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TRANSVERSE COHERENT RESONANCE EFFECTS IN STORAGE DEVICES

by

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Interactions of a charged particle beam with an external system having a finite Q may lead to both growth and damping of the coherent motion of the particles. In the papers so far available on this subject [1),2)] the resonances when the interactions become the most intense have not been thoroughly studied.

The present paper deals with the study of the dynamics of the small coherent excitations of the beam near the resonances $\omega_K + m\omega_\delta = n\omega_o$, where ω_K is the natural frequency of the resonator, ω_δ the betatron frequency, ω_o the revolution frequency and m and n are integers. In order to study the motion in the vicinity of a resonance, one may use the averaging method, regarding the oscillation of the resonator field and the beam particles as a coupled oscillator system. On the basis of the overall characteristics of the coupling resonances [3)] instabilities can be expected to appear only in the case of sum resonances $(m > 0)$. In the paper, estimates are made of the threshold of the appearance of instabilities. A study of the difference resonance is interesting from the point of view of obtaining rapid damping of coherent excitations. In the paper, estimates are made of the damping constants and the conditions are found under which interactions prevent the transition of the energy of the coherent excitations of the beam into heat. Let us take the Hamiltonian form :

$$H = \omega_{i\delta} J_i + \sum_K \frac{P_K^2 + \omega_K^2 Q_K^2}{2} - e\, Q_K \vec{A}_K \vec{V} + e\varphi$$

where $\omega_{i\delta}$ is the frequency, J_i the action of the betatron oscillations, P_k and Q_k canonical variables of the oscillator field, A_k the eigenfunctions of the resonator, $\vec{V}$ the velocity of the particle, φ the scalar potential; the particle co-ordinate in terms of action and phase is expressed in the following form :

$$Z = a\cos(\omega_{i\delta}t + \varphi_i) = \sqrt{\frac{2Ji}{m\omega_{\delta i}}}\cos\varphi .$$

$$\psi = \omega_{i\delta}t + \varphi_i \quad . \qquad \int A_k^2\, dv = 4\pi$$

In the case of the resonance $\omega_k + m\,\omega_{i\delta} = n\,\omega_o$ it is convenient to switch to slowly changing variables $\phi_i = \psi_i + \frac{\varepsilon}{m}t + \frac{n\varphi_c}{m}$. $q = \frac{\omega_k Q_k - i P_k}{\sqrt{2\omega_k}} e^{-i\omega_k t}$,

$p = -iqt$. Here $\varepsilon = m\omega_{i\delta} + \omega_k - n\omega_o$, and φ_c is the slow phase of the synchrotron oscillations. Then, the Hamiltonian averaged for the new variables is :

$$H = \frac{\varepsilon_i}{m} J_i + (p\, e^{im\phi_i} - iqe^{-im\phi_i})\, V(J) + V_o(J) \tag{1}$$

where

$$V(J) = \left| \frac{V_{mn}}{\sqrt{2\omega_k}} \right| = \left| e\,\frac{1}{\sqrt{2\omega_k}} \int \vec{V}\,\vec{A}_k\, e^{in\theta - im\psi}\,\frac{d\theta}{2\pi}\,\frac{d\psi}{2\pi} \right|$$

$$\theta = \omega_o t + \varphi_c$$

Hence, taking into account the friction, we obtain the equations of motion :

$$\dot{p} = \langle i\ e^{-im\phi}\ V(J)\rangle - \lambda_k p$$
$$\dot{q} = \langle\ e^{im\phi}\ V(J)\rangle - \lambda_k q \qquad (2)$$

where $\langle\ \rangle$ denotes the average according to the particle distribution of the beam $f = f(J,\phi,t)$ which satisfies the equation :

$$\frac{\partial f}{\partial t} + \left\{ H\ ;\ f \right\} = 0$$

assuming that the perturbation is small $f = f_o(J)+\tilde{f}(J,\phi,t)$ and that $\tilde{f} \sim e^{-i\omega t}$ we obtain the dispersion equation :

$$1 = \frac{m\ N}{\omega+i\lambda_k} \langle \frac{\partial}{\partial J} \frac{V^2(J)}{\omega-\varepsilon} \rangle = 0 \qquad (3)$$

where N is the number of particles.

Let us note that because it is a resonance that is being studied the transition to the slowly variable is possible and therefore the kinetic equation for the bunched beam does not differ from that for an unbunched beam. First let us examine the solution of equation (3) without frequency spread of the betatron oscillations, where the dispersion equation will be :

$$(\omega + i\lambda_k)(\omega - \varepsilon + i\lambda) + mN \langle V^{2'}(J)\rangle = 0 \qquad (4)$$

the solution of which in the absence of dissipation is as follows :

$$\omega'_{\pm} = (\omega'_{1,2} - \varepsilon) = -\frac{\varepsilon}{2} \pm \sqrt{\frac{\varepsilon^2}{4} - mN\langle V^{2'}(J)\rangle} = -\frac{\varepsilon}{2} \pm \frac{\Omega}{2} \quad (5)$$

$\omega_{1,2} - \varepsilon$ is the frequency shift of the beam.

Let us note that the value $V^{2'}(J)$ is expressed in terms of the impedance of the resonator in the following way :

$$V^{2'}(J) = \frac{r_e \omega_o^2 \lambda_{12}}{4\pi^2 \omega_\delta \omega_n \gamma} \frac{\partial}{\partial a^2} \left[V_o^2 \, Z_{kmn}^{yy} + \frac{a^2 \omega_\delta^2}{4} Z_{kmn}^{zz} + a\omega_\delta V_o Z_{knm}^{yz} \right]$$

The general solution is the superposition of normal oscillations, and Ω signifies the frequency of phase oscillations or the frequency of amplitude beats. It is easy to see that for $m < 0$ solutions of (5) are always real, whereas for $m > 0$, i.e. the sum resonance, an unstable solution is possible if $\varepsilon^2 < 4mN \langle V^{2'} \rangle$. This instability is referred to as dynamic.

Let us note that when $m > 1$ instability may occur when the beam is of fairly large dimensions, as must also be the case with multi-polar instabilities.

Dissipation in the difference resonance leads to damping, whereas in the sum resonance instability occurs if :

$$\Omega^2 + 4\,\lambda_k \lambda < \varepsilon^2 \left(\frac{\lambda_k - \lambda}{\lambda_k + \lambda} \right)^2$$

We shall call "dissipative" an instability appearing as a result of friction, when $\Omega^2 > 0$. Here it should be pointed out that the system will be unstable if at least friction is equal to 0. If $(\lambda - \lambda_k)^2 \ll \Omega^2$ the real part of the solution of equation (4) does not differ from (5), and the imaginary part takes the form :

$$\mathrm{Im}(-\omega) = \frac{\lambda}{2}\left(1 \pm \frac{|\varepsilon|}{\Omega}\right) + \frac{\lambda_k}{2}\left(1 \mp \frac{|\varepsilon|}{\Omega}\right) \tag{6}$$

In the opposite extreme case $(\lambda - \lambda_k)^2 \gg \Omega^2$ we find :

$$(\omega_{12}-\varepsilon) = -\frac{\varepsilon}{2} \pm \varepsilon\left(1 - \frac{mN\langle V^{2'}\rangle}{(\lambda-\lambda_k)^2}\right) - i\left[\frac{\lambda+\lambda_k}{2} \pm \left(\frac{|\lambda-\lambda_k|}{2} + \frac{mN\langle V^{2'}\rangle}{(\lambda-\lambda_k)^2}\right)\right]$$

In the difference resonance it is interesting to study the case where $\lambda = 0$, (i.e. the case when damping of the beam is due to the coupling with the resonator). It is evident that the maximum damping constant of the collective excitation of the beam is $\lambda_k/2$ when $|\varepsilon| \ll \Omega$, i.e. strong coupling, and the energy of the excitation in the absence of frequency spread is fully dissipated in the resonator. If the coupling is weak, $|\varepsilon| \gg N\,|m\,\langle V^{2'}\rangle|$, then the damping constant depends on the number of particles and is $\frac{N\lambda km\,|\langle V^{2'}\rangle|}{\varepsilon^2}$. For the schematic curves δ of λ, N, see figs. 1 and 2. The frequency spread of the oscillations in the beam results in part of the energy of the collective motion being changed into heat energy of the particles. Therefore, it is interesting to investigate the contribution of the spread to the speed of damping of the collective motion. It is clear that the contribution depends essentially on the relationship between the coherent frequency shift $\omega'_{\pm} = \pm\frac{\Omega}{2}\left(1 \mp \frac{|\varepsilon|}{\Omega}\right)$ and the width of the distribution function Δ . If $|\omega'| \ll \Delta$ and $\lambda_k \ll \Omega$ then the character of the damping depends on the shape of the distribution function and not on the coupling with the resonator; if $|\omega'| \gg \Delta$ then, solving the dispersion equation (3), we obtain :

$$\delta^{\pm} = \frac{|\omega'|}{\Omega}\left[\lambda_k + \pi N|m|\left(-V^2 \frac{\partial f_o}{\partial J}\right)\Big|_{J=\frac{\omega'}{\partial\omega_\delta/\partial J}}\right] \qquad (7)$$

from which it can be seen that the contribution of the spread depends essentially on the behaviour of the tail of the distribution function. With the assumptions made above, the fraction of energy dissipated is :

$$\eta = \lambda_k \frac{N|m|\langle V^{2'}\rangle}{\Omega^2}\left(\frac{1}{\delta^+} + \frac{1}{\delta^-}\right) \qquad (8)$$

For $\lambda_k \gg \Omega$ the relation $\frac{\omega'}{\delta} = \frac{|\varepsilon|}{\lambda_k}$ is small and therefore the contribution of the spread will also be small if $\delta \gg \Delta$. It should be noted that the longitudinal momentum spread for a bunched beam does not lead to Landau damping, since the spectrum of coherent oscillations is then discrete and equidistant. It is evident that the spread can appear only as a result of non-linearity of the synchrotron oscillations and only in a synchro-betatron resonance.

In the sum resonance the imaginary part of the solution of the dispersion equation under the condition $\omega' \gg \Delta$ is as follows :

$$\mathrm{Im}(-\omega) = \frac{1}{2}\left(1 \mp \frac{|\varepsilon|}{\Omega}\right)\left[\lambda_k - \frac{\pi N}{\partial\omega_\delta/\partial J}\left(V^{2'}\left|\frac{\partial f}{\partial J}\right|\right)\Big|_{J=\frac{\omega'}{\partial\omega_\delta/\partial J}}\right] \qquad (9)$$

from which it is easy to obtain the stability condition for $(\lambda - \lambda_k) \ll \Omega$

$$\frac{\pi N}{\frac{\partial \omega_\delta}{\partial J}} \left(|V^2| \left| \frac{\partial f_o}{\partial J} \right| \right) \Bigg|_{J = \frac{\omega_-'}{m \frac{\partial \omega_\delta}{\partial J}}} + \frac{\omega_+'}{\omega_-'} \lambda < \lambda_k < \lambda \frac{\omega_-'}{\omega_+'} + \tag{10}$$

$$+ \frac{\pi N}{\frac{\partial \omega_\delta}{\partial J}} \left(|V^2| \left| \frac{\partial f_o}{\partial J} \right| \right) \Bigg|_{J = \frac{\omega_+'}{m \frac{\partial \omega_\delta}{\partial J}}}$$

It can be seen from (9) that the spread may not only stabilise the dissipative instability but also leads to anti-damping, which can also appear when $\lambda_k = \lambda = 0$.

The physical significance of this instability is easy to understand from formula (9) : the particles of the tail of the distribution function remove energy from the resonator, when they are in the resonance, and the main part of the beam then damps or grows as if there was friction in the resonator. However, it should be noted that the energy of the resonator beam system does not vary.

In the general case, the boundary of the stable region is easy to find in the absence of self-phasing in the azimuthal direction and assuming a spread only on the limiting pulses we have

$$\begin{aligned} &\lambda - \pi m N \langle V^{2'} \rangle f(\omega') = 0 \\ &\varepsilon + \omega' + m N \langle V^{2'} \rangle \int_0^\infty d\tau \, f(\tau) e^{i\omega'\tau} = 0 \end{aligned} \tag{11}$$

The schematic curve of the stable region is given in Fig. 2. The distribution function is single-humped and symmetric.

In conclusion, we should like to point out that the character of the dependence of the speed of damping or growing is a function of the number of particles N and depends essentially on the coupling coefficient $\frac{N\ m\ \lambda_k |\langle V^2\rangle|}{\varepsilon^2}$ i.e. if it is small the increment and decrement $\sim N$, if it is large then, in the sum resonance, the increment $\sim \sqrt{N}$ and in the difference resonance, the maximum decrement is $\lambda_k/2$. If Landau anti-damping does not take place, then for a small coupling coefficient the decrement $\sim N^2$.

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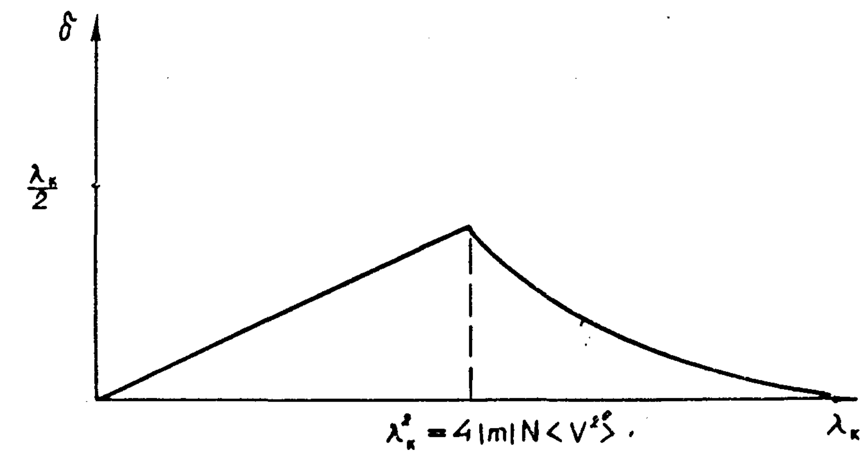
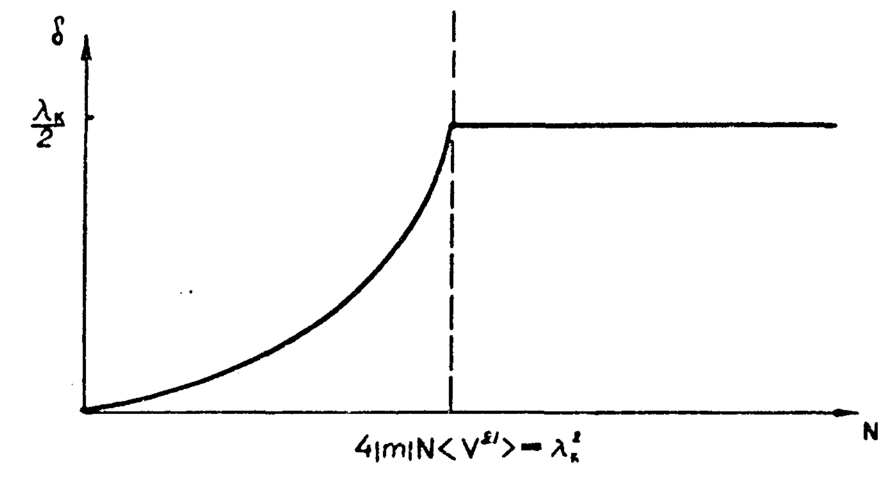
FIGURE CAPTIONS

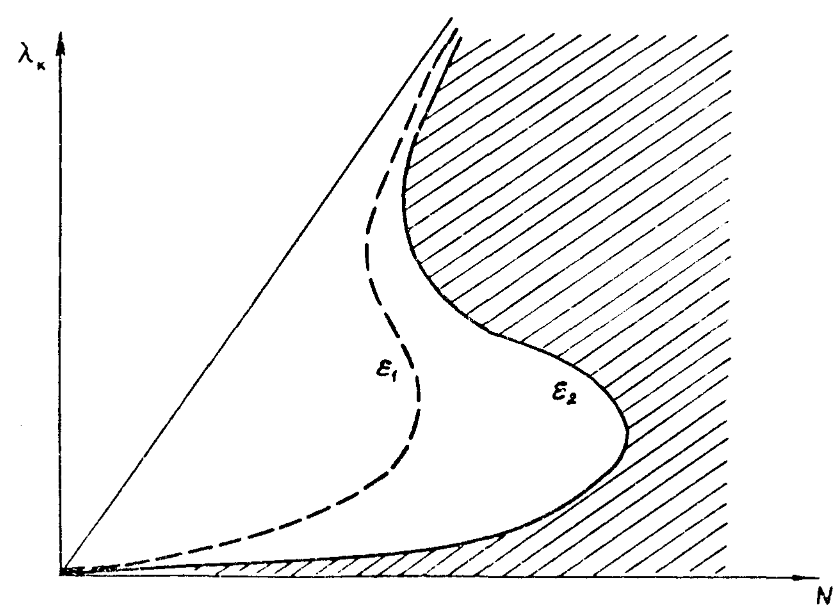
Fig. 1. Damping constant of coherent oscillations in a difference resonance when $\lambda = 0$

Fig. 2. Damping constant of coherent oscillation in a difference resonance depending on the number of particles in the beam.

Fig. 3. The unshaded portion represents a stable region corresponding to condition (11). The dotted line shows the boundary curve for $\varepsilon_1 < \varepsilon_2$.


δ
$\frac{\lambda_K}{2}$
$\lambda_K^2 = 4|m|N\langle V^{2}\rangle$
λ_K


δ
$\frac{\lambda_K}{2}$
$4|m|N\langle V^{2}\rangle = \lambda_K^2$
N


λ_K
ε_1
ε_2
N