' \sin k_0' f(y) \tag{67}$$

where

$$f(y) \;=\; \frac{1}{y}\left[1 - \frac{(1-y)}{\sqrt{y}}\arcsin\frac{1}{\sqrt{1+(1/y)}}\right] \tag{68}$$

and we have defined

$$y = \frac{k^2}{4m^2} = \frac{k_0'^2 + \epsilon^2\vec{k}^2}{4\epsilon^2 m^2} \tag{69}$$

One can now see that $y \to \infty$ for $\epsilon \to 0$ and fixed m. Then, expanding in powers of $1/y$ one has $f(y) \sim \pi/(2\sqrt{y})$ and then using distribution theory to define the integral over k_0' one finds

$$< [j_0(\vec{x},t), j_i(\vec{y},t] >= -\frac{1}{8\pi}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\partial_i\delta^{(2)}(\vec{x}-\vec{y}) \tag{70}$$

This result for the equal-time current commutator, evaluated within the bosonized theory, shows exactly the infinite Schwinger term that is found, using the BJL method, for free fermions in $d = 3$ dimensions [35]. As it happens in $d = 4$ dimensions [34], we see from eq.(70) that the commutator at *unequal* times is well defined: divergencies appear only when one takes the equal-time limit.

One can evaluate also the next order vanishing in the equal-time current commutator so as to compare it with the result from the original fermion model reported in the literature [35]. The answer is [4]

$$\begin{aligned} < [j_0(\vec{x},t), j_i(\vec{y},t)] > \;&=\; -\frac{1}{8\pi}\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\partial_i\delta^{(2)}(\vec{x}-\vec{y})\right. \\ &\left. -\frac{\epsilon}{\Lambda}[4m^2\partial_i\delta^{(2)}(\vec{x}-\vec{y}) - \frac{1}{2}\partial_i\Delta\delta^{(2)}(\vec{x}-\vec{y})]\right) \end{aligned} \tag{71}$$

where we have defined

$$\frac{1}{\Lambda} = \int_0^\infty dk'_0 \frac{1}{k'^2_0} \sin k'_0 \tag{72}$$

In order to compare with ref.[35] where current commutators were computed using dimensional regularization, we define, coming back to the original variable $k_0 = k'_0/\epsilon$

$$A[d] = \frac{1}{2} \int d^{d-2} k_0 \frac{1}{k_0^2} sin k_0 \epsilon \tag{73}$$

so that $A[d=3] = \epsilon/\Lambda$. One can now perform the analytically continued integral to find, near $d = 3$, the behavior

$$A[d] \sim -\epsilon \times \frac{\epsilon^{3-d}}{3-d} \tag{74}$$

The same ambiguous result for free fermions is obtained in ref.[35] near $d = 3$. This ambiguity can be however removed, the pole in dimensional regularization corresponding as usual to a logarithmic divergence. It is also interesting to note that if one uses the nice approximation $\tilde{F}_{appr}$ for $\tilde{F}$ proposed in ref.[17], one can well check the correctness of our previous analysis [4].

From the analysis above, we see that not only the infinite Schwinger term, analogous to that arising in $d = 4$ [34] is obtained in the bosonized version of our $d = 3$ fermion theory but also the mass-dependent second term as well as the triple derivative third term, both vanishing in the equal time limit. Our analysis should be compared with that in ref.[39] where the fermionic commutator algebra is inferred from the Maxwell-Chern-Simons algebra for electric and magnetic fields using a bosonization recipe which is valid in the large mass limit. One can see that in the large mass regime, terms depending on the product $\epsilon m = \lambda$ will produce ambiguities according to the way both limits ($\epsilon \to 0$ and $m \to \infty$) are taken into account, a problem which is not present in the limit of small masses. To see this in more detail, let us come back to (65) and consider the case in which λ is kept fixed while $\epsilon \to 0$ (so that $m \to \infty$). In this case, taking the limit before integrating out k'_0, one finds for I^ϵ

$$I^\epsilon(\vec{x}) \sim |m| h(\lambda) \partial_i \delta^{(2)}(\vec{x}) \tag{75}$$

where

$$h(\lambda) = \frac{1}{2\pi} \int_0^\infty dz z \sin(2\lambda z) f(z) \tag{76}$$

with f given by eq.(68). Let us note that using the approximate $\tilde{F}_{appr}$ of ref.[17] and taking the limit after the exact integration over k_0', we recover the same behavior (75). We see that for $\lambda = \epsilon m$ fixed, h just gives a normalization factor so that one reproduces from I^ϵ in the form (75) a commutator algebra at equal times and large mass that coincides with that to be infered from a Maxwell-Chern-Simons theory,

$$< [j_0(\vec{x}, t), j_i(\vec{y}, t] > \longrightarrow c|m|\partial_i \delta^{(2)}(\vec{x} - \vec{y}) \quad (m \to \infty) \tag{77}$$

with c a normalization constant. Again, currents appearing in the l.h.s. of eq.(77) are bosonic currents which can be written in terms of the electric and magnetic fields thus reproducing the MCS gauge invariant algebra [24],[28]. One should note however that the free fermion - MCS mapping is valid in the large mass limit of the original fermionic theory, this meaning the large-distances regime for fermion fields. Since current commutators test the short-distance regime, one should not take the MCS gauge-invariant algebra as a starting point to reproduce the fermion current commutators.

6 The non-Abelian case in $d = 3$

In three dimensional space-time, the bosonic action (14) takes the form

$$\exp(-S_{bos}[A]) = \int \mathcal{D}b_\mu \mathcal{D}\bar{c}_\mu \mathcal{D}c_\mu \exp\left(-tr\int d^3x \left(L[b] \pm \frac{i}{8\pi}\varepsilon_{\mu\nu\alpha}\bar{c}_\mu D_\nu[b] c_\alpha \right.\right.$$
$$\left.\left. \mp \frac{i}{16\pi}(A_\mu - b_\mu)^*\!f_\mu[b]\right)\right) \quad (78)$$

Here ghost fields $\bar{c}_\mu$ and c_μ were introduced to represent the Faddeev-Popov like determinant $\Delta[b]$. Again, we have written

$$tr\int d^3x\, L[b] = -\log\det(i\partial\!\!\!/ + m + b\!\!\!/) \quad (79)$$

and we have chosen the arbitrary constant λ appearing in (14) so as to make contact with the conventions of ref.[7], $\lambda = \frac{i}{16\pi}$. Moreover, we have shifted the bosonic field $A_\mu \to A_\mu - b_\mu$ (this amounting to a trivial Jacobian) for reasons that will become clear below.

It was observed in ref.[6] that when $L[b]$ is approximated by its first term in the $1/m$ expansion, a set of BRST transformations can be defined so that the corresponding BRST invariance allows to obtain the (approximate) bosonic action. We shall explicitly prove here that this invariance is present in (78) where no approximation for $L[b]$ is assumed. To this end, we introduce a set of auxiliary fields h_μ (taking values in the Lie algebra of G), l and $\bar{\chi}$ so that one can rewrite (78) in the form

$$\exp(-S_{bos}[A]) = \int \mathcal{D}b_\mu \mathcal{D}\bar{c}_\mu \mathcal{D}c_\mu \mathcal{D}h_\mu \mathcal{D}l \mathcal{D}\bar{\chi} \exp(-S_{eff}[A, b, h, l, \bar{c}, c, \bar{\chi}]) \quad (80)$$

with

$$S_{eff}[A, b, h, l, \bar{c}, c, \bar{\chi}] = tr\int d^3x \left(L[b-h] \pm \frac{i}{8\pi}\varepsilon_{\mu\nu\alpha}\bar{c}_\mu D_\nu[b] c_\alpha \right.$$
$$\left. \mp \frac{i}{16\pi}((A_\mu - b_\mu)^*\!f_\mu[b] + l h_\mu^2 - 2\bar{\chi} h_\mu c_\mu)\right) \quad (81)$$

where $\bar{\chi}$ is an anti-ghost field. Written in the form (81), the bosonic action has a BRST invariance under the following nilpotent off-shell BRST transformations

$$\delta\bar{c}_\mu = A_\mu - b_\mu, \qquad \delta b_\mu = c_\mu, \qquad \delta A_\mu = c_\mu, \qquad \delta c_\mu = 0, \qquad \delta\bar{\chi} = l$$

$$\delta h_\mu = c_\mu, \qquad \delta l = 0 \tag{82}$$

In view of this BRST invariance, one could add to S_{eff} a BRST exact form without changing the dynamics defined by $S_{bos}[A]$. Exploiting this, we shall see that one can factor out the A_μ dependence in the r.h.s. of eq.(81) so that it completely decouples from the path-integral over b_μ auxiliary and ghost fields exactly as we did in the Abelian case. Although complicated, this integral then becomes irrelevant for the definition of the bosonic action for A_μ. Indeed, let us add to S_{eff} the BRST exact form δG,

$$S_{eff}[A, b, h, l, \bar{c}, c, \chi] \to S_{eff}[A, b, h, l, \bar{c}, c, \chi] + \delta G[A, b, h, \bar{c}] \tag{83}$$

with

$$G[A, b, h, \bar{c}] = \mp \frac{i}{16\pi} tr \int d^3x \, \varepsilon_{\mu\nu\alpha} \bar{c}_\mu H_{\nu\alpha}[A, b, h] \tag{84}$$

and $H_{\nu\alpha}[A, b, h]$ a functional to be determined in order to produce the decoupling. Then, consider the change of variables (analogous to (21) for the Abelian case)

$$b_\mu = 2b'_\mu - A_\mu + V_\mu[A] \tag{85}$$

where $V_\mu[A]$ is some functional of A_μ changing covariantly under gauge transformations,

$$V_\mu[A^g] = g^{-1} V_\mu[A] g \tag{86}$$

so that b'_μ is, like A_μ and b_μ, a gauge field. Integrating over l in (80) and imposing the resulting constraint, $h_\mu = 0$, one sees that if one imposes on $H_{\nu\alpha}[A, b, h]$ the condition

$$\varepsilon_{\mu\nu\alpha} \int d^3y \left(\frac{\delta H_{\nu\alpha}}{\delta b^a_\rho(y)} + \frac{\delta H_{\nu\alpha}}{\delta A^a_\rho(y)} + \frac{\delta H_{\nu\alpha}}{\delta h^a_\rho(y)} \right) c^a_\rho(y) \,|_{h=0} = \varepsilon_{\mu\nu\rho} [A_\nu - b_\nu - V_\nu[A], c_\rho] \tag{87}$$

then, when written in terms of the new b'_μ variable, the ghost term becomes

$$S_{ghost}[b', c, \bar{c}] = \pm \frac{i}{8\pi} tr \int d^3x \, \varepsilon_{\mu\nu\alpha} \bar{c}_\mu D_\nu[b'] c_\alpha \tag{88}$$

so that its contribution is still A_μ independent. Then, we can write the effective action in the form

$$S_{eff}[b', A] + S_{ghost}[b', c, \bar{c}] \tag{89}$$

with

$$\begin{aligned} S_{eff}[b', A] &= \tilde{S}[b, A] \\ &= tr \int d^3x \left(L[b] \mp \frac{i}{16\pi} (A_\mu - b_\mu)({}^*f_\mu[b] +{}^* H_\mu[A, b, 0]) \right) \end{aligned} \tag{90}$$

where ${}^*H_\mu = \varepsilon_{\mu\nu\alpha} H_{\nu\alpha}$.

Condition (87) made the ghost term independent of the bosonic field A_μ. We shall now impose a second constraint in order to completely decouple the auxiliary field b'_μ from A_μ in S_{eff}. Indeed, consider the conditions

$$\frac{\delta^2 S_{eff}[b', A]}{\delta A^a_\rho(y) \delta\, b'^b_\sigma(x)} = 0 \tag{91}$$

In terms of the original auxiliary field b_μ these equations read

$$\frac{\delta^2 \tilde{S}[b, A]}{\delta A^a_\rho(y) \delta b^b_\sigma(x)} - \frac{\delta^2 \tilde{S}[b, A]}{\delta b^a_\rho(y) \delta b^b_\sigma(x)} + \int d^3u \frac{\delta^2 \tilde{S}[b, A]}{\delta b^c_\beta(u) \delta b^b_\sigma(x)} \frac{\delta V^c_\beta(u)}{\delta A^a_\rho(y)} = 0 \tag{92}$$

Eqs.(92) can be easily written in terms of L, H and V as a lengthy equation that we shall omit here.

The strategy is now as follows: once a given approximate expression for the fermion determinant is considered, one should solve eq.(92) in order to determine functionals V in eq.(85) and G in eq.(84), taking also in account the condition (87). In particular, if one considers the $1/m$ expansion for the fermion determinant, equations (87) and (92) should determine the form of V and G as a power expansion in $1/m$. In ref.[25] the $1/m$ expansion for the fermion determinant was shown to give

$$\ln \det(i\partial\!\!\!/ + m + b\!\!\!/) = \pm \frac{i}{16\pi} S_{CS}[b] + I_{PC}[b] + O(\partial^2/m^2), \tag{93}$$

where the Chern-Simons action S_{CS} is given by

$$S_{CS}[b] = \varepsilon_{\mu\nu\lambda}\, \mathrm{tr} \int d^3x\, (f_{\mu\nu} b_\lambda - \frac{2}{3} b_\mu b_\nu b_\lambda). \tag{94}$$

Concerning the parity conserving contributions, one has

$$I_{PC}[b] = -\frac{1}{24\pi m}\, \mathrm{tr} \int d^3x\, f^{\mu\nu} f_{\mu\nu} + \cdots, \tag{95}$$

To order zero in this expansion, solution of eqs.(87),(92) is very simple. Indeed, in this case the fermion determinant coincides with the CS action and one can easily see that the solution is given by

$$V_\mu^{(0)}[A] = 0 \tag{96}$$

$$G^{(0)}[A,b,h,\bar{c}] = \pm\frac{i}{16\pi} tr\int d^3x\, \bar{c}_\mu \,(\frac{1}{2}\,{}^*f_\mu[A] + \frac{1}{2}\,{}^*f_\mu[b] - 2\,{}^*D_{\mu\alpha}[A]h_\alpha) \tag{97}$$

With this, the change of variables (85) takes the simple form

$$b_\mu = 2b'_\mu - A_\mu \tag{98}$$

and the decoupled effective action reads

$$S_{eff}^{(0)}[b,A,\bar{c},c] = \mp\frac{i}{16\pi}(2S_{CS}[b'] - S_{CS}[A]) + S_{ghost}[b'] \tag{99}$$

We then see that the path-integral defining the bosonic action $S_{bos}[A]$, factors out so that one ends with a bosonic action in the form

$$S_{bos}^{(0)}[A] = \pm\frac{i}{16\pi}S_{CS}[A] \tag{100}$$

as advanced in [2],[6]. Let us remark that in finding the solution for G one starts by writing the most general form compatible with its dimensions,

$$\begin{aligned} G^{(0)}[A,b,h,\bar{c}] = tr\int d^3x\, \varepsilon_{\mu\nu\alpha}\bar{c}_\mu\,(d_1 b_\nu A_\alpha + d_2 A_\nu b_\alpha + d_3 b_\nu b_\alpha + d_4 A_\nu A_\alpha \\ + d_5 b_\nu h_\alpha + d_6 h_\nu b_\alpha + d_7 A_\nu h_\alpha + d_8 h_\nu A_\alpha + d_9 \partial_\nu A_\alpha + d_{10}\partial_\nu b_\alpha + d_{11}\partial_\nu h_\alpha) \end{aligned} \tag{101}$$

All the arbitrary parameters d_i are determined by imposing the conditions (87) and (92) with ${}^*H_\mu$ transforming covariantly (as h_μ does) under gauge transformations which leads, together with a gauge invariant action, to the solution (97).

To go further in the $1/m$ expansion one uses the next to the leading order in the fermion determinant as given in eq.(93). Again, starting from the general form of G and after quite lengthy calculations that we shall not reproduce here, one can find a unique solution for V_μ and $H_{\nu\alpha}$ leading to a gauge invariant action,

$$V_\mu^{(1)}[A] = \pm\frac{2i}{3m}\,{}^*f_\mu[A] \tag{102}$$

$$G^{(1)}[A,b,h,\bar{c}] = G^{(0)}[A,b,h,\bar{c}] \mp \frac{1}{96\pi m} tr \int d^3x \, \bar{c}_\mu \varepsilon_{\mu\nu\alpha} \varepsilon_{\nu\rho\sigma}$$
$$\Big(\frac{1}{2} [\, f_{\rho\sigma}[A-h] + 3 f_{\rho\sigma}[b-h] - 2 D_\rho[A-h](A_\sigma - b_\sigma) \, , \, (A_\alpha - b_\alpha) \,]$$
$$+ 4 [\, f_{\rho\sigma}[A-h] \, , \, h_\alpha \,] \Big) \tag{103}$$

The corresponding change of variables (85) takes now the form

$$b_\mu = 2b'_\mu - A_\mu \pm \frac{2i}{3m} {}^*f_\mu[A] \tag{104}$$

and the decoupled effective action reads

$$S^{(1)}_{eff}[b,A,\bar{c},c] = S^{(0)}_{eff}[b,A,\bar{c},c] + tr \int d^3x \left(\frac{1}{6\pi m} f^2_{\mu\nu}[b'] + \frac{1}{24\pi m} f^2_{\mu\nu}[A] \right) \tag{105}$$

so that one can again integrate out the completely decoupled ghosts and b' fields ending with the bosonic action

$$S^{(1)}_{bos}[A] = \pm \frac{i}{16\pi} S_{CS}[A] + \frac{1}{24\pi m} tr \int d^3x \, f^2_{\mu\nu}[A] \tag{106}$$

This result extends to order $1/m$ the bosonization recipe presented in refs. [2],[6].

In this way, from the knowledge of the $1/m$ expansion of the fermion determinant one can systematically find order by order the decoupling change of variables and construct the corresponding action for the bosonic field A_μ. One finds for the change of variables

$$b_\mu = 2b'_\mu - A_\mu \pm \frac{2i}{3m} {}^*f_\mu[A] + \frac{1}{m^2} C^{(2)} D_\rho[A] f_{\mu\rho}[A] + \ldots \tag{107}$$

Here $C^{(2)}$ is a (dimensionless) constant to be determined from the $1/m^2$ term in the fermion determinant expansion, which should be proportional to ${}^*f_\mu D_\rho f_{\rho\mu}$. Evidently, finding the BRST exact form becomes more and more involved and so is the form of the bosonic action which however, can be compactly written as

$$S_{bos}[A] = tr \int d^3x \, \Big(\, L[-A + V[A]]$$
$$\mp \frac{i}{16\pi} (2A_\mu - V_\mu[A]) ({}^*f_\mu[-A + V[A]] + {}^*H_\mu[-A + V[A], A, 0]) \Big) \tag{108}$$

Let us end this section by writing the bosonization recipe for the fermion current accompanying this result for the bosonic action. From eq.(15) we have, in $d=3$

$$\bar{\psi}^i \gamma_\mu t^a_{ij} \psi^j \to \pm \frac{i}{8\pi} \varepsilon_{\mu\nu\alpha} \partial_\nu A^a_\alpha \tag{109}$$

7 Wilson loops

I will transcribe in this section some results obtained in [2] concerning the evaluation of Wilson loops in the framework of our bosonization approach. In the Chern-Simons theory, they measure topological invariants determined by the linkings of the loops and by the topology of the base manifold [31]. For one loop Γ,

$$W[\Gamma] = \mathrm{tr}\, P \exp(i \oint_\Gamma dx^\mu\, A_\mu) \tag{110}$$

where P denotes the path ordering of the exponential, and the trace is taken in the representation carried by the loop. According to the bosonization prescription, to relate this operator to the fermionic theory we must express $W[\Gamma]$ in terms of the field strength $F_{\mu\nu}$ rather than the potential A_μ. In the abelian case this can always be done by means of Stokes theorem. As discussed in ref. [1], this leads to an explicit mapping between abelian Wilson loops and non-local fermionic operators. Hence, in this way, the latter are related to the linking of loops and thus probe the generalized statistics of the external particles that propagate along those loops. One way to extend that analysis to the non-abelian case is to use the non-abelian extension of Stokes theorem developed in [40]. For an arbitrary loop $\Gamma = \partial\Sigma$, the boundary of a surface Σ, one has

$$W[\partial\Sigma] = \mathrm{tr}\, P_t \exp\{i \int_0^1 dt \int_0^1 ds \frac{\partial\Sigma^\mu}{\partial s}\frac{\partial\Sigma^\nu}{\partial t} W^{-1}[{}_s\Sigma(t)_0] F_{\mu\nu}(\Sigma(t,s)) W[{}_s\Sigma(t)_0]\} \tag{111}$$

Here Σ is looked upon as a sheet, that is, a one parameter family of paths parametrized by t, $0 \le t \le 1$. For each t, $\Sigma(t)$ is a path, itself parametrized by s, $0 \le s \le 1$, with fixed end-points: $\partial\Sigma(t,s)/\partial t = 0$ at $s = 0,1$. For a given t, ${}_s\Sigma(t)_0$ denotes the segment of the path $\Sigma(t)$ connecting the points $\Sigma(t,0)$ and $\Sigma(t,s)$, and $W[{}_s\Sigma(t)_0]$ is the corresponding (open) Wilson line.

Finally, P_t in eq. (111) denotes ordering of the t integral, while the s integral is not ordered (although there is an s-ordering inside each $W[{}_s\Sigma(t)_0]$.)

In the abelian case the two open Wilson lines $W[{}_s\Sigma(t)_0]$ in eq. (111) cancel each other and one recovers the usual Stokes theorem, involving only the gauge field strength. In the non-abelian case, however, the factors $W[{}_s\Sigma(t)_0]$ are needed for gauge invariance, and introduce an explicit dependence of the Wilson loop operator on the gauge potential A_μ. Thus, as opposed to the abelian case, the non-abelian Wilson loop operator cannot be mapped in a straightforward way to a fermionic operator through the bosonization rule in eq. (109).

For planar loops this difficulty is only apparent. Indeed, consider $W[\partial\Sigma]$, with Σ contained, say, in the $(1,2)$ plane. Imposing the $A_3 = 0$ gauge condition, there is a remnant gauge freedom for the A_1 and A_2 components in the $(1,2)$ plane, which is the symmetry of a 2-dimensional gauge theory in that plane. As discussed in [40], one can use that gauge symmetry, together with the freedom of parametrization of the surface Σ, so the open Wilson line elements in the right hand side of eq. (111) become the identity. More precisely, choosing the gauge condition $A_2 = 0$ on the Σ-plane, eq. (111) can be simplified to

$$W[\partial\Sigma] = \mathrm{tr}\, P_t \exp\{i\int_0^1 dt \int_0^1 ds \frac{\partial\Sigma^\mu}{\partial s}\frac{\partial\Sigma^\nu}{\partial t} F_{\mu\nu}(\Sigma(t,s))\} \tag{112}$$

provided that Σ is parametrized so as to have $\partial\Sigma/\partial t$ and $\partial\Sigma/\partial s$ parallel to the x_1 and x_2 axis, respectively. This apparent breaking of rotational invariance, which includes the presence of t-ordering but not of s-ordering, is a consequence of the $A_2 = 0$ gauge condition on the Σ-plane, and will be removed by the functional integral over the gauge fields. One more ingredient is need: by an appropriate addition of a BRST exact form, the bosonization rule (109) can be written in the covariant way

$$\bar\psi^i \gamma_\mu t^a_{ij} \psi^j \to \pm \frac{i}{16\pi} \varepsilon_{\mu\nu\alpha} F^a_{\nu\alpha} \tag{113}$$

Then, writing

$$\mathcal{J}[\Sigma] = \mathrm{tr}\, P_t \exp\{\pm 16\pi \int_0^1 dt \int_0^1 ds \frac{\partial\Sigma^\mu}{\partial s}\frac{\partial\Sigma^\nu}{\partial t} \epsilon_{\mu\nu\lambda} j_\lambda[\Sigma(t,s)]\} \tag{114}$$

with j_μ the fermionic current , the bosonization formula gives

$$\langle \mathcal{J}[\Sigma]\rangle_{ferm} = \langle W[\partial\Sigma]\rangle_{CS} \tag{115}$$

where in the left hand side the subindex 'ferm' stands for free fermions. This is the non-abelian generalization of the result obtained in [1]. It relates a suitably defined non-abelian flux of the fermionic current through a flat surface, and the Wilson loop associated to the boundary of that surface, with both quantities in the same representation of the group.

It should be stressed that the bosonic side of this relation is, by definition, independent of the surface Σ and its parametrization. In the fermionic side, however, this is not obvious. The relation was derived assuming a flat surface Σ, and it is tempting to assume that this may be extended to smooth deformations away from the plane. But more relevant is the apparent breaking of rotational invariance in the fermionic side due to the remaining t-ordering in eq. (114). This should certainly be expected to be taken care of by the particular parametrization assumed above for Σ. Indeed, one should expect that the very need of a parametrization and of a matching ordering of the surface integral of the fermionic current, is just a limitation of our present analysis. In addition, as is well known, the expectation value of the Wilson loop is singular and must be regularized. A natural and consistent regularization scheme is provided by the framing of the loop [31]. In the case of a non-intersecting loop on a plane, considered here, that framing can be chosen also as a plane loop, not intersecting itself nor the original loop. Again, it is not clear at this point how these singularities in the bosonic side will show up in the (free) fermionic side, and what role will the framing play from the fermionic point of view.

It is natural to ask whether this analysis can be extended to several loops and their possible linkings, as done in [1] for the Abelian case. In the bosonic side one is interested in the expectation value $\langle W[\Gamma_1]W[\Gamma_2]\rangle_{CS}$ or, better yet, in the ratio

$$\frac{\langle W[\Gamma_1]W[\Gamma_2]\rangle_{CS}}{\langle W[\Gamma_1]\rangle_{CS}\langle W[\Gamma_2]\rangle_{CS}} \tag{116}$$

For non intersecting loops this is a well defined, non singular object in the Chern-Simons theory, which depends only on the linking of the two loops Γ_1 and Γ_2 [31]. Assuming this to be non-trivial (and non-singular), the two loops cannot be flat and lying on the same plane, so the previous construction fails. But once the ratio (116) has been computed in the Chern-Simons theory, we can take the limit in which the two loops collapse onto a single plane. This is a singular limit in which the loops necessarily intersect each other. Their linking is not well defined any more, and the value of (116) depends

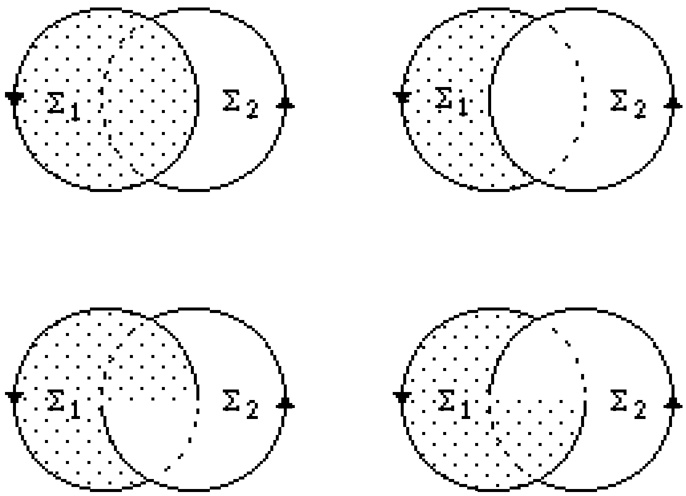

Figure 1: Different overlaps of the surfaces Σ_1 and Σ_2 on a plane, determined by the possible liftings of the loops $\partial\Sigma_1$ and $\partial\Sigma_2$ away from the plane.

on the initial non-singular loops used in the computation. However, at the classical level, before the functional integral is performed, we can repeat the previous construction with no difficulties for any arrangement of loops on the plane [40]. Thus, formally we can write

$$\frac{\langle \mathcal{J}[\Sigma_1 \cup \Sigma_2] \rangle_{ferm}}{\langle \mathcal{J}[\Sigma_1] \rangle_{ferm} \langle \mathcal{J}[\Sigma_2] \rangle_{ferm}} = \frac{\langle W[\partial\Sigma_1] W[\partial\Sigma_2] \rangle_{CS}}{\langle W[\partial\Sigma_1] \rangle_{CS} \langle W[\partial\Sigma_2] \rangle_{CS}} \tag{117}$$

where both surfaces Σ_1 and Σ_2 are contained in the same plane. As we just stated, the bosonic side of this relation will be ill defined in general. But it can be given a well defined meaning by lifting the loops $\partial\Sigma_i$ from the plane to non intersecting three-dimensional loops Γ_i. This can be done in different ways, specifying different linkings of the loops Γ_i compatible with the intersections of their projections $\partial\Sigma_i$ onto the plane. Correspondingly, in the fermionic side, the surface $\Sigma_1 \cup \Sigma_2$ must be complemented with a prescription stating the way in which the two surfaces Σ_i overlap. The different possible liftings of the loops specify different overlaps of the surfaces, as ilustrated in Fig. (1). In this way, relation (117) (and its generalizations) can be viewed as a defining relation, through bosonization, of the vaccuum expectation value of the flux of the fermionic current through surfaces with foldings.

8 Bosonization in $d = 4$

In ref.[5] we have applied our bosonization approach to the study of vector and axial-vector currents in $d = 4$ dimensions. I will briefly describe the main results of this work. Consider the generating functional for a massless Dirac field in $3+1$ (Euclidean) dimensions, coupled to Abelian vector (s_μ) and axial-vector (t_μ) external sources

$$\begin{aligned} Z(s_\mu, t_\mu) &= \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left[-S(\bar{\psi}, \psi; s_\mu, t_\mu)\right] \\ S(\bar{\psi}, \psi; s_\mu, t_\mu) &= -i \int d^4x \, \bar{\psi} \, (i\partial\!\!\!/ - s\!\!\!/ - \gamma_5 t\!\!\!/) \, \psi \end{aligned} \tag{118}$$

where we have adopted for the γ-matrices the following conventions:

$$\gamma_\mu^\dagger = \gamma_\mu \ , \ \gamma_5^\dagger = \gamma_5 \ , \ \{\gamma_\mu \, , \, \gamma_\nu\} = 2\,\delta_{\mu\nu} \ . \tag{119}$$

The addition of the source s_μ is due to the fact that, in four dimensions, the vector and axial currents are *independent* fermionic bilinears. Thus not all the information provided by $Z(s_\mu, t_\mu)$ can be obtained from, say, $Z(s_\mu, 0)$. In two dimensions, because of the smaller number of generators for the Dirac algebra, these two currents are related, and the bosonization rule for one of the currents also implies the proper rule for the other.

We now consider the following change of variables

$$\psi(x) = e^{i\theta(x) - i\gamma_5\alpha(x)}\psi'(x) \ , \ \bar{\psi}(x) = \bar{\psi}'(x)e^{-i\theta(x) - i\gamma_5\alpha(x)} \ . \tag{120}$$

In terms of the new variables, the generating functional (118) reads

$$\mathcal{Z}(s_\mu, t_\mu) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, J(\alpha; s_\mu, t_\mu) \, \exp\left[-S(\bar{\psi}, \psi \, ; \, s_\mu + \partial_\mu\theta, t_\mu + \partial_\mu\alpha)\right] \tag{121}$$

where the primed fermionic fields have been renamed as unprimed for the sake of simplicity, and J is the anomalous Jacobian corresponding to this fermionic change of variables, a well-known consequence of the chiral anomaly [41].

This Jacobian is evaluated by using the standard Fujikawa's recipe accomodated so as to get the consistent anomaly. The answer is [5]

$$J(\alpha; s_\mu, t_\mu) \;=\; \exp\left[\frac{1}{4\pi^2}\int d^4x\, \alpha(x)\, \epsilon_{\mu\nu\rho\sigma}(\partial_\mu s_\nu \partial_\rho s_\sigma \;+\; \frac{1}{3}\partial_\mu t_\nu \partial_\rho t_\sigma)\right] \;. \quad (122)$$

Now, as (120) is a change of variables, Z cannot depend on either θ or α. Thus θ and α can be integrated out without other effect than the introduction of an irrelevant constant factor in Z, which we ignore. Integration over θ and α is equivalent to integration over two flat Abelian vector fields θ_μ and α_μ:

$$\begin{aligned} \theta_\mu \;&=\; \partial_\mu\theta \quad , \quad \alpha_\mu \;=\; \partial_\mu\alpha \\ f_{\mu\nu}(\theta) \;\equiv\; \partial_\mu\theta_\nu - \partial_\nu\theta_\mu \;&=\; 0 \;\; , \;\; f_{\mu\nu}(\alpha) \;\equiv\; \partial_\mu\alpha_\nu - \partial_\nu\alpha_\mu \;=\; 0 \;. \end{aligned} \quad (123)$$

Writing

$$J(\alpha_\mu; s_\mu, t_\mu) \;=\; \exp\left[-\frac{1}{4\pi^2}\int d^4x\, \alpha_\mu(x)\, \epsilon_{\mu\nu\rho\sigma}(s_\nu \partial_\rho s_\sigma \;+\; \frac{1}{3}t_\nu \partial_\rho t_\sigma)\right] \;. \quad (124)$$

eq.(121) becomes

$$\begin{aligned} \mathcal{Z} \;=\; \int \mathcal{D}\theta_\mu\, \mathcal{D}\alpha_\mu\, \mathcal{D}\bar\psi\, \mathcal{D}\psi\; \delta[f_{\mu\nu}(\theta)]\, \delta[f_{\mu\nu}(\alpha)]\; J(\alpha_\mu; s_\mu, t_\mu) \\ \exp\left[-S(\bar\psi, \psi; s_\mu + \theta_\mu, t_\mu + \alpha_\mu)\right] \;. \end{aligned} \quad (125)$$

Formally integrating out the fermionic fields and making the shift of variables

$$\theta_\mu \;\to\; \theta_\mu \;-\; s_\mu \quad , \quad \alpha_\mu \;\to\; \alpha_\mu \;-\; t_\mu \;, \quad (126)$$

(125) leads to

$$\begin{aligned} Z(s_\mu, t_\mu) \;=\; \int \mathcal{D}\theta_\mu\, \mathcal{D}\alpha_\mu\, \delta[f_{\mu\nu}(\theta - s)]\delta[f_{\mu\nu}(\alpha - t)]\; J(\alpha_\mu - t_\mu; s_\mu, t_\mu) \\ \times \det(\partial\!\!\!/ + i\; \theta\!\!\!/ + i\gamma_5\; \alpha\!\!\!/) \;. \end{aligned} \quad (127)$$

As before, we exponentiate the two functional delta functions in (127) using two antisymmetric tensor fields $A_{\mu\nu}$ and $B_{\mu\nu}$ as Lagrange multipliers

$$
\begin{aligned}
Z(s_\mu, t_\mu) \ &= \ \int \mathcal{D}A_{\mu\nu}\, \mathcal{D}B_{\mu\nu}\, \mathcal{D}\theta_\mu\, \mathcal{D}\alpha_\mu \ \ J(\alpha_\mu - t_\mu; s_\mu, t_\mu) \\
\times \exp\Big(i \int d^4x [\epsilon_{\mu\nu\rho\sigma} A_{\mu\nu}(\partial_\rho \theta_\sigma - \partial_\rho s_\sigma) \ &+ \ \epsilon_{\mu\nu\rho\sigma} B_{\mu\nu}(\partial_\rho \alpha_\sigma - \partial_\rho t_\sigma)]\Big) \\
&\times \ \det(\not{\partial} + i \not{\theta} + i\gamma_5 \not{\alpha}) \, . \qquad (128)
\end{aligned}
$$

The bosonized form of Z can then be obtained by integrating out θ_μ and α_μ in (128). This produces a generating functional with the tensor fields $A_{\mu\nu}$ and $B_{\mu\nu}$ as dynamical variables. This step requires the evaluation of the fermionic determinant, which of course is necessarily non-exact.

At this stage we can already derive the rules that map the vector and axial-vector currents into functions of the bosonic fields $A_{\mu\nu}$ and $B_{\mu\nu}$. This correspondence requires no approximation and may well be called 'exact'. These rules follow from elementary functional differentiation

$$
j_\mu = \langle \bar{\psi}\gamma_\mu\psi \rangle = -i\frac{\delta}{\delta s_\mu} \log \mathcal{Z}|_{s_\mu=0} = -\epsilon_{\mu\nu\rho\sigma}\partial_\nu A_{\rho\sigma} \qquad (129)
$$

$$
j_{5\mu} = \langle \bar{\psi}\gamma_5\gamma_\mu\psi \rangle = -i\frac{\delta}{\delta t_\mu} \log \mathcal{Z}|_{t_\mu=0} = -\epsilon_{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma} - \frac{i}{4\pi^2}\, \epsilon_{\mu\nu\rho\sigma} s_\nu \partial_\rho s_\sigma \ . \qquad (130)
$$

From the antisymmetry of the tensors $A_{\mu\nu}$ and $B_{\mu\nu}$, we are entitled to derive the equations for the divergencies of the currents:

$$
\begin{aligned}
\partial_\mu j_\mu \ &= \ 0 \\
\partial_\mu j^5_\mu \ &= \ -\frac{i}{8\pi^2}\, \tilde{F}_{\mu\nu}(s) F_{\mu\nu}(s) \, . \qquad (131)
\end{aligned}
$$

with $\tilde{F}_{\mu\nu} = (1/2)\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$. We then see that the bosonization rule (130) correctly reproduces the axial anomaly.

As before, although the bosonization recipe (129)-(130) for associating the fermionic currents with expressions written in terms of bosonic fields is exact, the bosonic action governing the boson field dynamics cannot be evaluated in an exact form in $d > 2$ dimensions. Different approximations for computing the fermionic determinant would yield alternative effective bosonic actions valid in different regimes. In ref.[5] the fermionic determinant in (128) was evaluated to second order in the fields θ_μ and α_μ. The use of this quadratic approximation can be motivated by the same kind of arguments (see in particular the 'quasi-theorem') used in ref. [16]. Calling

$$\det(\not{\partial} + i\,\not{\theta} + i\gamma_5\,\not{\alpha}) \;=\; \exp\left[W(\theta_\mu,\alpha_\mu)\right] \;. \tag{132}$$

the answer for the renormalized W is

$$W(\theta_\mu,\alpha_\mu) \;=\; -\frac{1}{2}\int d^4x d^4y \,\Big[\theta_\mu(x)\delta^{\perp}_{\mu\nu}F(x-y)\theta_\nu(y)$$

$$+\,\alpha_\mu(x)\delta^{\perp}_{\mu\nu}G(x-y)\alpha_\nu(y) \,+\, m^2\,\alpha_\mu(x)\delta^{\|}_{\mu\nu}\delta(x-y)\alpha_\nu(y)\Big] \tag{133}$$

where

$$G(x-y) \;=\; F(x-y) \,+\, m^2\delta(x-y)\;. \tag{134}$$

With this result, one easily finds for the generating functional

$$\begin{aligned}
&Z(s_\mu,t_\mu) \;=\; \exp[\mathcal{C}(s_\mu,t_\mu)]\int \mathcal{D}A_{\mu\nu}\,\mathcal{D}B_{\mu\nu}\,\times\\
&\exp\left\{-i\int d^4x[s_\mu\epsilon_{\mu\nu\rho\sigma}\partial_\nu A_{\rho\sigma} + t_\mu(\epsilon_{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma} + \frac{i}{4\pi^2}\epsilon_{\mu\nu\rho\sigma}s_\nu\partial_\rho s_\sigma)]\right\}\times\\
&\exp\left\{-\frac{1}{3}\int d^4xd^4y[A_{\mu\nu\rho}(x)F^{-1}(x-y)A_{\mu\nu\rho}(y)+\right.\\
&\left.B_{\mu\nu\rho}(x)G^{-1}(x-y)B_{\mu\nu\rho}(y)]\right\}\times\exp\left\{-\frac{i}{4\pi^2}\int d^4xd^4y\partial_\mu B_{\nu\rho}(x)\times\right.\\
&\left.G^{-1}(x-y)\delta_{\mu\nu\rho,\alpha\beta\gamma}(s_\alpha\partial_\beta s_\gamma + \frac{1}{3}t_\alpha\partial_\beta t_\gamma)\right\}
\end{aligned} \tag{135}$$

where

$$\begin{aligned}
A_{\mu\nu\rho} &= \partial_\mu A_{\nu\rho} + \partial_\nu A_{\rho\mu} + \partial_\rho A_{\mu\nu}\\
B_{\mu\nu\rho} &= \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}\\
\delta_{\mu\nu\rho,\alpha\beta\gamma} &= \det\begin{pmatrix} \delta_{\mu\alpha} & \delta_{\mu\beta} & \delta_{\mu\gamma}\\ \delta_{\nu\alpha} & \delta_{\nu\beta} & \delta_{\nu\gamma}\\ \delta_{\rho\alpha} & \delta_{\rho\beta} & \delta_{\rho\gamma}\end{pmatrix}
\end{aligned} \tag{136}$$

and

$$\mathcal{C}(s_\mu,t_\mu) \;=\; \frac{1}{2(2\pi)^4}\int d^4xd^4y\,\Big\{[s_\mu(x)\partial_\nu s_\lambda(x) \,+\, \frac{1}{3}t_\mu(x)\partial_\nu t_\lambda(x)]$$

$$
\begin{aligned}
&\delta_{\mu\nu\rho,\alpha\beta\gamma}G^{-1}(x-y)[s_\alpha(y)\partial_\beta s_\gamma(y)\;+\;\frac{1}{3}t_\alpha(y)\partial_\beta t_\gamma(y)]\\
&\quad+\frac{1}{2(2\pi)^4}\int d^4xd^4y\,\mathcal{G}(x)\partial^{-2}G^{-1}(x-y)\mathcal{G}(y)\\
&\quad+\frac{1}{2m^2(2\pi)^4}\int d^4xd^4y\,\mathcal{G}(x)\partial^{-2}(x-y)\mathcal{G}(y)\Bigg\}
\end{aligned}
\tag{137}
$$

where $\mathcal{G}=\epsilon_{\mu\nu\rho\lambda}(\partial_\mu s_\nu\partial_\rho s_\lambda\;+\;\frac{1}{3}\partial_\mu t_\nu\partial_\rho t_\lambda)$.

In conclusion, we have applied our bosonization technique to the case of massless Dirac fermions in four dimensions in the presence of both vector and axial-vector sources. This has allowed us to find the bosonization rules for both fermionic currents, eqs.(129)-(130), in terms of Kalb-Ramond bosonic fields. While the bosonization rule for the vector current can be written in a natural and compact form, reminiscent of the well-known two-dimensional bosonization rule,

$$
\bar{\psi}\gamma_\mu\psi\to-\epsilon_{\mu\nu\rho\sigma}\partial_\nu A_{\rho\sigma}\;, \tag{138}
$$

the result for the axial current is more involved and includes the vector source

$$
\bar{\psi}\gamma_5\gamma_\mu\psi\to-\epsilon_{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma}-\frac{i}{4\pi^2}\,\epsilon_{\mu\nu\rho\sigma}s_\nu\partial_\rho s_\sigma\;. \tag{139}
$$

In our approach, this is a consequence of the anomalous behaviour of the fermionic measure under axial gauge transformations and in this way the bosonic form of the axial current correctly yields its anomalous divergence. We want to comment on the possibility of considering the particular case of a purely chiral external source ($s_\mu\equiv\pm t_\mu$), and obtaining a bosonized version for this model. The Kalb-Ramond field then corresponds to a particular 'chiral' combination of A and B, namely $A_{\mu\nu}\pm B_{\mu\nu}$.

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