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Transformation of Black-Hole Hair under Duality and Supersymmetry

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ABSTRACT

We study the transformation under the full String Theory duality group of the observable charges (including mass, angular momentum, NUT charge, electric, magnetic and different scalar charges) of four dimensional point-like objects whose asymptotic behavior constitutes a subclass closed under duality.

We find that those charges fall into two complex four-dimensional representations of the duality group. T duality (including Buscher's transformations) has an $O(1,2)$ action on them and S duality a $U(1)$ action. The generalized Bogomol'nyi bound is an $U(2,2)$-invariant built out of one representations while the other representation (which includes the angular momentum) never appear in it. The bound is manifestly duality-invariant.

Consistency between T duality and supersymmetry seems to require that *primary scalar hair* is included in the generalized Bogomol'nyi bound. We also find that all known four-dimensional supersymmetric massless black holes are the T duals in the time direction of the usual massive supersymmetric black holes. Non-extreme massless "black holes" can be found as the T duals of the non-extreme black holes. All of them have primary scalar hair and naked singularities.

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Introduction

Black-hole physics is probably the only non perturbative problem in gravity in which non trivial progress has recently been made owing to the different perspective afforded by "string dualities" (for a recent review see e.g. [1]).

In a previous work [2] a systematic analysis was made of the behavior of asymptotic charges under T duality (see e.g. [3, 4] for four-dimensional non-rotating black holes[5].

The main goal of the present work is to extend their results, by essentially widening the class of metrics considered both by allowing a more general asymptotic behavior and by including more non-trivial fields. This simultaneously widens the subgroup of the duality group that acts on that class preserving the asymptotic behavior.

Therefore, we will define the asymptotic behavior considered ("TNbh") and we will identify the subgroup that preserves it (the "ADS"). We will find that the charges naturally fit in multiplets under the action of this subgroup and that the Bogomol'nyi bound can be written as a natural invariant of this subgroup. This was to be expected since duality transformations in general respect unbroken supersymmetries, but, since duality transformations in general transform conserved charges that appear in the Bogomol'nyi bound into non-conserved charges (associated to primary scalar hair) that, in principle, do not, the consistency of the picture will require us to include those non-conserved charges into the generalized Bogomol'nyi bound. A by-product of our study will be the identification of the known supersymmetric massless black holes as the T duals in the time direction of the usual supersymmetric massive black holes. These are our main results.

One of the motivations for this work was to try to constrain the angular momentum of black holes using duality and supersymmetry in such a way that the extreme limit would never be surpassed. As it is well known the striking difference between the black-hole extremality bound and the supersymmetry (or Bogomol'nyi) bound: although teh angular momentum appears in the extremality bound, it does not enter the supersymmetry bound. This difference is even more surprising in view of the fact that in presence of only NUT charge (that is, for some stationary, non-static, cases) both bounds still coincide; the NUT charge squared must simply be added to the first member in the two bounds [5]. On the other hand, it is also known that some T duality transformations seem to break spacetime supersymmetry making

[5]Since some of the objects studied are singular, as opposite to black holes with a regular horizon, the name black hole will be used in a generalized sense for (usually point-like) objects described by asymptotics such that a mass, angular momentum etc. can be assigned to them.

it non-manifest [6]. These two facts could perhaps give rise to an scenario in which extremal Kerr-Newman black holes (which are not supersymmetric) could be dual to some supersymmetric configuration. At the level of the supersymmetry bounds one would see the angular momentum transforming under a non-supersymmetry-preserving duality transformation into a charge that does appear in the supersymmetry bound (like the NUT charge). In this way, the constraints imposed by supersymmetry on the charge would constraint equally the angular momentum.

Although this scenario has been disproved by the calculations that we are going to present[6] the transformation of black-hole charges and the corresponding Bogomol'nyi bounds under general string duality transformations remains an interesting subject in its own right and its study should help us gain more insight into the physical space-time meaning of duality.

Thus, in the sequel, the transformation of asymptotic observables (as multipoles of the metric or of other physical fields) of four-dimensional "black holes" under the T duality and S duality groups will be systematically analyzed, from the four-dimensional effective action point of view[7].

We are going to consider for simplicity a consistent (from the point of view of the equations of motion and of the supersymmetry transformations) truncation[8] of the four-dimensional heterotic string effective action including the metric, dilaton and two-form field plus two Abelian vector fields. This truncation is, however, rich enough to contain solutions with 1/4 of the supersymmetries unbroken [9, 7].

Since all the configurations we are going to consider are stationary and axially symmetric, the theory can be reduced to two dimensions were the T dualities due to the presence of isometries in four dimensions become evident. This we will do in Section 1 getting manifest $O(2,4)$ due to the presence of the two Abelian vector fields in four dimensions [10]. We will also find the S duality transformations in their four-dimensional form.

Then, in Section 2 we will define the asymptotic behavior of those fields in the configurations we are interested in: (charged, rotating) black holes, Taub-NUT metrics etc. which are stationary and axially-symmetric. This class of asymptotic behavior will be referred to as *TNbh asymptotics*. Any configuration in this class will be characterized by a set of parameters (*charges*) such as the electric and magnetic charges with respect to the gauge fields,

[6]In fact, the angular momentum is part of a set of charges which transform amongst themselves under duality and never appear in the Bogomol'nyi bound.

[7]The transformation of some of the charges we are going to consider here was studied previously in Refs. [7, 8]. Here we will extend that study to other sets of charges.

[8]This truncation is also invariant under duality transformations in the compact six-dimensional space directions.

the ADM mass, the angular momentum, the NUT charge and some other charges forced upon us by duality.

The rest of the paper is organized as follows: In Section 3 we study the transformation of the charges under different elements of the T and S duality groups and show explicitly the transformations that preserve TNbh asymptotics including the effect of constant terms in the asymptotics of the vector fields (Section 3.4). In Section 4 we define the *Asymptotic Duality Subgroup* as the subgroup of the duality group that preserves TNbh asymptotics and study the transformation of the Bogomol'nyi bound under duality. We will find full agreement with the preservation of unbroken supersymmetry if we admit the presence of primary scalar hair in the generalized Bogomol'nyi bound. We illustrate this with several examples in Section 4.3 where we also find that the known massless supersymmetric black holes are the T duals of the common massive supersymmetric ones. Section 5 contains our conclusions.

1 The Derivation of the Duality Transformations

In this paper we are going to consider a consistent truncation of the four-dimensional heterotic string effective action in the string frame including the metric, axion 2-form and two Abelian vector fields given by[9]:

$$S = \int d^4x \sqrt{|\hat{g}|}\, e^{-\hat{\phi}} \left[R(\hat{g}) + \hat{g}^{\hat{\mu}\hat{\nu}} \partial_{\hat{\mu}}\hat{\phi}\partial_{\hat{\nu}}\hat{\phi} - \tfrac{1}{12}\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}} - \tfrac{1}{4}\hat{F}^{I}{}_{\hat{\mu}\hat{\nu}}\hat{F}^{I\hat{\mu}\hat{\nu}} \right] , \tag{1.1}$$

where $I = 1, 2$ sums over the Abelian gauge fields $\hat{A}^{I}{}_{\hat{\mu}}$ with standard field strengths

$$\hat{F}^{I}{}_{\hat{\mu}\hat{\nu}} = 2\partial_{[\hat{\mu}}\hat{A}^{I}{}_{\hat{\nu}]} \, . \tag{1.2}$$

and the two-form field strength is

$$\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} = 3\partial_{[\hat{\mu}}\hat{B}_{\hat{\nu}\hat{\rho}]} - \tfrac{3}{2}\hat{F}^{I}{}_{[\hat{\mu}\hat{\nu}}\hat{A}^{I}{}_{\hat{\rho}]} \, . \tag{1.3}$$

Before proceeding, an explanation of the origin of this action is in order. This action can be obtained from the ten-dimensional heterotic string

[9]Our signature is $(-,+,+,+)$. All hatted symbols are four-dimensional and so $\hat{\mu}, \hat{\nu} = 0, 1, 2, 3$. The relation between the four-dimensional Einstein metric $\hat{g}_{E\hat{\mu}\hat{\nu}}$ and the string-frame metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ is $\hat{g}_{E\hat{\mu}\hat{\nu}} = e^{-\hat{\phi}}\hat{g}_{\hat{\mu}\hat{\nu}}$.

effective action by first considering only the lowest order in α' terms (so the Yang-Mills fields and R^2 terms are consistently excluded), then compactifying the theory on T^6 to four dimensions following essentially Ref. [10] and afterwards setting to zero all the scalars and identifying the six Kaluza-Klein vector fields with the six vector fields that come from the ten-dimensional axion. This last truncation is done in the equations of motion and it is perfectly consistent with them and with the supersymmetry transformation rules. The result of this truncation is the action of $N = 4, d = 4$ supergravity [11] in the string frame and with the axion 2-form. Setting to zero four of the six vector fields one gets the above action.

The above action is invariant under Buscher's T duality transformations in the six compact directions because these interchange the vector fields whose origin is the ten-dimensional metric with the the vector fields whose origin is the ten-dimensional axion and we have identified these two sets of fields. There are still some trivial T duality transformations which correspond to rotations in the internal compact space. They correspond to global $O(2)$ rotations of the two vector fields ($O(6)$ rotations in the full $N = 4$ supergravity theory).

The reason why we consider two vector fields instead of six or just one is that the generating solution for black-hole solutions of the full $N = 4, d = 4$ theory only needs two non-trivial vector fields. Starting from this generating solution and performing T duality transformations in the compact space and S duality transformations (to be described later) which do not change the Einstein metric one can generate the most general black-hole solution (if the no-hair theorem holds). Also, the minimal number of vector fields required in this theory for allowing solutions with $1/4$ of the $N = 4$ supersymmetries unbroken, is two [9, 7].

As announced in the Introduction, it will be assumed that the metric has a timelike and a spacelike rotational isometry[10]. The former is physically associated to the stationary (but not static, in general) character of the spacetime and the other to the axial symmetry[11]. They comute with each other and, thus, one can find two coordinates, in this case the time t and the angular variable φ, adapted to them, such that the background does not depend on them. This, then, implies that the theory can be dimensionally reduced. Using the standard technique [12] the resulting dimensionally reduced, Euclidean, action turns out to be

[10]The action of rotational isometries has fixed points, while translational isometries act with no fixed points.

[11]The axis corresponds obviously to the set of fixed points of the isometry.

$$
\begin{aligned}
S \;&=\; \int d^2x\sqrt{|g|}\; e^{-\phi}\Big[R(g) + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \tfrac{1}{8}\mathrm{Tr}\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1} \\
&\qquad\qquad -\tfrac{1}{4}W^i_{\mu\nu}(\mathcal{M}^{-1})_{ij}W^{j\mu\nu}\Big]\,,
\end{aligned}
\tag{1.4}
$$

Now the spatial indices $\mu\nu = 2,3$ for simplicity and we also have internal indices $\alpha,\beta = 0,1$. The two-dimensional fields are the metric $g_{\mu\nu}$, six vector fields $\mathcal{K}^i{}_\mu = (K^{(1)\alpha}{}_\mu, K^{(2)}{}_{\alpha\mu}, K^{(3)I}{}_\mu)$ with the standard Abelian field strengths $W^i{}_{\mu\nu}$ $(i = 1,\ldots,6)$ and a bunch of scalars $G_{\alpha\beta}, \hat{B}_{\alpha\beta}, \hat{A}^I{}_\alpha$ that appear combined in the 6×6 matrix $\mathcal{M}_{ij}$. They are given by

$$
\begin{array}{lcllcl}
G_{\alpha\beta} & = & \hat{g}_{\alpha\beta}\,, & \phi & = & \hat{\phi} - \tfrac{1}{2}\log|\det G_{\alpha\beta}|\,,\\
K^{(1)\alpha}{}_\mu & = & \hat{g}_{\mu\beta}(G^{-1})^{\beta\alpha}\,, & C_{\alpha\beta} & = & \tfrac{1}{2}\hat{A}^I{}_\alpha\hat{A}^I{}_\beta + \hat{B}_{\alpha\beta}\,,\\
g_{\mu\nu} & = & \hat{g}_{\mu\nu} - K^{(1)\alpha}{}_\mu K^{(1)\beta}{}_\nu G_{\alpha\beta}\,, & K^{(3)I}_\mu & = & \hat{A}^I{}_\mu - \hat{A}^I{}_\alpha K^{(1)\alpha}{}_\mu\,,\\
K^{(2)}{}_{\alpha\;\mu} & = & \hat{B}_{\mu\alpha} + \hat{B}_{\alpha\beta}K^{(1)\beta}{}_\mu + \tfrac{1}{2}\hat{A}^I{}_\alpha K^{(3)I}{}_\mu\,, & & &
\end{array}
\tag{1.5}
$$

and

$$
(\mathcal{M}_{ij}) \;=\; \begin{pmatrix} G^{-1} & -G^{-1}C & -G^{-1}A^T \\ -C^TG^{-1} & G + C^TG^{-1}C + A^TA & C^TG^{-1}A^T + A^T \\ -AG^{-1} & AG^{-1}C + A & \mathbb{I}_2 + AG^{-1}A^T \end{pmatrix}, \tag{1.6}
$$

A being the 2×2 matrix with entries $\hat{A}^I{}_\alpha$. If B stands for the 2×2 scalar matrix $(\hat{B}_{\alpha\beta})$, then the 2×2 scalar matrix C is given by

$$
C = \tfrac{1}{2}A^TA + B\,. \tag{1.7}
$$

Any explicit contribution from the three-form automatically vanishes in two dimensions, which explains why it does not occur in Eq. (1.4). On the other hand, the dynamics of a two-dimensional vector field is trivial[12] and this seems to suggest that we can safely ignore it. However, the correct procedure to eliminate the vector fields is to solve their equation of motion

[12] The equation of motion of a two-dimensional vector field implies that the single independent field-strength component is a constant.

and then substitute the solution into the equations of motion of the other fields. The equations of motion for the vector fields in the action above tell us that the components of the fields $(\mathcal{M}^{-1})_{ij}W^{j}{}_{\mu\nu}$ are constant. In Ref. [13] the constants were chosen to be zero by setting the vector fields themselves to zero, which can be consistently done at the level of the action. This is obviously an additional restriction on the backgrounds considered[13]. This restriction was also made (in the purely gravitational sector) in the original paper by Geroch [14] and it has been done in all the subsequent literature on this subject in the form proposed in Refs. [15] where it was expressed as the requirement that the background possess "orthogonal insensitivity", i.e. it is invariant under $(t,\varphi) \to (-t,-\varphi)$.

This restriction is crucial to obtain an infinite-dimensional algebra of invariances of the equations of motion of the two-dimensional system. As we are going to explain, though, in this paper we are not interested in the infinite-dimensional algebra but only in its zero-mode subalgebra and so we will not impose this restriction. Nevertheless, all the configurations that we will explicitly consider will obey it.

1.1 T Duality Transformations

The matrix $\mathcal{M}$ satisfies $\mathcal{MLML} = \mathbb{I}_6$, with

$$\mathcal{L} \equiv \begin{pmatrix} 0 & \mathbb{I}_2 & 0 \\ \mathbb{I}_2 & 0 & 0 \\ 0 & 0 & \mathbb{I}_2 \end{pmatrix} . \tag{1.8}$$

It can be immediately seen from Eq. (1.4) that the dimensionally reduced action, is invariant under the global transformations given by

$$\mathcal{M} \to \Omega\ \mathcal{M}\ \Omega^T , \qquad \mathcal{K}^{i}{}_{\mu} \to \Omega^{i}{}_{j}\ \mathcal{K}^{j}{}_{\mu} , \tag{1.9}$$

if the transformation matrices Ω satisfy the identity

$$\Omega\ \mathcal{L}\ \Omega^T = \mathcal{L} . \tag{1.10}$$

The matrix $\mathcal{L}$ given in Eq. (1.8) can be diagonalized and put into the form $\eta = \mathrm{diag}(-,-,+,+,+,+)$ by a change of basis associated to the orthogonal matrix ρ:

$$\rho\ \mathcal{L}\ \rho^T = \eta , \qquad \rho = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{I}_2 & -\mathbb{I}_2 & 0 \\ \mathbb{I}_2 & \mathbb{I}_2 & 0 \\ 0 & 0 & \sqrt{2}\mathbb{I}_2 \end{pmatrix} , \qquad \rho\ \rho^T = \mathbb{I}_6 , \tag{1.11}$$

[13]Other choices could lead to two-dimensional cosmological terms.

where now η is the diagonal metric of $O(2,4;\mathbb{R})$ which implies that the Ω's are $O(2,4;\mathbb{R})$ transformations in a non-diagonal basis. The transformations in the diagonal (Ω_η) and non-diagonal basis are related by

$$\Omega_\eta = \rho\ \Omega\ \rho^T\,, \qquad \Omega_\eta\ \eta\ \Omega_\eta^T = \eta\,. \tag{1.12}$$

This symmetry group corresponds to the classical T duality group. From the quantum-mechanical point of view, $O(2,4;\mathbb{R})$ is broken to $O(2,4;\mathbb{Z})$ and this group is an exact perturbative symmetry of string theory.

We must stress at this point that no S duality transformations are included in this group. S duality is a non-local symmetry while T duality consists only on local transformations[14]. So, where are the S duality transformations that were present in four dimensions?

It is well-known [16, 13] that this finite symmetry group can be extended to the infinite algebra $o\widehat{(2,4)}$. The zero-mode subalgebra corresponds to the algebra $o(2,4;\mathbb{R})$ of the symmetry we just described. The S duality transformations are included in this algebra as non-local transformations which are not in the zero-mode subalgebra.

Observe that we could have proceeded in a completely different way: we could have started by reducing the theory in the time direction to three dimensions and we could have dualized in three dimensions all vectors into scalars (as in Ref. [17]). In this way we would have gotten two scalars from each vector field: one would be the electrostatic potential $\hat{A}^I{}_t$ and the other would be the magnetostatic potential $\tilde{\hat{A}}^I{}_t$, non-locally related to the other three components of the vector. In this three-dimensional theory, S duality would be realized by local transformations rotating the electrostatic and magnetostatic potentials into each other. Further reduction to two dimensions would give us a different ("dual") version of the two-dimensional theory related to the one we have obtained and we are going to study by a non-local transformation. The dual theory has also a $o\widehat{(2,4)}_2$ invariance but now the S duality transformations are in the zero-mode subalgebra $o(2,4;\mathbb{R})_2$ [13].

Another possibility is to study the S duality transformations directly in four dimensions.

1.2 S Duality Transformations

The $N=4, d=4$ supergravity equations of motion [11] have another duality symmetry nowadays called S duality that consists of electric-magnetic duality rotations accompanied of the inversion of the dilaton (the string coupling

[14] At the level of the effective action, of course.

constant) and constant shifts of pseudoscalar axion (see e.g. Ref. [18] for a review with references). This symmetry is only manifest in the Einstein frame and with the pseudoscalar axion. To study it we have to rewrite the action Eq. (1.1) in the Einstein frame and then trade the axion two-form $\hat{B}_{\hat{\mu}\hat{\nu}}$ by the pseudoscalar axion $\hat{a}$ by means of a Poincaré duality transformation. (One would get an inconsistent result if one replaced $\hat{H}$ by its dual field strength directly in the action). Thus, we consider first the above action as a functional of $\hat{H}$ which is now unrelated to $\hat{B}$. Then, we have to introduce a Lagrange multiplier ($\hat{a}$) to enforce the Bianchi identity of $\hat{H}$. Eliminating $\hat{H}$ in the action by using its equation of motion one finally gets the following action

$$S = \int d^4x\sqrt{|\hat{g}_E|}\left[\hat{R}(\hat{g}_E) - \tfrac{1}{2}(\partial\hat{\phi})^2 - \tfrac{1}{2}e^{2\hat{\phi}}(\partial\hat{a})^2 - \tfrac{1}{4}e^{-\hat{\phi}}\hat{F}^I\hat{F}^I + \tfrac{1}{4}\hat{a}\hat{F}^{I\star}\hat{F}^I\right]. \tag{1.13}$$

It is important for our purposes to have a very clear relation between the fields in both formulations since we have to identify the same charges in both and track them after their transformation. The (non-local) relation between $\hat{a}$ and $\hat{B}$ and the relation between the Einstein- and string-frame metric are given by

$$\left\{\begin{array}{rcl}\partial_{\hat{\mu}}\hat{a} & = & \frac{1}{3!\sqrt{|\hat{g}_E|}}e^{-2\hat{\phi}}\hat{\epsilon}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\hat{H}^{\hat{\nu}\hat{\rho}\hat{\sigma}}\,, \\ & & \\ \hat{g}_{E\hat{\mu}\hat{\nu}} & = & e^{-\hat{\phi}}\;\hat{g}_{\hat{\mu}\hat{\nu}}\,. \end{array}\right. \tag{1.14}$$

Defining now the complex scalar $\hat{\lambda}$ and the S dual vector field strengths $\tilde{\hat{F}}^I$ [8]

$$\hat{\lambda} \equiv \hat{a} + ie^{-\hat{\phi}}\,, \qquad \tilde{\hat{F}}^I \equiv e^{-\hat{\phi}}\,{}^{\star}\hat{F}^I + \hat{a}\hat{F}^I = \hat{\lambda}\hat{F}^{I\,+} + \text{c.c.}\,, \tag{1.15}$$

where

$$\hat{F}^{I\,\pm} \equiv \tfrac{1}{2}\left(\hat{F}^I \mp i\,{}^{\star}\hat{F}^I\right), \qquad {}^{\star}\hat{F}^{I\,\pm} = \pm i\hat{F}^{I\,\pm}\,, \tag{1.16}$$

one gets the action

$$S = \int d^4x\sqrt{|g_E|}\left[\hat{R}(\hat{g}_E) - \tfrac{1}{2}\frac{\partial_{\hat{\mu}}\hat{\lambda}\partial^{\hat{\mu}}\bar{\hat{\lambda}}}{\left(\Im\mathrm{m}\hat{\lambda}\right)^2} + \tfrac{1}{4}\hat{F}^I\,{}^{\star}\tilde{\hat{F}}^I\right]. \tag{1.17}$$

The equations of motion plus the Bianchi identities for the vector field strengths can be written in the following convenient form

$$\hat{G}_{E\hat{\mu}\hat{\nu}} + \frac{2}{(\hat{\lambda}-\bar{\hat{\lambda}})}\left[\partial_{(\hat{\mu}}\hat{\lambda}\partial_{\hat{\nu})}\bar{\hat{\lambda}} - \tfrac{1}{2}\hat{g}_{E\hat{\mu}\hat{\nu}}\partial\hat{\lambda}\partial\bar{\hat{\lambda}}\right]$$

$$-\tfrac{1}{4}\left({}^{\star}\tilde{\hat{F}}{}^{I}{}_{\hat{\mu}}{}^{\hat{\rho}}\;\; {}^{\star}\hat{F}^{I}{}_{\hat{\mu}}{}^{\hat{\rho}}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{\hat{F}}{}^{I}{}_{\hat{\nu}\hat{\rho}} \\ \hat{F}^{I}{}_{\hat{\nu}\hat{\rho}} \end{pmatrix} = 0\,, \tag{1.18}$$

$$\nabla^2\hat{\lambda} - 2\frac{(\partial\hat{\lambda})^2}{(\hat{\lambda}-\bar{\hat{\lambda}})} + \tfrac{i}{8}(\hat{\lambda}-\bar{\hat{\lambda}})^2\left(\hat{F}^{I-}\right)^2 = 0\,, \tag{1.19}$$

$$\nabla_{\hat{\mu}}\begin{pmatrix} {}^{\star}\tilde{\hat{F}}{}^{I\,\hat{\mu}\hat{\nu}} \\ {}^{\star}\hat{F}^{I\,\hat{\mu}\hat{\nu}} \end{pmatrix} = 0\,. \tag{1.20}$$

In this way, it is easy to see that the last equation is covariant under linear combinations of the vector fields and the S dual vector fields

$$\begin{pmatrix} \tilde{\hat{F}}{}^{I\prime\,\hat{\mu}\hat{\nu}} \\ \hat{F}^{I\prime\,\hat{\mu}\hat{\nu}} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tilde{\hat{F}}{}^{I\,\hat{\mu}\hat{\nu}} \\ \hat{F}^{I\,\hat{\mu}\hat{\nu}} \end{pmatrix}, \tag{1.21}$$

with the only requirement that the transformation matrix is non-singular. However, these vector fields are not independent and consistency implies the following non-linear transformations for the complex scalar $\hat{\lambda}$

$$\hat{\lambda}' = \frac{a\hat{\lambda}+b}{c\hat{\lambda}+d}\,. \tag{1.22}$$

The Einstein equation and the scalar equations are invariant if the constants a, b, c, d are the entries of an $SL(2,\mathbb{R})$ ($Sp(2,\mathbb{R})$) matrix i.e.

$$ad - bc = 1\,. \tag{1.23}$$

These transformations do not act on the Einstein metric. Observe that, although they are local transformations of the vector field strengths they are in fact non-local transformations in terms of the true variables; the vector fields themselves. Observe that the equations of motion of the vector fields are nothing but the Bianchi identities for the dual vector fields $\tilde{\hat{F}}{}^{I}{}_{\hat{\mu}\hat{\nu}}$ implying the local existence of the dual vector fields $\tilde{\hat{A}}{}^{I}{}_{\hat{\mu}}$ such that

$$\tilde{\hat{F}}{}^{I}{}_{\hat{\mu}\hat{\nu}} = 2\partial_{[\hat{\mu}}\tilde{\hat{A}}{}^{I}{}_{\hat{\nu}]}\,, \tag{1.24}$$

which justifies the definition of the $\tilde{\hat{F}}^{I}$'s. $\tilde{\hat{A}}^{I}$ depends non-locally on $\hat{A}^{I}$ and the pair $\tilde{\hat{A}}^{I}$, $\hat{A}^{I}$ transforms as an $SL(2,\mathbb{R})$ doublet.

$SL(2,\mathbb{R})$ is generated by three types of transformations[15]: rescalings of $\hat{\lambda}$

$$\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \qquad \hat{\lambda}' = a^2\hat{\lambda}\,, \tag{1.25}$$

continuous shifts of the axion

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \qquad \hat{\lambda}' = \hat{\lambda} + b\,, \tag{1.26}$$

and the discrete transformation

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \hat{\lambda}' = -1/\hat{\lambda}\,. \tag{1.27}$$

2 TNbh Asymptotics

In this section we will present the asymptotic behavior that we will assume for the solutions of the equations of motion originating from the action (1.1).

As advertised in the Introduction we are going to consider generalizations of asymptotically flat Einstein metrics. The asymptotic behavior of four-dimensional asymptotically flat metrics is completely characterized to first order in $1/r$ by only two charges: the ADM mass M and the angular momentum J. However, duality transforms asymptotically flat metrics into non-asymptotically flat metrics which need different additional charges to be asymptotically characterized. One of them [2] is the NUT charge N and closure under duality forces us to consider it. We will not need any more charges in the metric but, for completeness we define a possible new charge u which we will simply ignore in what follows.

With these conditions on the asymptotics of the four-dimensional metric it is always possible to choose coordinates such that the Einstein metric in the $t-\varphi$ subspace has the following expansion in powers of $1/r$:

$$(\hat{g}_{E\alpha\beta}) = \begin{pmatrix} -1+2M/r & 2N\cos\theta + \left[2J\sin^2\theta - 4M(N+u)\cos\theta\right]/r \\ & \\ 2N\cos\theta + \left[2J\sin^2\theta - 4M(N+u)\cos\theta\right]/r & (r^2+2Mr)\sin^2\theta \end{pmatrix}$$

[15] $SL(2,\mathbb{Z})$ can be generated by the discrete versions of the last two.

$$
+\begin{pmatrix} \mathcal{O}(r^{-2}) & \mathcal{O}(r^{-2}) \\ & \\ \mathcal{O}(r^{-2}) & \mathcal{O}(1) \end{pmatrix}. \tag{2.1}
$$

We will assume the following behavior for the dilaton

$$
e^{-\hat{\phi}} = 1 - 2\mathcal{Q}_d/r + 2\mathcal{W}\cos\theta/r^2 - 2\mathcal{Z}/r^2 + \mathcal{O}(r^{-3})\,, \tag{2.2}
$$

where $\mathcal{Q}_d$ is the dilaton charge, $\mathcal{W}$ is a charge related to the angular momentum that will be forced upon us by S duality and $\mathcal{Z}$ is a charge which is not independent, but a function of the electric and magnetic charges (see below) and is also forced upon us by S duality . This implies for the two-dimensional scalar matrix G:

$$
(G_{\alpha\beta}) = \begin{pmatrix} -1 + 2(M-\mathcal{Q}_d)/r & 2N\cos\theta + \left[2J\sin^2\theta - 4N(M-\mathcal{Q}_d)\cos\theta\right]/r \\ & \\ 2N\cos\theta + \left[2J\sin^2\theta - 4N(M-\mathcal{Q}_d)\cos\theta\right]/r & \left[r^2 + (M+\mathcal{Q}_d)r\right]\sin^2\theta \end{pmatrix}
$$

$$
+\begin{pmatrix} \mathcal{O}(r^{-2}) & \mathcal{O}(r^{-2}) \\ & \\ \mathcal{O}(r^{-2}) & \mathcal{O}(1) \end{pmatrix}. \tag{2.3}
$$

where we have already set $u = 0$.

Observe that we have fixed its constant asymptotic value equal to zero using the same reasoning as Burgess *et.al.* [2], i.e. rescaling it away any time they arise. The time coordinate, when appropriate, will be rescaled as well, in order to bring the transformed Einstein metric to the above form (i.e. to preserve our coordinate (gauge) choice), but in a duality-consistent way.

Sometimes it will also be necessary to rescale the angular coordinate φ in order to get a metric looking like (2.1). Conical singularities are then generically induced, and then the metric is not asymptotically TNbh in spite of looking like (2.1).

The objects we will consider will generically carry electric ($\mathcal{Q}_e^I$) and magnetic ($\mathcal{Q}_m^I$) charges with respect to the Abelian gauge fields $\hat{A}^I{}_{\hat{\mu}}$. Since we allow also for angular momentum, they will also have electric ($\mathcal{P}_e^I$) and magnetic ($\mathcal{P}_m^I$) dipole momenta. This implies for the two-dimensional scalar matrix A the following asymptotic behavior

$$
\left(\hat{A}^{I}{}_{\alpha}\right) = -2\begin{pmatrix} \mathcal{Q}_e^1/r - \mathcal{P}_e^I\cos\theta/r^2 & \mathcal{Q}_m^1\cos\theta + \mathcal{P}_m^1\sin^2\theta/r \\ \mathcal{Q}_e^2/r - \mathcal{P}_e^I\cos\theta/r^2 & \mathcal{Q}_m^2\cos\theta + \mathcal{P}_m^2\sin^2\theta/r \end{pmatrix}
$$

$$
+\begin{pmatrix} \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-2}) \\ \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-2}) \end{pmatrix}. \tag{2.4}
$$

Electric dipole momenta appear at higher order in $1/r$ and it is not strictly necessary to consider them from the point of view of T duality, since it will not interchange them with any of the other charges we are considering and that appear at lower orders in $1/r$. However, S duality will interchange the electric and magnetic dipole momenta and we cannot in general ignore them.

The different behavior of T and duality is due to the fact that T duality acts on the potential's components and S duality acts on the field strengths. Thus, for the purpose of performing T duality transformations the electric charge and the magnetic momentum terms in the potentials are of the same order in $1/r$. From the point of view of S duality, the electric and magnetic charge terms are of the same order in $1/r$.

To the matrix A in (2.4) we could have added a constant 2×2 matrix which would be the constant value of the t, φ components of the vector fields at infinity. Usually these constants are not considered because they can be removed by a four-dimensional gauge transformation with gauge parameters depending linearly on t and φ.

In [2] it was claimed those constants (in particular a constant term in the asymptotic expansion of $\hat{A}^{I}{}_{t}$), although pure gauge, do have an influence on physical characteristics of the dual solutions (actually this fact was interpreted there as evidence against the possibility of performing duality with respect to isometries with non-compact orbits).

However, a glance at the steps necessary to derive the $O(2,4)$ invariance of the dimensionally reduced theory [10] immediately reveals the necessity of not only staying in an adapted coordinate system, but also that the allowed four-dimensional gauge transformations are those which correspond to two-dimensional gauge transformations which are obviously independent of cyclic coordinates (in this case t and φ) and keep the matrix A invariant.

In other words: a constant shift in the matrix A, is not a symmetry of the two-dimensional theory but relates two inequivalent vacua[16]. The situation

[16]In any case, one should not be too dogmatic in this issue. After all, we are studying

from the point of view of S duality is not different: the result of the same classical S duality transformation (i.e. $SL(2,\mathbb{R})$ transformation) depends on the asymptotic constant values of the dilaton and axion. These can always be absorbed by further classical S duality transformations, but they do not relate equivalent vacua in general.

Thus, at least from the point of view adopted here, constant terms are indeed physically meaningful. From the point of view of obtaining a closed class of solutions under duality they are necessary because they are generated by duality transformations. Setting the constant terms to zero is just a specific gauge choice (as much as the coordinate choice made for the metric is also a coordinate choice). Duality transformations do not respect these gauge choices. In the next section we will study the inclusion of these constant terms in a consistent way by performing gauge transformations and coordinate changes in all the fields. However, the transformations with constant terms become very clumsy and we will consider most transformations on the configurations we are describing in this section, with zero constant terms. Only in Section 3.4, we will briefly consider a discrete duality transformation on the most general configuration.

The two-index form will have the usual charge $\mathcal{Q}_a$. Closure under duality again demands the introduction of a new extra parameter ("charge") that we denote by $\mathcal{F}$ and which will play an important role in what follows. At the same order in $1/r$ it is possible to define another charge $\mathcal{H}$ which is not independent, but a function of the electric and magnetic charges, as we will show. Its presence, is required by closure of duality but it transforms as a dependent charge and it does not play a relevant role. The asymptotic expansion are, then

$$
\begin{aligned}
\left(\hat{B}_{\alpha\beta}\right) &= 2\begin{pmatrix} 0 & \mathcal{Q}_a\cos\theta + \mathcal{F}\sin^2\theta/r + \mathcal{H}\cos\theta/r \\ -\mathcal{Q}_a\cos\theta - \mathcal{F}\sin^2\theta/r - \mathcal{H}\cos\theta/r & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & \mathcal{O}(r^{-2}) \\ \mathcal{O}(r^{-2}) & 0 \end{pmatrix}. \qquad (2.5)
\end{aligned}
$$

only the massless spectrum of four-dimensional string theory and performing dimensional reduction to two dimensions disregarding all the massive Kaluza-Klein modes which are associated to specific functional dependences on the coordinates t and φ. A full answer on whether t- or φ-dependent gauge transformations are allowed and their effect on the two-dimensional theory can only be obtained from the study of the full theory and it is beyond the scope of the effective theory that describes the massless spectrum.

As we will show in Section 3.3, the necessity of the new charge $\mathcal{F}$ becomes clear when looking at discrete duality subgroups checking that the subgroup's multiplication table is satisfied. From the physical point of view it is clear that the presence of angular momentum should induce such a charge.

Observe that we could have added a constant term to $\hat{B}_{t\varphi}$ as well which could be reabsorbed by four-dimensional gauge transformations which are not allowed from the two-dimensional point of view. We choose them to be initially zero as well for simplicity[17].

Now we have to show that the asymptotics that have been assumed on the gauge fields $\hat{A}^I$, and on the two-index field $\hat{B}$ correspond to the gauge-invariant charges that one can define by looking directly in the asymptotic expansions of the field strengths $\hat{F}^I$, or $\hat{H}$. The field strengths corresponding to the above potentials are

$$\begin{aligned}
\hat{F}^I &= 2\left(\mathcal{Q}_e^I - 2\frac{\mathcal{P}_e^I}{r}\cos\theta\right)\frac{1}{r^2}dr \wedge dt + 2\mathcal{P}_m^I \sin^2\theta \frac{1}{r^2} dr \wedge d\varphi \\
&\quad +2\left(\mathcal{Q}_m^I - 2\frac{\mathcal{P}_m^I}{r}\cos\theta\right)\sin\theta\; d\theta \wedge d\varphi - 2\mathcal{P}_e^I \sin\theta \frac{1}{r^2} d\theta \wedge dt \\
&\quad +\mathcal{O}(r^{-3})\,,
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
\hat{H} &= -2\left\{\mathcal{Q}_a - \left[\left(\mathcal{Q}_e^I \mathcal{Q}_m^I + \mathcal{H}\right) - 2\mathcal{F}\cos\theta\right]\frac{1}{r}\right\}\sin\theta\; d\theta \wedge dt \wedge d\varphi \\
&\quad -2\left[\mathcal{F}\sin^2\theta - \left(\mathcal{Q}_e^I \mathcal{Q}_m^I - \mathcal{H}\right)\cos\theta\right]\frac{1}{r^2}\; dr \wedge dt \wedge d\varphi \\
&\quad +\mathcal{O}(r^{-2})\,.
\end{aligned} \tag{2.7}$$

Observe that the effect of taking the Hodge dual of $\hat{F}^I$ is equivalent to replacing $(\mathcal{Q}_e^I, \mathcal{P}_e^I)$ by $(\mathcal{Q}_m^I, \mathcal{P}_m^I)$ and $(\mathcal{Q}_m^I, \mathcal{P}_m^I)$ by $(-\mathcal{Q}_e^I, -\mathcal{P}_e^I)$.

Now we have to identify $\mathcal{H}$. A convenient way of doing this is to dualize the three-form field strength to find the asymptotics of the pseudoscalar axion $\hat{a}$ defined in Eq. (1.14). The partial-differential equation $\partial_{\hat{\mu}}\hat{a}$ for $\hat{a}$ the consistency condition $\partial_{[\hat{\mu}}\partial_{\hat{\nu}]}\hat{a} = 0$ (which is the Bianchi identity for $\hat{a}$ and,

[17] A constant term in $\hat{B}_{t\varphi}$ implies via duality a constant term in $G_{t\varphi}$ which we have also initially set to zero for the same reason. We will consider both kinds of constant terms in the next section

therefore, the equation of motion for $\hat{B}$) has to be satisfied and this implies that the combination $\mathcal{Q}_e^I \mathcal{Q}_m^I - \mathcal{H}$ vanishes, so

$$\mathcal{H} = \mathcal{Q}_e^I \mathcal{Q}_m^I \,, \tag{2.8}$$

and we find

$$\begin{aligned} \hat{H} &= -2\left\{\mathcal{Q}_a - 2\mathcal{F}\cos\theta\frac{1}{r} + 2\mathcal{Q}_e^I\mathcal{Q}_m^I\frac{1}{r}\right\}\sin\theta\; d\theta\wedge dt\wedge d\varphi \\ & \quad -2\mathcal{F}\sin^2\theta\frac{1}{r^2}dr\wedge dt\wedge d\varphi + \mathcal{O}(r^{-2})\,. \end{aligned} \tag{2.9}$$

From this expression and (2.6) we see that all charges considered have a gauge-invariant meaning. The asymptotic expansion of the pseudoscalar axion $\hat{a}$ is (allowing for a constant value at infinity $\hat{a}_0$ that we will set to zero in the initial configuration)

$$\hat{a} = \hat{a}_0 + 2\mathcal{Q}_a/r - 2\mathcal{F}\cos\theta/r^2 + 2\mathcal{Q}_e^I\mathcal{Q}_m^I/r^2 + \mathcal{O}(r^{-3})\,, \tag{2.10}$$

which shows that $\mathcal{Q}_a$ is the standard axion charge defined, for instance, in Ref. [7]. With the pseudoscalar axion and the dilaton we find the asymptotic expansion of the complex scalar $\hat{\lambda}$ (allowing also for a non-vanishing asymptotic value for the dilaton $\hat{\phi}_0$)

$$\hat{\lambda} = \hat{\lambda}_0 + 2e^{-\hat{\phi}_0}\Upsilon/r - 2e^{-\hat{\phi}_0}\chi\cos\theta/r^2 + 2e^{-\hat{\phi}_0}\Theta/r^2 + \mathcal{O}(r^{-3})\,, \tag{2.11}$$

where

$$\begin{aligned} \hat{\lambda}_0 &= \hat{a}_0 + ie^{-\hat{\phi}_0}\,, \\ \Upsilon &= \mathcal{Q}_a - i\mathcal{Q}_d\,, \\ \chi &= \mathcal{F} - i\mathcal{W}\,, \\ \Theta &= \mathcal{Q}_e^I\mathcal{Q}_m^I - i\mathcal{Z}\,. \end{aligned} \tag{2.12}$$

2.1 Inclusion of Constant Terms

The inclusion of constant terms in the asymptotics of the matrices G, A and B in a consistent way is trickier than it seems at first sight. Let us start by discussing the modifications needed to include constant terms in A.

In the presence of constant terms in A one has to be very careful when identifying the *right* axion charges. If we consider the presence of only constant terms v^I in $\hat{A}^I{}_t$ for the moment

$$
\begin{aligned}
\left(\hat{A}^I{}_\alpha\right) &= \begin{pmatrix} v^1 - 2\mathcal{Q}_e^1/r + 2\mathcal{P}_e^1 \cos\theta/r^2 & -2\mathcal{Q}_m^1 \cos\theta - 2\mathcal{P}_m^1 \sin^2\theta/r \\ v^2 - 2\mathcal{Q}_e^2/r + 2\mathcal{P}_e^2 \cos\theta/r^2 & -2\mathcal{Q}_m^2 \cos\theta - 2\mathcal{P}_m^2 \sin^2\theta/r \end{pmatrix} \\
&\quad + \begin{pmatrix} \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-2}) \\ \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-2}) \end{pmatrix} . \qquad (2.13)
\end{aligned}
$$

the above expression for the axion field strength changes due to the Chern-Simons terms to

$$
\begin{aligned}
\hat{H} &= -2\left\{\left(\mathcal{Q}_a - \tfrac{1}{2}v^I\mathcal{Q}_m^I\right) - 2\left(\mathcal{F} - \tfrac{1}{2}v^I\mathcal{P}_m^I\right)\cos\theta/r \right. \\
&\quad \left. +2\mathcal{Q}_e^I\mathcal{Q}_m^I\frac{1}{r}\right\}\sin\theta\ d\theta\wedge dt\wedge d\varphi \\
&\quad -2\left(\mathcal{F} - \tfrac{1}{2}v^I\mathcal{P}_m^I\right)\sin^2\theta/r^2 dr\wedge dt\wedge d\varphi + \mathcal{O}(r^{-2})\,. \qquad (2.14)
\end{aligned}
$$

Now, the *right* charges are no longer $\mathcal{Q}_a$ and $\mathcal{F}$ but the combinations $\mathcal{Q}_a - \frac{1}{2}v^I\mathcal{Q}_m^I$ and $\mathcal{F} - \frac{1}{2}v^I\mathcal{P}_m^I$ that appear in $\hat{H}$. This really means that in presence of constant terms in $\hat{A}^I{}_t$ as above, the asymptotic expansion of B that gives the right charges as in Eq. (2.9), and the one that on has to use is (setting $\mathcal{H} = 0$)

$$
\begin{aligned}
\hat{B}_{t\varphi} &= 2\left(\mathcal{Q}_a + \tfrac{1}{2}v^I\mathcal{Q}_m^I\right)\cos\theta + 2\left(\mathcal{F} + \tfrac{1}{2}v^I\mathcal{P}_m^I\right)\sin^2\theta/r \\
&\quad +2\mathcal{Q}_e^I\mathcal{Q}_m^I\cos\theta/r + \mathcal{O}(r^{-2})\,, \qquad (2.15)
\end{aligned}
$$

Analogous results would have been obtained by creating the constant terms via a t-dependent gauge transformation of the gauge fields (which induces, due to the Chern-Simons term present in $\hat{H}$ a gauge transformation of the two-form field) of and looking for a gauge-independent definition of the axion charges. This way of thinking (i.e. that the terms arise because of gauge transformations that take us from the gauge in which we have written the asymptotic expansion of the potentials and metric in the previous section) is the most appropriate to study the inclusion of constant terms in G and B. For instance, let us now perform a φ-dependent gauge transformation of the gauge fields with parameter $\Lambda^I = w^I\varphi$ that induces a constant term in $\hat{A}^I{}_\varphi$

$$\delta\hat{A}^I{}_\varphi = w^I\,. \tag{2.16}$$

This transformation induces on the two-form field Eq. (2.15) (taking into account the constant terms v^I) a gauge transformation in B. To make the story short, we will simply say that if we consider a general matrix A with constant terms

$$\begin{aligned}\left(\hat{A}^I{}_\alpha\right) \;&=\; \begin{pmatrix} v^1 - 2\mathcal{Q}_e^1/r + 2\mathcal{P}_e^1\cos\theta/r^2 & w^1 - 2\mathcal{Q}_m^1\cos\theta - 2\mathcal{P}_m^1\sin^2\theta/r \\ v^2 - 2\mathcal{Q}_e^2/r + 2\mathcal{P}_e^2\cos\theta/r^2 & w^2 - 2\mathcal{Q}_m^2\cos\theta - 2\mathcal{P}_m^2\sin^2\theta/r \end{pmatrix} \\ &\quad + \begin{pmatrix} \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-2}) \\ \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-2}) \end{pmatrix}\,,\end{aligned} \tag{2.17}$$

we must consider a B matrix of the form (we only write the $\hat{B}_{t\varphi}$ entry)

$$\begin{aligned}\hat{B}_{t\varphi} \;&=\; x + \tfrac{1}{2}v^I w^I 2\left(\mathcal{Q}_a + \tfrac{1}{2}v^I\mathcal{Q}_m^I\right)\cos\theta - w^I\mathcal{Q}_e^I/r \\ &\quad +2\left(\mathcal{F} + \tfrac{1}{2}v^I\mathcal{P}_m^I\right)\sin^2\theta/r + 2\mathcal{Q}_e^I\mathcal{Q}_m^I\cos\theta/r + \mathcal{O}(r^{-2})\,,\end{aligned} \tag{2.18}$$

to get an axion field strength of the form (2.9) so the constants $\mathcal{Q}_a$ and $\mathcal{F}$ are still the axion charges. Now, a new t- or φ-dependent gauge transformation of the form $\Lambda = \delta v^I t + \delta w^I\varphi$ is reabsorbed in a redefinition of the constants x, v^I, w^I and does not affect the charges, that keep their gauge-invariant meaning. The constant x can also be generated or absorbed by a gauge transformation of the two-form field and it does not induce any other changes in the asymptotics of other fields.

Finally, we will see that duality sometimes creates a constant term in $\hat{g}_{Et\varphi}$. This term can be reabsorbed or induced by a reparametrization of the time coordinate $t \to t - q\varphi$. This transformation changes G and A to

$$(G_{\alpha\beta}) = \begin{pmatrix} -1 + 2(M-\mathcal{Q}_d)/r & (q + 2N\cos\theta)\left[1 - 2\left(M-\mathcal{Q}_d\right)/r\right] + 2J\sin^2\theta/r \\ (q + 2N\cos\theta)\left[1 - 2\left(M-\mathcal{Q}_d\right)/r\right] + 2J\sin^2\theta/r & \left[r^2 + (M+\mathcal{Q}_d)r\right]\sin^2\theta \end{pmatrix}$$

$$
+\begin{pmatrix} \mathcal{O}(r^{-2}) & \mathcal{O}(r^{-2}) \\ & \\ \mathcal{O}(r^{-2}) & \mathcal{O}(1) \end{pmatrix} . \tag{2.19}
$$

and

$$
\left(\hat{A}^I{}_\alpha\right) = \begin{pmatrix} v^1 - 2\mathcal{Q}_e^1/r + 2\mathcal{P}_e^1 \cos\theta/r^2 & (w^1 - qv^1) - 2\mathcal{Q}_m^1 \cos\theta \\ & +2q\mathcal{Q}_e^1/r - 2\mathcal{P}_m^1 \sin^2\theta/r \\ & \\ v^2 - 2\mathcal{Q}_e^2/r + 2\mathcal{P}_e^2 \cos\theta/r^2 & (w^2 - qv^2) - 2\mathcal{Q}_m^2 \cos\theta \\ & +2q\mathcal{Q}_e^2/r - 2\mathcal{P}_m^2 \sin^2\theta/r \end{pmatrix}
$$

$$
+\begin{pmatrix} \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-2}) \\ & \\ \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-2}) \end{pmatrix} . \tag{2.20}
$$

Observe that the electric dipole momenta do not appear in the right column because they are of higher order.

It is easy to see that there is no need to do further changes in B. Thus, the most general asymptotic expansions that we will consider are given by the matrices G in Eq. (2.19) A in Eq. (2.20) and B in Eq. (2.18) which define gauge-invariant charges in the sense that t and φ-dependent gauge transformations $\Lambda^I = \delta v^I t + (\delta w^I - q\delta v^I)$ and reparametrizations of the form $t \to t + \delta q \varphi$ which are the ones that do not take us out of the Kaluza-Klein ansatz become simple redefinitions of the constants $v^I \to v^I + \delta v^I$ etc. leaving the charges invariant (which justifies their name).

The class of asymptotic behavior just described, determined by the twelve charges

$$
\left\{M, J, N, \mathcal{Q}_a, \mathcal{F}, \mathcal{Q}_d, \mathcal{Q}_e^I, \mathcal{Q}_m^I, \mathcal{P}_m^I\right\} \tag{2.21}
$$

(with or without constant terms in the matrices G, B, A) will be referred to henceforth as *TNbh asymptotics*.

3 Transformation of the Charges under Duality

In this Section we are going to study the transformation of the charges of asymptotically TNbh configurations under the T and S duality transformations found in Section 1.

3.1 Deriving the Form for the T Duality Transformation Matrices

The problem that now will be tackled is how to generate the explicit transformations of the full $O(2,4,\mathbb{R})$ classical T duality group to find which subgroup maps TNbh asymptotics into TNbh asymptotics. $O(2,4)$ is a non-compact, non-connected group and our first task is to elucidate its structure.

It is known that every element from a group G, can be written as a sequence of operators, which are always part of the connected component containing the identity G_0 (which is itself a subgroup of G), and elements from the coset G/G_0. The action of these elements on any element of G is to take them from a connected part to a different connected part. This coset is called the *mapping-class group* $\pi_0(G)$.

$O(2,4)$ has four connected pieces: two correspond to matrices with determinant $+1$ and two to matrices with determinant -1. The former two connected pieces constitute the subgroup $SO(2,4)$ and are related to the other two by a discrete transformation that generates the group $O(2,4)/SO(2,4) = \mathbb{Z}_2^{(B)}$. The two connected components of the subgroup[18] $SO(2,4)$ differ by the sign of the $(1,1)$ component of the matrices of the defining representation. The component with positive sign contains the identity and is the subgroup $SO^{\uparrow}(2,4)$ and is related to the other connected component (which is not a subgroup and we denote by $SO^{\downarrow}(2,4)$) by a discrete transformation that generates another $\mathbb{Z}_2^{(S)} = SO(2,4)/SO^{\uparrow}(2,4)$ subgroup.

Thus, the mapping-class group of $O(2,4)$ is $O(2,4)/SO^{\uparrow}(2,4) = \mathbb{Z}_2^{(B)} \times \mathbb{Z}_2^{(S)}$.

We will study it in detail later. Now we are going to concentrate on describing the duality transformations in the component connected with the identity $SO^{\uparrow}(2,4)$.

Every element of the connected component of a group can be written as a sequence of its one-parameter-subgroups [20] and we are going to study these first.

In our case these are the exponentiated versions of the generators of the Lie algebra $so(2,4)$, which we write in the covariant form M_{ij}

$$\Omega_{ij}(\alpha_{ij}) = \exp\{-\alpha_{(ij)} M_{(ij)}\}\,, \tag{3.1}$$

and which satisfy the commutation relations

[18] All groups $SO(n,m)$ have two connected components [19].

$$[M_{ij}, M_{kl}] = \eta_{il}M_{jk} - \eta_{ik}M_{jl} + \eta_{jk}M_{il} - \eta_{jl}M_{ik} \ , \tag{3.2}$$

where, again, the indices $i, j, k, l = 1\ldots, 6$ and $\eta = \mathrm{diag}(-,-,+,+,+,+)$ is the diagonal metric of $O(2,4)$.

It should be noted that the action of $O(2,4)$ is 6-dimensional, which means that the group acts through its vector representation on the matrix $\mathcal{M}_{ij}$ and on the vectors $\mathcal{K}^i{}_\mu$. The generators of $so(2,4)$ in the vector representation, denoted by Γ, are given by

$$\Gamma\left(M_{ij}\right)^k{}_l = 2\eta_{l[i}\eta^k{}_{j]} \, . \tag{3.3}$$

Upon exponentiation of a single generator, one gets a one-parameter subgroup. In this way to get all the basic one-parameter subgroups of $SO^{\uparrow}(2,4)$ *in the diagonal basis* with metric η. Thus, we still need to transform the on-parameter subgroup transformations to the non-diagonal basis using Eq. (1.12) and finally we can study the effect of these transformations on the fields using Eq. (1.9).

3.2 The One-Parameter Subgroups of the T Duality Group

The one-parameter subgroups of $SO^{\uparrow}(2,4)$ are either boosts involving one of the indices $1,2$ and one of the indices $3,4,5,6$ or rotations involving the indices 1 and 2 or two of the indices $3,4,5,6$.

Boost matrices are taken to have the form

$$\Omega_\eta(\mathrm{boost}) = \begin{pmatrix} \mathrm{ch} & .. & \mathrm{sh} & .. \\ .. & .. & .. & .. \\ \mathrm{sh} & .. & \mathrm{ch} & .. \\ .. & .. & .. & .. \end{pmatrix} , \tag{3.4}$$

and generate a non-compact $SO^{\uparrow}(1,1) = \mathbb{R}^+$ subgroup and every rotation will be taken to have the form

$$\Omega_\eta(\mathrm{rotation}) = \begin{pmatrix} \cos & .. & \sin & .. \\ .. & .. & .. & .. \\ -\sin & .. & \cos & .. \\ .. & .. & .. & .. \end{pmatrix} , \tag{3.5}$$

and generates a compact $U(1)$ subgroup. Here the operators will be labelled by the Lie algebra generator that generates the operators. For instance, we have[19]

$$\Omega_{\eta 13} \equiv \exp\{-\alpha_{13} M_{\eta 13}\} = \left(\begin{array}{cccc} \mathrm{ch} & 0 & \mathrm{sh} & \\ 0 & 1 & 0 & \\ \mathrm{sh} & 0 & \mathrm{ch} & \\ & & & \mathbb{I}_3 \end{array} \right), \tag{3.6}$$

and in the non-diagonal basis

$$\Omega_{13} = \left(\begin{array}{cccc} \mathrm{ch}+\mathrm{sh} & 0 & 0 & \\ & & & \\ 0 & 1 & 0 & \\ & & & \\ 0 & 0 & \mathrm{ch}-\mathrm{sh} & \\ & & & \\ & & & \mathbb{I}_3 \end{array} \right), \tag{3.7}$$

etc. Therefore it is not difficult to compute the action of these subgroups on the background fields. It turns out that only a few of them (seven, but only five with a non-trivial action and just three with different actions on the charges) preserve TNbh asymptotics.

When the transformations that leave TNbh asymptotics intact are known the exact change in the asymptotic charges can be computed. The charges transform linearly. Actually, the T duality transformations close on sets of four charges and their effect can be described by matrices acting on three four-component charge vectors:

$$\vec{M} \equiv \begin{pmatrix} M \\ \mathcal{Q}_d \\ \mathcal{Q}_e^1 \\ \mathcal{Q}_e^2 \end{pmatrix}, \qquad \vec{N} \equiv \begin{pmatrix} N \\ \mathcal{Q}_a \\ \mathcal{Q}_m^1 \\ \mathcal{Q}_m^2 \end{pmatrix}, \qquad \vec{J} \equiv \begin{pmatrix} J \\ \mathcal{F} \\ \mathcal{P}_m^1 \\ \mathcal{P}_m^2 \end{pmatrix}, \tag{3.8}$$

which will be referred to, respectively, as *electric, magnetic* and *dipole* charge vectors.

There is a fourth charge vector that contains the electric dipole momenta $\mathcal{P}_e^I$, the dilaton dipole-type charge $\mathcal{W}$ and an unidentified geometrical charge that we denote by K

$$\vec{K} \equiv \begin{pmatrix} K \\ \mathcal{W} \\ \mathcal{P}_e^1 \\ \mathcal{P}_e^2 \end{pmatrix}. \tag{3.9}$$

[19]Throughout we shall use the abbreviations c $= \cos(\alpha_{ij})$, s $= \sin(\alpha_{ij})$, ch $= \cosh(\alpha_{ij})$ and sh $= \sinh(\alpha_{ij})$, the ij being the indices of the transformation.

The presence of this fourth charge vector is required by S duality, as we will explain later.

For each TNbh duality transformation there is a unique matrix action on the three vectors. This, for the moment, can be considered merely a convenient representation of the duality transformations. It will be shown later in the paper that the Bogomol'nyi bound can be rewritten in terms of our multiplets in an exceedingly convenient way.

Let us examine now examine the interesting duality transformations case by case.

3.2.1 Ω_{13}

The action of this subgroup is simply equivalent to a rescaling of the time coordinate and obviously it preserves TNbh asymptotics. Using the inverse rescaling to rewrite the metric in the gauge (2.1) we find the the action of this duality transformation is trivial.

3.2.2 Ω_{15}

This subgroup preserves TNbh asymptotics and our gauge choice for the metric (2.1) and for the matrices A, B. In particular it does not generate any constant term in the A, B, G matrices. Thus, one can proceed to compute the transformation of the charges. This transformation is described by the 4×4 symmetric matrix

$$\Omega_{15}^{(4)} \equiv \begin{pmatrix} \frac{1+\mathrm{ch}}{2} & \frac{1-\mathrm{ch}}{2} & \frac{\mathrm{sh}}{\sqrt{2}} & 0 \\ \frac{1-\mathrm{ch}}{2} & \frac{1+\mathrm{ch}}{2} & -\frac{\mathrm{sh}}{\sqrt{2}} & 0 \\ \frac{\mathrm{sh}}{\sqrt{2}} & -\frac{\mathrm{sh}}{\sqrt{2}} & \mathrm{ch} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.10}$$

so

$$\tilde{\vec{M}} = \Omega_{15}^{(4)} \vec{M}\,, \qquad \tilde{\vec{N}} = \Omega_{15}^{(4)} \vec{N}\,, \qquad \tilde{\vec{J}} = \Omega_{15}^{(4)} \vec{J}\,. \tag{3.11}$$

3.2.3 Ω_{16}

The effect of this transformation is identical to the previous one with the interchange of the labels $I = 1$ and $I = 2$. This, the matrix that describes it on the charges is

$$\Omega_{16}^{(4)} \equiv \begin{pmatrix} \frac{1+\mathrm{ch}}{2} & \frac{1-\mathrm{ch}}{2} & 0 & \frac{\mathrm{sh}}{\sqrt{2}} \\ \frac{1-\mathrm{ch}}{2} & \frac{1+\mathrm{ch}}{2} & 0 & -\frac{\mathrm{sh}}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ \frac{\mathrm{sh}}{\sqrt{2}} & -\frac{\mathrm{sh}}{\sqrt{2}} & 0 & \mathrm{ch} \end{pmatrix} . \tag{3.12}$$

3.2.4 Ω_{24}

This transformation is analogous to the transformation Ω_{13}: its effect is equivalent to a rescaling of the coordinate φ that preserved it periodicity, which is initially fixed to be 2π, i.e. all components of fields with indices φ are rescaled, but the coordinate itself is not rescaled. Now, to go back to our coordinate choice (2.1) we have to rescale φ, changing its periodicity and, thus, introducing conical singularities. Therefore, this transformation does not preserve TNbh asymptotics.

3.2.5 Ω_{35}

The result of this transformation is another asymptotically TNbh metric written in our gauge (2.1) up to a rescaling of the time coordinate and up to constant term in the matrix A:

$$v^1 = \sqrt{2}\sin\alpha_{35} \,, \tag{3.13}$$

and this has to be taken into account in the definitions of the axion charges that have to be identified in the transformed configurations using the expansion of B in Eq. (2.15). The rescaling of the time coordinate can be performed combining ω_{35} with an ω_{13} transformation with the right parameter. The result of this composition is a one-parameter subgroup of transformations that do preserve TNbh asymptotics and our gauge choice except for the non-vanishing v^1. The effect of this composite transformation on the charges can be described by the 4×4 symmetric matrix

$$
(\Omega_{13}\Omega_{35})^{(4)} \equiv \begin{pmatrix} \frac{\mathrm{c}+1}{2\mathrm{c}} & \frac{\mathrm{c}-1}{2\mathrm{c}} & \frac{1}{\sqrt{2}}\frac{\mathrm{s}}{\mathrm{c}} & 0 \\ \frac{\mathrm{c}-1}{2\mathrm{c}} & \frac{\mathrm{c}+1}{2\mathrm{c}} & -\frac{1}{\sqrt{2}}\frac{\mathrm{s}}{\mathrm{c}} & 0 \\ \frac{1}{\sqrt{2}}\frac{\mathrm{s}}{\mathrm{c}} & -\frac{1}{\sqrt{2}}\frac{\mathrm{s}}{\mathrm{c}} & \frac{1}{\mathrm{c}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.14}
$$

Now, this matrix is exactly the same as $\Omega_{15}^{(4)}$ with the replacement

$$
\cos\alpha_{35} = 1/\cosh\alpha_{15} , \tag{3.15}
$$

and, so, these transformations are identical on the charges.

3.2.6 Ω_{36}

This transformation is identical to the previous one with the interchange of the labels $I = 1$ and $I = 2$. Thus, it also generates a constant term in the matrix A which has to be taken care of when identifying the axion charges of the transformed configurations:

$$
v^2 = \sqrt{2}\sin\alpha_{36} , \tag{3.16}
$$

Therefore, although it does not preserve TNbh asymptotics, it can be combined with an Ω_{13} transformation into a TNbh-preserving one-parameter subgroup of transformations that can be described by the action of the 4×4 symmetric matrix

$$
(\Omega_{13}\Omega_{36})^{(4)} \equiv \begin{pmatrix} \frac{\mathrm{c}+1}{2\mathrm{c}} & \frac{\mathrm{c}-1}{2\mathrm{c}} & 0 & \frac{1}{\sqrt{2}}\frac{\mathrm{s}}{\mathrm{c}} \\ \frac{\mathrm{c}-1}{2\mathrm{c}} & \frac{\mathrm{c}+1}{2\mathrm{c}} & 0 & -\frac{1}{\sqrt{2}}\frac{\mathrm{s}}{\mathrm{c}} \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}}\frac{\mathrm{s}}{\mathrm{c}} & -\frac{1}{\sqrt{2}}\frac{\mathrm{s}}{\mathrm{c}} & 0 & \frac{1}{\mathrm{c}} \end{pmatrix} . \tag{3.17}
$$

on the three four-component charge vectors $\vec{M}, \vec{N}, \vec{J}$.

3.2.7 Ω_{56}

This transformation is just the $SO(2)$ subgroup acting on the gauge fields only and rotates the electric and magnetic charges and the magnetic dipole momenta.Thus, it can be described by the 4×4 antisymmetric matrix

$$\Omega_{56}^{(4)} \equiv \begin{pmatrix} \mathbb{I}_2 & & \\ & \mathrm{c} & \mathrm{s} \\ & -\mathrm{s} & \mathrm{c} \end{pmatrix} . \tag{3.18}$$

3.3 The Mapping-Class Group T Duality Transformations $\mathbb{Z}_2^{(B)}$

The last "elementary" duality transformations of $O(2,4)$ that we have to study are those in the coset $O(2,4)/SO^{\uparrow}(2,4) = \mathbb{Z}_2^{(B)} \times \mathbb{Z}_2^{(S)}$. What do these transformations correspond to? In Ref. [21] the simpler duality groups $O(1,1)$ and $O(1,2)$ where analyzed in detail and it was found that the subgroup $\mathbb{Z}_2^{(S)}$ is essentially generated by a reflection in all directions in the scalar σ-model target space. Here we can do the same and take the generator of $\mathbb{Z}_2^{(S)}$ as the total reflection $-\mathbb{I}_6$. In the same reference it was also found that the subgroup $\mathbb{Z}_2^{(B)}$ corresponds essentially to Buscher's duality transformations [22].

The generator of $\mathbb{Z}_2^{(B)}$ is not unique (it is a coset group). Two obvious choices correspond to the Buscher transformations in the directions t and φ. The Buscher transformation in the direction φ does not preserve TNbh asymptotics and so we will take as generator of $\mathbb{Z}_2^{(B)}$ the Buscher transformation in the direction t, that we denote by τ, with matrix

$$\Omega_{\eta}(\tau) = \begin{pmatrix} +1 & \\ & -\mathbb{I}_5 \end{pmatrix} . \tag{3.19}$$

Observe that there is no inconsistency in taking one and not the other as inequivalent representatives because from the point of view of the TNbh-preserving duality subgroup they are no related: only an infinite boost (Ω_{14}) will completely rotate t into φ and, in any case, this subgroup does not preserve TNbh asymptotics itself.

As was said in Section 2, the necessity of introducing additional "charges" like $\mathcal{F}$ becomes evident[20] when one studies discrete duality subgroups like $\mathbb{Z}_2^{(B)}$. If we analyze the τ-transformation explicitly, we see that the τ-transform of $\hat{g}_{t\varphi}$ is

$$\tilde{\hat{g}}_{t\varphi} = \frac{1}{2\Omega} \left\{ \hat{g}_{tt} \left[\hat{A}^I{}_t \hat{A}^I{}_\varphi - 2\hat{B}_{t\varphi} \right] - \hat{g}_{t\varphi} \hat{A}^I{}_t \hat{A}^I{}_t \right\} , \tag{3.20}$$

where Ω goes asymptotically as

[20] To see it in the continuous subgroups one has to study the invertibility of the transformations, which is much harder.

$$\Omega = 1 - \frac{4M}{r} + \mathcal{O}(r^{-2}) \ . \tag{3.21}$$

Looking at the asymptotic behavior of the terms involved, it is easy to see that to get a contribution to J, the initial configuration has to have a $r^{-1}\sin^2\theta$ term in its asymptotic expansion of the Kalb-Ramond field. This also shows that J transforms into the new charge $\mathcal{F}$ since $\tau^{-1} = \tau$.

The effect of τ on all the charges can be expressed in terms of the same symmetric 4×4 matrix $\Omega^{(4)}_\tau$

$$\Omega^{(4)}_\tau = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & \mathbb{I}_2 \end{pmatrix} , \tag{3.22}$$

acting on the charge vectors $\vec{M}, \vec{N}, \vec{J}$. The involutive property, that on the charges $\tau^2 = id$ is immediately apparent.

A natural worry at this point is whether a combination of transformations that do not preserve TNbh asymptotics, can result in a TNbh asymptotics-preserving transformation.

This is a complicated and time-consuming problem that can only be handled by computational methods for just products of two transformations. The result of our (CPU-limited) search is negative.

3.4 Constant Parts in the Gauge Fields and the Closed Set of Asymptotic "Charges" Under τ

We have to find the way in which the transformations of the charges change when we include constant terms in the matrices G, A, B. To do a general study would take to much CPU time. Thus, we will only perform a full check of only the τ transformation, although the general picture should become quite clear from our results for τ and other general arguments.

First of all, the consistency in the way we have defined charges and constant terms (which will be referred to as *moduli*) implies that the moduli transform non-linearly amongst themselves and, thus, their transformations can be studied by setting to zero the charges. For the τ transformation this allows us to immediately get[21]

[21] Simultaneous rescalings of the dilaton and the time coordinate t are necessary to eliminate the constant value of the dilaton at infinity and to get an asymptotically TNbh metric.

$$
\begin{cases} \tilde{v}^I = v^I\,, & \tilde{q} = \xi x\,, \\ \tilde{w}^I = w^I\,, & \tilde{x} = \xi^{-1} q\,. \end{cases} \tag{3.23}
$$

where we have used $\xi = (1 - \vec{v}\,^2/2)^{-1}$.

Next, we expect the multiplet structure of the duality transformations to remain valid in the presence of non-trivial moduli. (The multiplets contain multipole terms of the same order of different fields.) This can be checked explicitly, but it also allows us to set to zero all charges except for those in one multiplet and find their transformation more easily. The result is that we can describe in all of them the τ transformation with a unique moduli-dependent matrix $\Omega^{(4)}_\tau(x, q, v, w)$

$$
\Omega^{(4)}_\tau(x, q, v, w) = \begin{pmatrix} -\frac{1}{2}\xi\vec{v}\,^2 & \xi & \xi v^1 & \xi v^2 \\ \xi & -\frac{1}{2}\xi\vec{v}\,^2 & -\xi v^1 & -\xi v^2 \\ -\xi v^1 & \xi v^1 & 1 + \xi(v^1)^2 & \xi v^1 v^2 \\ -\xi v^2 & \xi v^2 & \xi v^1 v^2 & 1 + \xi(v^2)^2 \end{pmatrix}. \tag{3.24}
$$

Observe that this matrix indeed squares to the identity.

What happens to the other transformations in presence of non-trivial moduli? The rule is that now the 4×4 matrices $\Omega^{(4)}_{ij}$ will become moduli-dependent matrices $\Omega^{(4)}_{ij}(x, q, v, w)$ and the group multiplication table is satisfied in the following sense:

$$
\Omega^{(4)}_{T_2}(\tilde{x}, \tilde{q}, \tilde{v}, \tilde{w})\Omega^{(4)}_{T_1}(x, q, v, w) = \Omega^{(4)}_{T_2 \cdot T1}(x, q, v, w)\,, \tag{3.25}
$$

where $(\tilde{x}, \tilde{q}, tildev, \tilde{w})$ are the transformed moduli under T_1. In the case of τ we had, trivially

$$
\Omega^{(4)}_\tau(\tilde{x}, \tilde{q}, \tilde{v}, \tilde{w})\Omega^{(4)}_\tau(x, q, v, w) = \mathbb{I}_4\,. \tag{3.26}
$$

because $\Omega^{(4)}_\tau(x, q, v, w)$ only depends on the v^I and these are invariant under τ.

3.5 Transformation of the Charges under S Duality

The transformation of the electric, magnetic, dilaton and axion charges under S duality has been previously studied in Ref. [7, 8]. Here we are considering more charges and we are choosing initial configurations with vanishing asymptotic values of the axion and dilaton. In general, S duality generates non-vanishing values of these constants and we will remove them by applying further S duality transformations.

Let us first see the effect of general classical S duality transformations on arbitrary configurations. It is easy to see that the transformation (1.22) acts on the asymptotic value of $\hat{\lambda}$ as follows [7, 8]

$$\hat{\lambda}_0' = \frac{a\hat{\lambda}_0 + b}{c\hat{\lambda}_0 + d}\,, \tag{3.27}$$

and on its complex charge as follows

$$\Upsilon' = \left(\frac{c\bar{\hat{\lambda}}_0 + d}{c\hat{\lambda}_0 + d}\right)\Upsilon\,. \tag{3.28}$$

The factor multiplying Υ is just a $\hat{\lambda}_0$-dependent complex phase and, thus the axion and dilaton charges are simply rotated into one another. It is also easy to see that the additional complex charge that we are considering here $\chi = \mathcal{F} - i\mathcal{W}$ transforms exactly as Υ.

The effect on the electric and magnetic charges is a bit more difficult to explain because the electric and magnetic charges that transform in a natural way under S duality, and which are the ones conserved in the quantum theory when the Witten effect [23] is taken into account, are not the ones we have defined. To be precise, the equation of motion and the Bianchi identity tell us that the two charges that are well defined in the quantum theory and obey the Dirac-Schwinger-Zwanziger quantization condition are

$$\left\{\begin{array}{lclcl} q_e^I & \sim & \int_{S^2_\infty} \tilde{\hat{F}}^I & = & e^{-\hat{\phi}_0}\mathcal{Q}_e^I - \hat{a}_0\,\mathcal{Q}_m^I\,, \\ q_m^I & \sim & \int_{S^2_\infty} \hat{F}^I & = & \mathcal{Q}_m^I\,. \end{array}\right. \tag{3.29}$$

This pair of charges transform under (1.21) as an $SL(2,\mathbb{R})$ doublet

$$\begin{pmatrix} q_e^{I\,\prime} & q_m^{I\,\prime} \end{pmatrix} = \begin{pmatrix} q_e^I & q_m^I \end{pmatrix} \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}\,, \tag{3.30}$$

which ensures that the DSZ quantization condition, which can be written for two dyons in the form

$$\begin{pmatrix} q_e^{I\,(1)} & q_m^{I\,(1)} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q_e^{I\,(2)} \\ q_m^{I\,(2)} \end{pmatrix} = cn\,, \qquad n \in \mathbb{Z}\,, \tag{3.31}$$

where c is some constant, is S duality invariant. From the relation between the charges $\left(q_e^{I\,(1)} \quad q_m^{I\,(1)}\right)$ and the charges $\mathcal{Q}_e^I, \mathcal{Q}_m^I$ that we are using (3.29) one readily finds

$$\begin{aligned}
\mathcal{Q}_e^{I\,\prime} &= (c\hat{a}_0 + d)\mathcal{Q}_e^I + ce^{-\hat{\phi}_0}\mathcal{Q}_m^I\,, \\
\mathcal{Q}_m^{I\,\prime} &= -ce^{-\hat{\phi}_0}\mathcal{Q}_e^I + (c\hat{a}_0 + d)\mathcal{Q}_m^I\,.
\end{aligned} \tag{3.32}$$

It is easy to see that the electric and magnetic dipole momenta transform in exactly the same fashion.

Now we have to adapt these formulae to our case in which the original configuration has $\hat{\lambda}_0 = i$ and in which we want the transformed configuration to have also $\hat{\lambda}_0' = i$. This can be achieved by applying after the general $SL(2,\mathbb{R})$ transformation, two transformations (1.25,1.26) with the appropriate values of a and b to absorb the constant values of the axion and dilaton. This is equivalent to allow only an $SO(2)$ subgroup of $SL(2,\mathbb{R})$ to act on the charges. The result, expressed in terms of the entries of the original $SL(2,\mathbb{R})$ matrix is

$$\begin{pmatrix} \mathcal{Q}_e^{I\,\prime} \\ \\ \mathcal{Q}_m^{I\,\prime} \end{pmatrix} = \begin{pmatrix} \frac{d}{\sqrt{c^2+d^2}} & \frac{c}{\sqrt{c^2+d^2}} \\ \\ \frac{-c}{\sqrt{c^2+d^2}} & \frac{d}{\sqrt{c^2+d^2}} \end{pmatrix} \begin{pmatrix} \mathcal{Q}_e^I \\ \\ \mathcal{Q}_m^I \end{pmatrix}, \tag{3.33}$$

and similarly for the vector of dipole momenta $\left(\mathcal{P}_m^I\,, \mathcal{P}_e^I\right)$ and

$$\begin{pmatrix} \mathcal{Q}_d' \\ \\ \mathcal{Q}_a' \end{pmatrix} = \begin{pmatrix} \frac{d^2-c^2}{\sqrt{c^2+d^2}} & \frac{2cd}{\sqrt{c^2+d^2}} \\ \\ \frac{-2cd}{\sqrt{c^2+d^2}} & \frac{d^2-c^2}{\sqrt{c^2+d^2}} \end{pmatrix} \begin{pmatrix} \mathcal{Q}_d \\ \\ \mathcal{Q}_a \end{pmatrix}, \tag{3.34}$$

and, analogously for the charge vector $(\mathcal{W}\,, \mathcal{F})$. Observe that the last $SO(2)$ transformation matrix is precisely the square of the former.

It is now clear that the multiplet structure that we built for the T duality transformations is not respected by S duality: the last three components of the "electric" multiplet $\hat{M}$ are rotated into the last three components of the "magnetic" multiplet $\vec{N}$ and vice versa. The same happens with the multiplet $\vec{K}$ defined in Eq. (3.9), whose last three components are rotated into those of the multiplet $\vec{J}$ in exactly the same way, and vice versa (this is the reason why we introduced K and $\vec{K}$ in the first place). To respect the T duality multiplet structure and, at the same time incorporate the S duality multiplet structure it is useful to introduce the complexified multiplets

$$\vec{\mathcal{M}} \equiv \vec{M} + i\vec{N} = \begin{pmatrix} \mathcal{M} \\ i\Upsilon \\ \Gamma^1 \\ \Gamma^2 \end{pmatrix}, \tag{3.35}$$

where

$$\mathcal{M} \equiv M + iN\,, \qquad \Gamma^I \equiv \mathcal{Q}_e^I + i\mathcal{Q}_m^I\,, \tag{3.36}$$

and

$$\vec{\mathcal{J}} \equiv \vec{K} + i\vec{J} = \begin{pmatrix} \mathcal{J} \\ i\chi \\ \Pi^1 \\ \Pi^2 \end{pmatrix}, \tag{3.37}$$

where

$$\mathcal{J} \equiv K + iJ\,, \qquad \Pi^I \equiv \mathcal{P}_e^I + i\mathcal{P}_m^I\,. \tag{3.38}$$

These two complex vectors transform under T duality with exactly the same $\Omega^{(4)}_{ij}$ matrices as the real vectors and, under the above S duality transformations with the complex $\Sigma^{(4)}$ $SO(2)$ matrix

$$\Sigma^{(4)} = \begin{pmatrix} 1 & & & \\ & e^{2i\theta} & & \\ & & e^{i\theta} & \\ & & & e^{i\theta} \end{pmatrix} \qquad \theta = \mathrm{Arg}(d - ic)\,, \tag{3.39}$$

so

$$\vec{\mathcal{M}}' = \Sigma^{(4)}\vec{\mathcal{M}}\,, \qquad \vec{\mathcal{J}}' = \Sigma^{(4)}\vec{\mathcal{J}}\,. \tag{3.40}$$

4 The Asymptotic Duality Subgroup

We define the Asymptotic Duality Subgroup (ADS) as the subgroup of the full duality group that respects TNbh asymptotics. In the previous section we have identified several one-parameter subgroups of the T duality part of the ADS and we know that the full S duality group is a subgroup of the ADS. However these two subgroups do not commute and, together, generate a large

ADS. We proceed to identify it in the next section and later we will use it to study the invariance of the Bogomol'nyi bound relevant for the theory we are considering under it.

4.1 Identification of the Asymptotic Duality Subgroup

First, we are going to identify the T duality subgroup of the ADS. As we have seen, from the point of view of its action on the charges it has only three non-trivial one-parameter subgroups which we take to be the ones corresponding to the transformations $\Omega^{(4)}_{15}, \Omega^{(4)}_{16}, \Omega^{(4)}_{56}$. To find the group that they generate we first study the algebra of their infinitesimal generators $M^{(4)}_{ij}$

$$\Omega^{(4)}_{ij} = \mathbb{I}_4 - \alpha_{(ij)} M^{(4)}_{(ij)}\,, \tag{4.1}$$

which are given by

$$\begin{aligned} M^{(4)}_{15} &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M^{(4)}_{16} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}, \\ M^{(4)}_{56} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{4.2}$$

These infinitesimal generators obey the algebra

$$[M^{(4)}_{56}, M^{(4)}_{15}] = M^{(4)}_{16}\,, \;\; [M^{(4)}_{56}, M^{(4)}_{16}] = -M^{(4)}_{15}\,, \;\; [M^{(4)}_{15}, M^{(4)}_{16}] = -M^{(4)}_{56}\,. \tag{4.3}$$

A small calculation of the Killing metric then show that on the base $\{M^{(4)}_{15}, M^{(4)}_{16}, M^{(4)}_{56}\}$ the metric is diagonal with entries $\eta^{(3)} = \mathrm{diag}(+,-,-)$ thus proving that the algebra is $o(1,2)$ and the group generated by the one-parameter subgroups is $SO^{\uparrow}(1,2)$ and that the T duality part of the ADS (taking into account the discrete transformations) is $O(1,2)$.

This raises now the question as to what is the meaning of the four-component charge vectors. Clearly they transform in the four-dimensional reducible representation of $O(1,2)$ furnished by the matrices $\Omega^{(4)}$. The only representation of this kind is the direct sum of a singlet and a vector (three-dimensional) representation of $O(1,2)$, which in turn means that there is a linear combination of the charges in each charge vector that is invariant under the full T duality part of the ADS. It is easy to see that these combinations are

$$\tfrac{1}{\sqrt{2}}\left(M+\mathcal{Q}_d\right)\,,\qquad \tfrac{1}{\sqrt{2}}\left(N+\mathcal{Q}_a\right)\,,\qquad \tfrac{1}{\sqrt{2}}\left(J+\mathcal{F}\right)\,. \tag{4.4}$$

The triplets over which T duality acts in the vector representation of $SO(1,2)$ are

$$\begin{aligned}
\vec{M}^{(3)} &= \begin{pmatrix} \frac{1}{\sqrt{2}}\left(M-\mathcal{Q}_d\right) \\ \mathcal{Q}_e^1 \\ \mathcal{Q}_e^2 \end{pmatrix}, \qquad & \vec{N}^{(3)} &= \begin{pmatrix} \frac{1}{\sqrt{2}}\left(N-\mathcal{Q}_a\right) \\ \mathcal{Q}_m^1 \\ \mathcal{Q}_m^2 \end{pmatrix}, \\
\vec{J}^{(3)} &= \begin{pmatrix} \frac{1}{\sqrt{2}}\left(J-\mathcal{F}\right) \\ \mathcal{P}_m^1 \\ \mathcal{P}_m^2 \end{pmatrix},
\end{aligned} \tag{4.5}$$

and, on this representation the generators of the algebra are

$$\begin{aligned}
M_{15}^{(3)} &= \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad & M_{16}^{(3)} = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
M_{56}^{(3)} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.
\end{aligned} \tag{4.6}$$

We remark for future use that the four-dimensional matrices $\Omega^{(4)}$ of the $1\oplus 3$ representation of $O(1,2)$ respect the diagonal $O(2,2)$ metric $\eta^{(4)} = \mathrm{diag}(+,+,-,-)$ and are also automatically $O(2,2)$ matrices.

4.2 The Bogomol'nyi Bound and its Variation

In $N=4$ supergravity there are two Bogomol'nyi (B) bounds, of the form

$$M^2 - |Z_i|^2 \geq 0\,,\qquad i=1,2\,, \tag{4.7}$$

where the Z_i's are the complex skew eigenvalues of the central charge matrix and are combinations of electric and magnetic charges of the six graviphotons.

These two bounds can be combined into a single bound by multiplying them and then dividing by M^2. One gets, then, a *generalized* B bound

$$M^2 + \frac{|Z_1 Z_2|}{M^2} - |Z_1|^2 - |Z_2|^2 \geq 0\,. \tag{4.8}$$

In regular black-hole solutions the second term can be identified with scalar charges of "secondary" type. The identification is, actually (with zero value for the dilaton at infinity)

$$\frac{|Z_1 Z_2|}{M^2} = \mathcal{Q}_d^2 + \mathcal{Q}_a^2\,, \tag{4.9}$$

and, taking into account the expression of the central charges in terms of the $\mathcal{Q}^I_{e,m}$'s one gets the generalized B bound[22] [7]

$$M^2 + \mathcal{Q}_d^2 + \mathcal{Q}_a^2 - \mathcal{Q}_e^I \mathcal{Q}_e^I - \mathcal{Q}_m^I \mathcal{Q}_m^I \geq 0\,. \tag{4.10}$$

Note however that this bound is valid only for asymptotically flat spaces (i.e. with $N = 0$). This problem can however be overcome by the reasoning of Ref. [5] where it was observed that the NUT charge N does enter in the generalized B bound. With our definitions the B bound for asymptotically TNbh spaces takes the form

$$M^2 + N^2 + \mathcal{Q}_d^2 + \mathcal{Q}_a^2 - \mathcal{Q}_e^I \mathcal{Q}_e^I - \mathcal{Q}_m^I \mathcal{Q}_m^I \geq 0\,. \tag{4.11}$$

Now we want to study the invariance of this bound under the T and S duality pieces of the ADS that preserves TNbh asymptotics. We will not make distinctions between primary and secondary scalar charges since all we are interested in are the transformation rules of the scalar charges which are the same for primary- or secondary-type scalar charges. We will focus on this distinction in the next section.

Before perform a direct check, let us analyze what we can expect the result to be. The T duality piece of the ADS preserves in general unbroken supersymmetries of the low-energy string effective action: one can prove that if one solution admits Killing spinors the dual solution does as well. Equivalent properties can be checked from the world-sheet point of view. The only instances in which T duality seems not to respect unbroken supersymmetries (at least in a *manifest* fashion from the spacetime point of view) is when a Buscher T duality transformation is performed with respect to an isometry with fixed points, like the isometry in the direction φ in our axially-symmetric case [6]. However, this transformation does not respect TNbh asymptotics

[22] A general expression of the same kind for black holes with regular horizons in genera; theories with scalars non-minimally coupled to vector fields has been found in [24].

and therefore it does not belong to the ADS. S duality is known to always preserve unbroken supersymmetry [7] and, thus, we can expect the B bound to be invariant under the full ADS.

To study the transformation properties of the B bound under the physical TNbh asymptotics-preserving duality group it is convenient to use the diagonal metric of $SO(2,2)$ $\eta^{(4)} = \text{diag }(1,1,-1,-1)$ already introduced at the beginning of this section. Using this metric and the charge vectors defined in Eqs. (3.35,3.37) the B bound can be easily rewritten in this form:

$$\vec{\mathcal{M}}^{\dagger}\eta^{(4)}\vec{\mathcal{M}} \geq 0\,. \tag{4.12}$$

In this form the B bound of $N=4, d=4$ supergravity is manifestly $U(2,2)$-invariant. Observe that $U(2,2) \sim O(2,4)$, although it is not clear if this fact is a mere coincidence or it has a special significance. The T duality piece of the ADS is an $O(1,2)$ subgroup of the $O(2,2)$ canonically embedded in $U(2,2)$ and obviously preserves the B bound. The S duality piece of the ADS is a $U(1)$ subgroup diagonally embedded in $U(2,2)$ through the matrices $\Sigma^{(4)}$ defined in Eq. (3.39) and obviously preserve the B bound.

The charges in the vector $\vec{\mathcal{J}}$ do not appear in the B bound and neither T nor S duality change this fact. It is not possible to constrain the values of any of the charges it (in particular J) by using duality and supersymmetry, as was suggested in the Introduction.

Although we are not going to study the full ADS generated by the T duality and the S duality pieces, it is clear that there are transformations in it that rotate the mass M into the NUT charge N and J into K: it is enough to perform first a τ transformation to interchange the first and second components of the $U(2,2)$ vectors $\vec{\mathcal{M}}$ and $\vec{\mathcal{J}}$, then perform an S duality transformation that interchanges the real and imaginary parts of the second component of those vectors and a further τ-transformation to bring this rotated component back to the first position.

4.3 Primary Scalar Hair and Unbroken Supersymmetry

So far we have not discussed in detail the physical meaning of the charges that define TNbh asymptotics. In particular, we have considered completely unrestricted charges $\mathcal{Q}_d$ and $\mathcal{Q}_a$.

The dilaton charge $\mathcal{Q}_d$, not being protected by a gauge symmetry, is not a conserved charge. In four dimensions the Kalb-Ramond two-form is dual to the pseudoscalar axion and the charge $\mathcal{Q}_a$ is just its charge. Again, $\mathcal{Q}_a$ is not a conserved charge. This may seem contradictory because in the two-form version there is indeed a gauge symmetry. However, the two-form conserved charge is actually associated to one-dimensional extended objects, not to the point-like objects we are considering here. Thus both charges can be considered non-conserved scalar charges (*hair*).

If these scalars were minimally-coupled scalars the standard no-hair theorems would apply to them and any non-vanishing value of $\mathcal{Q}_d$ and $\mathcal{Q}_a$ would imply the presence of naked singularities. The prototype of this kind of singular solution with non-trivial scalar hair (called *primary hair*) is the one given in Refs. [25] for the theory with a massless scalar and action

$$S = \int d^4x\sqrt{|\hat{g}_E|}\ \left[\hat{R}(\hat{g}_E) + \tfrac{1}{2}\partial_{\hat{\mu}}\hat{\phi}\partial^{\hat{\mu}}\hat{\phi}\right] . \tag{4.13}$$

The solutions take the form

$$\left\{ \begin{array}{rcl} d\hat{s}_E^2 & = & W^{\frac{M}{r_0}-1}W dt^2 - W^{1-\frac{M}{r_0}}\left[W^{-1}dr^2 + r^2 d\Omega^2\right] , \\ \\ \hat{\phi} & = & \hat{\phi}_0 - \frac{\mathcal{Q}_d}{r_0}\ln W , \end{array} \right. \tag{4.14}$$

where

$$\left\{ \begin{array}{rcl} W & = & 1 - 2r_0/r , \\ \\ r_0^2 & = & M^2 + \mathcal{Q}_d^2 . \end{array} \right. \tag{4.15}$$

The three fully independent parameters that characterize each solution are the mass M, the scalar charge[23] $\mathcal{Q}_d$ and the value of the scalar at infinity ϕ_0. Only when $\mathcal{Q}_d = 0$ one has a regular solution (Schwarzschild). In all other cases there is a singularity at $r = r_0$, where the area of 2-spheres of radius r vanishes.

Before continuing with our discussion a couple of remarks should be made: first, this whole family of solutions belong to the TNbh class and, second, observe that the above family of solutions includes a non-trivial *massless* solution. Setting $M = 0$ above we find

$$\left\{ \begin{array}{rcl} d\hat{s}_E^2 & = & dt^2 - dr^2 - W r^2 d\Omega^2 , \\ \\ \hat{\phi} & = & \hat{\phi}_0 - \ln W , \qquad e^{\hat{\phi}-\hat{\phi}_0} = W^{-1} , \end{array} \right. \tag{4.16}$$

with

[23] We use the symbol of the dilaton charge because these solutions (which are written in the Einstein frame) are also solutions of the equations of motion of the low-energy string-effective action Eq. (1.1) with $\hat{\phi}$ identified with the dilaton.

$$W = 1 - \frac{2\mathcal{Q}_d}{r}\,. \tag{4.17}$$

In the full low-energy string effective action, the dilaton and the axion are non-minimally coupled scalars, though, and the existence of black-hole solutions with regular horizons in theories with non-minimally coupled scalars is known [26, 27]. In these solutions, the scalar (dilaton) charge is identical to a certain fixed combinations of the other, conserved, charges:

$$\mathcal{Q}_d \sim \frac{\mathcal{Q}_m^I \mathcal{Q}_m^I - \mathcal{Q}_e^I \mathcal{Q}_2^I}{2M}\,. \tag{4.18}$$

The same is true of the axion charge in solutions with non-trivial axion hair and regular horizons [28, 7, 29, 30]. The axion charge is in those cases given by

$$\mathcal{Q}_a \sim \frac{\mathcal{Q}_e^I \mathcal{Q}_m^I}{2M}\,. \tag{4.19}$$

This kind of scalar hair, whose existence does not imply the presence of naked singularities is called *secondary hair*. It is clear that the existence of secondary hair does not preclude the existence of primary hair. In fact, the solutions above can be interpreted in the framework of string theory with primary but no secondary hair and there are solutions which have both kinds of hair at the same time [31].

Primary scalar hair always seems to imply the presence of naked singularities, and the no-hair theorem (if it existed such a general theorem) should probably be called *no-primary hair theorem*.

So, what can duality and supersymmetry tell us about primary scalar hair? At first sight, nothing. In the standard derivations of the different B bound formulae only conserved electric and magnetic charges appear and only when all the scalar hair is secondary and given by the above formulae one can derive the generalized B bounds of the previous section in which the scalar charges appear.

Nevertheless, let us consider a simple example: Schwarzschild's solution (given above just by setting $\mathcal{Q}_d = 0$). This solution has no unbroken supersymmetries, which can be understood in terms of non-saturation of the B bound ($M \geq 0$). A Buscher T duality transformation in the time direction belongs to the physical duality group and should preserve the supersymmetry properties and asymptotic behavior of the solution and so it should yield a new solution with no unbroken supersymmetries and TNbh asymptotics. A short calculation shows that the dual solutions is exactly the massless solution with primary scalar hair written above in Eqs. (4.16,4.17)! It is easy to check that this solution admits no $N = 4$ Killing spinors and so it has no unbroken supersymmetries[24]. However, the fact that this solution has no unbroken supersymmetries would not have been clear from the B bound point of view , had we used the once-standard form in which primary hair should not added to it, since its mass and all the other conserved charges are zero, meaning that the bound would be trivially saturated.

All that happened in this transformation is that the mass M, which does appear in the B bound has completely transformed in primary dilaton charge $\mathcal{Q}_d$ which in principle does not.

After our study of the transformation of charges it is clear that to reconcile these two results one has to admit that the generalized B bound formula Eq. (4.11) does apply to all kinds of scalar charge and not only to the secondary-type one. Only in this way the invariance of the B bound becomes consistent with the covariance of the Killing spinor equations.

Although our reasoning is completely clear when we look on specific solutions one should be able to derive B bounds including primary scalar charges using a Nester construction based on the supersymmetry transformation laws of the fermions of the supergravity theory under consideration. To be able to do this one has to be able to manage more general boundary conditions including the seemingly unavoidable naked singularities that primary hair implies.

Although we have kept this discussion strictly four-dimensional it is easy to generalize these arguments to higher dimensions. In fact, solutions generalizing the one above to higher (d) dimensions can be straightforwardly found

$$\left\{ \begin{array}{rcl} ds^2 & = & -W^{\frac{M}{r_0}-1} W dt^2 + W^{\frac{1}{d-3}\left(1-\frac{M}{r_0}\right)} \left[W^{-1} d\rho^2 + \rho^2 d\Omega^2_{(d-2)}\right] \, , \\ & & \\ \phi & = & \phi_0 + \frac{\mathcal{Q}_d}{r_0} \ln W \, . \end{array} \right. \tag{4.20}$$

where

$$W = 1 - \frac{2r_0}{\rho^{d-3}} \, , \tag{4.21}$$

and now

$$r_0^2 = M^2 + 2\left(\tfrac{d-3}{d-2}\right) \mathcal{Q}_d \, . \tag{4.22}$$

[24] The dilatino supersymmetry transformation rule would be equal to $\delta_\epsilon \lambda^I \sim \not\partial \hat{\phi} \epsilon^I$ which only vanishes for $\epsilon^I = 0$. (I is an $SU(4)$ index here).

For $\mathcal{Q}_d = 0$ we recover the d-dimensional Schwarzschild solution. In all other cases we have metrics with naked singularities either at $\rho = 0$ or $\rho^{d-3} = 2r_0$.

A further example can be useful to fix these ideas.

Using our conventions, it is possible to write the stringy RN solution in the following form:

$$\left\{ \begin{array}{rcl} d\hat{s}_E^2 & = & -H^{-2}Wdt^2 + H^2\left[W^{-1}dr^2 + r^2d\Omega^2\right] , \\ \\ e^{-\hat{\phi}} & = & H/H = 1 , \\ \\ \hat{A}^{(1)}{}_t & = & 2\alpha_1 \frac{|Q|}{M-r_0}\left(H^{-1} - 1\right) , \\ \\ \hat{A}^{(2)}{}_\varphi & = & -2\alpha_2|Q|\cos\theta , \end{array} \right. \tag{4.23}$$

where H and W are (not independent) harmonic functions

$$H = 1 + \frac{M - r_0}{r} , \qquad W = 1 - \frac{2r_0}{r} , \tag{4.24}$$

and the constants are:

$$\alpha_i^2 = 1 , \qquad r_0^2 = M^2 - 2Q^2 , \tag{4.25}$$

where we have set

$$\mathcal{Q}_e^1 = \alpha_1|Q| , \qquad \mathcal{Q}_m^2 = \alpha_2|Q| . \tag{4.26}$$

The dilaton charge is identically zero for this family. Observe also that $M - r_0 \geq 0$ always, and thus H never vanishes and so it never gives rise to any singularities in the metric apart from the one at $r = 0$, which is the curvature singularity. The metric is also singular at the horizon $r = 2r_0 > 0$ where W vanishes, covering the physical singularity at $r = 0$.

The extremal limit is $r_0 = 0$, $M = \sqrt{2}|Q|$, which makes W disappear and H becomes an unrestricted harmonic function (we could describe many BHs if we wanted). In this limit the horizon is placed at $r = 0$, which is th locus of a two-sphere instead of a point, as can be seen by a coordinate change. The curvature singularity is not covered by these coordinates.

Th B bound for this family of solutions is

$$M^2 - 2Q^2 = M^2 - (\mathcal{Q}_e^1)^2 - (\mathcal{Q}_m^2)^2 \geq 0 , \tag{4.27}$$

with the equality satisfied in the extreme $r_0 = 0$ limit. Performing the τ transformation on the above family of solutions we get the dual family of solutions

$$
\left\{
\begin{array}{rcl}
d\tilde{\hat{s}}^2_E & = & -H^{-1}K^{-1}W dt^2 + HK\left[W^{-1}dr^2 + r^2 d\Omega^2\right] \,, \\
 & & \\
e^{\tilde{\hat{\phi}}} & = & H/K \,, \\
 & & \\
\tilde{\hat{A}}^1{}_t & = & 2\alpha_1 \frac{|Q|}{M-r_0}\left(K^{-1}-1\right) \,, \\
 & & \\
\tilde{\hat{A}}^2_\varphi & = & -2\alpha_2 |Q| \cos\theta \,,
\end{array}
\right.
\tag{4.28}
$$

where

$$
K = 1 - \frac{M+r_0}{r} \,, \tag{4.29}
$$

The above metric has several singularities: there is a curvature singularity at $r=0$ and the would-be horizon singularity at $r=2r_0$ but both lie beyond another physical singularity at $r = M + r_0 \geq 2r_0$ which is where the function K vanishes and where 2-spheres of radius r have zero area. This is, therefore, a naked singularity.

Now the mass of the dual solution is clearly equal to the dilaton charge of the original RN solution $\tilde{M} = \mathcal{Q}_d = 0$ and vice-versa $\tilde{\mathcal{Q}}_d = M$. The electric and magnetic charges have the same values.

This is a non-extreme massless "black hole" where the non-extremality is provided by primary scalar hair.

Now, if one takes the "extreme limit" $r_0 = 0$ (that is, the extreme limit in the original solution) which is also the limit in which all the primary scalar hair vanishes and all the dilaton charge is completely determined by the electric and magnetic charges[25] $\tilde{\mathcal{Q}}_d^2 = 2\mathcal{Q}^2$ so the B bound is saturated

$$
\left\{
\begin{array}{rcl}
d\tilde{\hat{s}}^2_E & = & -H^{-1}K^{-1} dt^2 + HK\left[dr^2 + r^2 d\Omega^2\right] \,, \\
 & & \\
e^{\tilde{\hat{\phi}}} & = & H/K \,, \\
 & & \\
\tilde{\hat{A}}^1{}_t & = & -\sqrt{2}\alpha_1 \left(K^{-1}-1\right) \,, \\
 & & \\
\tilde{\hat{A}}^2_\varphi & = & -2\alpha_2 |Q| \cos\theta \,,
\end{array}
\right.
\tag{4.30}
$$

[25]The situation parallels the usual situation in which there is unconstrained "primary mass" and "secondary mass" which is completely fixed by the electric and magnetic charges through the B bound.

which is one of the extreme massless black holes in Refs. [32], identified as composite objects in the sense of Ref. [33] in Ref. [34] and further studied in Refs. [35].

Observe that, while primary scalar hair should be included in the B bound, the primary scalar hair completely disappears in the saturated B bound. Thus, unbroken supersymmetry acts as a *cosmic hairdresser* and it is not possible to find solutions with unbroken supersymmetry and primary scalar hair.

As a last example we consier the well-known Kerr spacetime metric which in Boyer-Lindquist coordinates reads:

$$
\begin{aligned}
d\hat{s}_E^2 \;=\; & -\frac{r^2-2Mr+a^2}{r^2+a^2\cos^2\theta}(dt-a\sin^2\theta d\phi)^2 \\
& +\frac{\sin^2\theta}{r^2+a^2\cos^2\theta}\left[(r^2+a^2)d\phi-adt\right]^2 \\
& +\frac{r^2+a^2\cos^2\theta}{r^2-2Mr+a^2}dr^2+(r^2+a^2\cos^2\theta)d\theta^2\,,
\end{aligned}
\tag{4.31}
$$

where $a=J/M$. This metric belongs to a more general class of metrics which can be written in appropriate coordinates as:

$$
d\hat{s}_E^2=-G(dt-\omega d\phi)^2+Adr^2+Bd\theta^2+Cd\phi^2\,, \tag{4.32}
$$

where G,ω,A,B and C are arbitrary functions of r and θ, conveying the adapted character of the coordinates employed.

The T dual with respect to the isometry with Killing vector $\frac{\partial}{\partial t}$ is easily found to be, in the Einstein frame,

$$
d\tilde{\hat{s}}_E^2=-G^{-1}dt^2+Adr^2+Bd\theta^2+Cd\phi^2\,, \tag{4.33}
$$

There is also a two-form present, given by

$$
\hat{B}=-\omega dt\wedge d\phi\,, \tag{4.34}
$$

as well as a dilaton, namely

$$
\hat{\phi}=-\tfrac{1}{2}\log|G|\,, \tag{4.35}
$$

It is well known that in the static Schwarzschild case [2] what appear as horizons in one metric, look as singularities in the T dual of it. In the more general, stationary case considered here, there are two related concepts: the infinite redshift surface, (also called the "static limit") that is, the stationary limit surface bordering the region in which the Killing $\frac{\partial}{\partial t}$ is timelike; and the event horizon; that is the hypersurface where $r = constant$ becomes null ; the region between those two surfaces being the ergosphere.

In the $\tilde{Kerr}$ metric presented above, there is no "infinite redshift surface", and before the surface $r = const$ becomes null, a singularity develops, located at

$$G \equiv \frac{r^2 + a^2 \cos^2\theta - 2mr}{r^2 + a^2\cos^2\theta} = 0\,. \tag{4.36}$$

The metric is easily seen to be asymptotically flat, and the 2-form goes to zero at infinity as

$$\hat{B} = 2ma \sin^2\theta\, \frac{1}{r}\left[1 + \frac{2m}{r} + \mathcal{O}(r^{-2})\right] dt \wedge d\phi\,. \tag{4.37}$$

5 Conclusions

The results of the present paper concerning the transformation of the charges under duality leave unanswered the question posed in the Introduction: why the angular momentum appears in the definition of extremality (defining the borderline between regular horizon and a naked singularity, with zero Hawking temperature) but not in the Bogomol'nyi bound (whose saturation guarantees absence of quantum corrections, as well as a "zero force condition", allowing superposition of static solutions).

We now believe this is due to the fact that stationary (as opposed to static) black holes possess a specific decay width, which can even be seen classically by scattering waves off the black hole. This process is known as "superradiance" ([36]; see also [37]) in the black hole literature.

The way this appears is that the amplitude for reflected waves is greater than the corresponding incident amplitude, for low frequencies, up to a given frequency cutoff, $m\Omega_H$, depending on the angular momentum of the hole, and such that $\Omega_H(a = 0) = 0$. The angular momentum of the hole decreases by this mechanism until a static configuration is reached. The physics underlying this process is similar to the one supporting Penrose's energy extraction mechanism, namely, the fact that energy can be negative in the ergosphere. This, in turn, is an straightforward consequence of the mathematical fact that the Energy of a test particle is defined as $E = p.k$, where p is the momentum of the particle, and k is the Killing vector (which has spacelike character precisely in the ergosphere); and the product of a spacelike vector with a timelike one does not have a definite sign.

Quantum mechanically, this means that there are two competing mechanisms of decay for a rotating (stationary) black hole: spontaneous emission (the quantum effect associated to the superradiance), which is not thermal (and disappears when the angular momentum goes to zero) and Hawking radiation, which is thermal.

The first one is most efficient for massive black holes, but its width is never zero even for small masses, until the black hole has lost all its angular momentum.

This clearly shows that even if the black hole is extremal, it cannot be stable quantum mechanically as long as its angular momentum is different from zero. This argument taken literally would suggest that it is not possible to have BPS states with non zero angular momentum, unless they are such that no ergosphere exists. This is the case of the supersymmetric Kerr-Newman solutions which are singular and, therefore, do not have ergosphere. What is not clear is why supersymmetry signals as special that singular case and not the usual extremal Kerr-Newman black hole[26].

It could well be that Supergravity is not capable to give that answer but String Theory is: from the String Theory point of view, given an extreme Reissner-Nordström black hole, if we want to add angular momentum, we can only do it at the expense of adding mass at the same time. Thus, according to the String Theory black-hole building rules, one can get extreme Kerr-Newman black holes but never a supersymmetric (singular) object with non-zero angular momentum. In this sense, while Supergravity acts as a cosmic censor only in static cases, String Theory seems to act as a true cosmic censor in all cases. The singular solutions cannot be built in the theory.

A similar argument could also be enough to prove a no-hair theorem in String Theory: it could happen that it is impossible to build String Theory states with primary scalar hair because there is no primary source for scalar hair in it. In this sense String Theory would act as a *cosmic hairdresser*. Here the situation is, though, a bit different. First of all, we are clearly a long way from proving that there are no microscopic configurations in String Theory that result in macroscopic primary scalar hair. In fact, the situation resembles a bit the situation of the "primary mass" (the mass that exceeds the Bogomol'nyi identity) since it is not clear what the microscopic configuration that manifests itself as that primary mass is and, thus, there is no String Theory model for the Schwarzschild black hole. It is, in fact, conceivable that both quantities have the similar origins, as T duality seems

[26]This argument seems to be valid only in four dimensions, though, since rotating charged black holes which are BPS states exist in five dimensions [38]. The existence of two Casmirs for the five-dimensional angular momentum seems to play an important role.

to be indicating. This would be a more attractive scenario since then we would have a tool ins String Theory to understand no-hair theorems from first principles.

There is, yet, another, more speculative, possibility that we could like to mention. Since extreme and non-extreme massless "black holes" seem to have the same kind of singularities as their regular T dual counterparts (null and spacelike, respectively) one could, in principle, use the spacetime of the massless black hole to patch up the spacetime of the massive one, gluing them at the singularity. This would be a non-analytic continuation through the singularity with the help of T duality much in the same spirit as T duality at finite temperature can relate high and low-temperature regimes of the heterotic string even though, in between the free energy diverges at the Hagedorn temperature [39].

From the point of view of String Theory this possibility looks more plausible when one takes into account the lower sensitivity of strings to spacetime singularities, as compared to point particles [40],

Work in this direction is in progress.

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