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Preprint

# THE LIMITS FOR PROTON CURRENTS IN AN ACCELERATOR TUBE WITH STRONG FOCUSING

by

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Novosibirsk - 1967

Translated at CERN by E. Fischer

(Original : Russian)

(CERN Trans. 67-4)

Geneva

July, 1967

## A B S T R A C T

The motion of an intense proton beam in an accelerator tube with strong focusing quadrupole lenses has been studied.

The maximum current $I_{max}$ passing through the tube is calculated as function of the phase space volume $V_n$ occupied by the beam. Furthermore are calculated the conditions for injection into the tube necessary in order to realize the maximum current.

The basic method used for the investigation is that of random search.

The results can be used for chosing the working conditions of a tube, of the proton injector and of the ion-optical matching system.

The maximum proton currents accelerated up to an energy of several MeV with accelerators of a new type as, for instance, the transformer type accelerator (Ref. 1), are determined to a large degree by the electrical stability and the focusing properties of the accelerating tubes. It became, therefore, necessary to calculate the limit transmission characteristics for given focusing conditions and for concrete accelerating tubes and to calculate, furthermore, the injection conditions necessary for the acceleration of those limit currents.

When investigating the motion of an intense proton beam in an accelerator tube most authors limit themselves to the construction of the trajectories of "border" particles of the beam while the initial conditions at the entrance into the tube are chosen on the basis of the performance of the proton injector (Ref. 2).

It has been shown in Ref. 3 that the trajectory of a "border" particle coincides with the envelope of the beam only if the phase space volume is zero, that is if the defocusing influence of the thermal velocities are neglected. This neglection and with it the hypothesis of linearity are inadmissible for the treatment of real proton beams.

The maximum beam current $I_{max}$ captured into acceleration depends on the phase space volume $V_n$ of the beam. For the choice of the working conditions and of the injector parameters one must at first determine the transmission characteristics of the tube $I_{max} = f(V_n)$.

Actually, in order to reach the maximum possible current one must, furthermore, determine the exact conditions for the injection of the beam into the tube. The knowledge of these conditions is essential for the choice of the injector optic and for the design of a beam matching system.

In the present investigation a proton accelerator tube is considered with strong focusing magnetic quadrupole lenses (the use of electrostatic quadrupole lenses in the tube is described in Ref. 1). Strong focusing is very useful, because it allows the acceleration of intense proton beams (Refs. 1, 2) and prevents the formation of electron avalanches, which can reduce considerably the electrical stability of the tube (Ref. 3).

An essential feature of the operation of such tubes is the high energy increase of the accelerated particles over the length of one element of periodicity of the strong focusing channel. This feature makes it imperative to use mainly numerical methods for the solution of the equations of motion.

We use the following system of relative units (see, for instance, Ref. 4):

a) The unit of the length is an arbitrary linear dimension $\ell_o$(m):

$$y_1 = \frac{y}{\ell_o}\,, \quad y_2 = y_1'\,, \quad y_3 = \frac{y}{\ell_o}\,, \quad y_4 = y_3'\,, \quad s = \frac{z}{\ell_o}\,, \qquad (1)$$

$$(\ )' \equiv \frac{d}{ds}(\ )\,;$$

x, y, z are the Cartesian co-ordinates of the trajectory of a particle. z is measured along the axis of the tube and the co-ordinate of the entrance into the tube is z = 0.

b) The unit of the potential is:

$$\varphi_o = \frac{m_o c^2}{e}\,(V)\,, \quad \varphi(s) = -\frac{U(s)}{\varphi_o}\,; \qquad (2)$$

U(s) is the accelerating potential on the tube axis. For protons $e = 1.6 \times 10^{-19}$ Cb and $\varphi_o = 9.38 \times 10^{8}$ V. The potential of the emitter is chosen to be zero.

c) The unit of the current is:

$$I_o = 4\pi\, c\, \boldsymbol{\varepsilon}_o\, \varphi_o \text{ (A)} , \quad J = \frac{I}{I_o} ; \tag{3}$$

$\varepsilon_o = 10^{-9}/36\ \pi$ (F/m). For protons $I_o = 3.14 \times 10^7$ A.

d) The unit of the field gradient of the electromagnetic quadrupole lenses is:

$$G_o = \frac{\varphi_o}{\ell_o^2 c} \text{ (T/m)} , \quad g(s) = \frac{G(s)}{G_o} ; \tag{4}$$

e) The unit of the momentum is:

$$P_o = m_o c \text{ (eV sec/m)} , \quad \mathcal{P}(s) = \frac{P(s)}{P_o} = \sqrt{\varphi(2+\varphi)} ; \tag{5}$$

If these units are used, the differential equations for the co-ordinates $y_1$ and $y_3$ of the trajectory of a particle of an intense, non-relativistic ($\varphi \ll 1$) ion beam accelerated in a tube with electromagnetic quadrupole lenses have in linear approximation the form:

$$\begin{aligned} y_1'' + y_1' \frac{d}{ds} \ell n\, \mathcal{P} + \left[\frac{g(s)}{\sqrt{2\varphi}} + \frac{\varphi''}{4\varphi}\right] y_1 &= \frac{\sqrt{2}\, J}{\varphi^{3/2}} \frac{y_1}{Y_1(Y_1+Y_3)} ; \\ y_3'' + y_3' \frac{d}{ds} \ell n\, \mathcal{P} - \left[\frac{g(s)}{\sqrt{2\varphi}} - \frac{\varphi''}{4\varphi}\right] y_3 &= \frac{\sqrt{2}\, J}{\varphi^{3/2}} \frac{y_3}{Y_3(Y_1+Y_3)} . \end{aligned} \tag{6}$$

Here $Y_1(s)$, $Y_3(s)$ describe the beam envelope. The expressions on the right hand side of the Eqs. (6) describe the action of the "self-matching field" of the beam (Ref. 3).

Instead of the vector $y = (y_1, y_2, y_3, y_4)$ we introduce a new vector $x = (x_1, x_2, x_3, x_4)$ by using the following relationships:

$$\begin{aligned}
x_1(s) &= y_1(s)\left[\frac{\wp(s)}{\wp(o)}\right]^{1/2} = y_1(s)\left[\frac{\varphi(s)}{\varphi(o)}\right]^{1/4} ; \\
x_2(s) &= x_1'(s) = \left[y_2(s) + \frac{y_1(s)}{4}\frac{\varphi'(s)}{\varphi(s)}\right]\times\left[\frac{\varphi(s)}{\varphi(o)}\right]^{1/4} ; \\
x_3(s) &= y_3(s)\left[\frac{\varphi(s)}{\varphi(o)}\right]^{1/4} ; \\
x_4(s) &= x_3'(s) = \left[y_4(s) + \frac{y_3(s)}{4}\frac{\varphi'(s)}{\varphi(s)}\right]\times\left[\frac{\varphi(s)}{\varphi(o)}\right]^{1/4} .
\end{aligned} \tag{7}$$

The expressions (7) couple also the envelopes of the x and y motions, making it possible, in particular, to calculate from the vector $Y(o) = \left[Y_1(o),\ Y_2(o) = Y_1'(o),\ Y_3(o),\ Y_4(o) = Y_3'(o)\right]$ at the entrance into the tube the corresponding vector $X(o) = \left[X_1(o),\ X_2(o) = X_1'(o),\ X_3(o),\ X_4(o) = X_3'(o)\right]$.

If the ion current is symmetrical with respect to the two pairs of phase co-ordinates $y_1$, $y_2$ and $y_3$, $y_4$ the surface areas of the two phase ellipses are equal and propotional to the emittence of the beam at a cross-section with the co-ordinate s:

$$E_y(s) = \frac{1}{\pi}\int dy_1(s)\, dy_2(s) = \frac{1}{\pi}\int dy_3(s)\, dy_4(s) . \tag{8}$$

It is easy to obtain with the help of the Eqs. (7) the corresponding phase ellipses in the planes $x_1$, $x_2$ and $x_3$, $x_4$. Also here the two surface areas are the same, do not depend on s and are equal to $\pi\, E_y(o) = \text{const}$ .

From Eqs. (6) and (7) we find:

$$\begin{aligned}
x_1'' + \left[\frac{g(s)}{\sqrt{2\varphi}} + \frac{3}{16}\left(\frac{\varphi'}{\varphi}\right)^2\right] x_1 &= \frac{J}{\varphi}\sqrt{\frac{2}{\varphi(o)}}\,\frac{x_1}{X_1(X_1+X_3)} ; \\
x_3'' - \left[\frac{g(s)}{\sqrt{2\varphi}} - \frac{3}{16}\left(\frac{\varphi'}{\varphi}\right)^2\right] x_3 &= \frac{J}{\varphi}\sqrt{\frac{2}{\varphi(o)}}\,\frac{x_3}{X_3(X_1+X_3)} .
\end{aligned} \tag{9}$$

The solutions of the system of Eqs. (9) can be written in the form:

$$x_\nu(s) = \frac{C_\nu}{2}\,\sigma_\nu(s)\,\exp i\,\psi_\nu(s) + \text{c.c.} \quad \nu = 1,3\ ; \tag{10}$$

with the usual normalization:

$$\sigma_\nu(\sigma_\nu^{*\prime} - i\psi_\nu'\sigma_\nu^{*}) - \text{c.c.} = -2i\ .$$

If the ellipses of the quadrupole channel (the equations of which can be obtained from the expressions $x_\nu(s)$ and $x_\nu'(s)$ by eliminating the phase $\psi_\nu(s)$) coincide with the phase ellipses in the planes $x_\nu x_\nu'$, the relationship holds (see Ref. 3):

$$X(s) = \sqrt{E_y(o)}\ \sigma(s), \qquad |C_\nu|^2 = E_y(o)\ ; \tag{11}$$

Here is $\sigma(s) = \left[\sigma_1(s),\ \sigma_2(s) = \sigma_1'(s),\ \sigma_3(s),\ \sigma_4(s) = \sigma_3'(s)\right]$ the vector of the modules of the solution Eq. (10).

The equation for $\sigma(s)$ can be obtained from the Eqs. (9) by taking into account the Eqs. (11):

$$\begin{aligned}
\sigma_1'' + \left[\frac{g(s)}{\sqrt{2\varphi}} + \frac{3}{16}\left(\frac{\varphi'}{\varphi}\right)^2\right]\sigma_1 - \frac{1}{\sigma_1^3} &= \frac{J}{E_y(o)}\sqrt{\frac{2}{\varphi(o)}}\,\frac{1}{\varphi}\,\frac{1}{\sigma_1 + \sigma_3}\ ;\\
\sigma_3'' - \left[\frac{g(s)}{\sqrt{2\varphi}} - \frac{3}{16}\left(\frac{\varphi'}{\varphi}\right)^2\right]\sigma_3 - \frac{1}{\sigma_3^3} &= \frac{J}{E_y(o)}\sqrt{\frac{2}{\varphi(o)}}\,\frac{1}{\varphi}\,\frac{1}{\sigma_1 + \sigma_3}\ .
\end{aligned} \tag{12}$$

A system of equations of the type of Eqs. (12) has also be obtained in Ref. 3 and has been carefully studied for the case of long channels described by differential equations with periodical coefficients. Here the optimal (matched) transmission of the beam through the channels is granted if $\sigma(s)$ coincides with the vector of the modules of the Floquet function: $\sigma(s) \equiv \rho(s)$.

The system of Eqs. (9) has in general no solutions of Floquet's form. Because of the acceleration the coefficients of the equation of motion are essentially non-periodical. For the determination of the transmission characteristic of a tube and of the matched injection conditions it is convenient to use a search method made possible by electronic computers.

Concrete values of the ratio $\eta = J/E_y(o)$ and the injection vector $\sigma(o) = A$ determine the solution $\sigma(s, \eta, A)$ of the system of the Eqs. (12). With Eqs. (11) and (7) one can find the maximum values of the functions $Y_1(s)/\sqrt{E_y(o)}$ and $Y_3(s)/\sqrt{E_y(o)}$ within the limits of the tube.

A scalar function $Q(A)$ is constructed by taking - while A is varied - always the bigger one of those two values

$$Q(A) = \max_{o \leq s \leq L} \left\{ \left[Y_1(s)/\sqrt{E_y(o)}\right]_{max}, \quad \left[Y_3(s)/\sqrt{E_y(o)}\right]_{max} \right\} \qquad (13)$$

where L is the full length of the tube.

The aim of the search is to find for every fixed value $\eta$ such an injection vector $A^* = \sigma^*(o)$ which gives the minimum value $Q(A^*) = Q_{min}$ of Eq. (13).

One can, in order to reach this aim, regard the components of the vector A as four independently variable parameters and apply the method of random search, as described in Ref. 5. The programmation of the latter for the given case is identical with the one described in Ref. 6.

We call $r_n$ the maximum radius of the channel which can be reached by incoherent beam oscillations. The value $Q_{min}$ allows then the calculation of the injection emittence $E_y(o)$ of the beam, of the current J, of the phase space volume $V_n$ of the beam and of the phase space density $j_\varphi$:

$$E_y(o) = \left(\frac{r_n}{Q_{min}}\right)^2, \quad J = \eta E_y(o), \quad V_n = E_y(o)\ \beta(o), \quad j_\varphi = \frac{J}{V_n}. \qquad (14)$$

Apart from this, the vector $A^* = \sigma(o)$ corresponding to a given $Q_{min}$ presents for the values J and $V_n$ calculated from Eq. (14) the matched injection conditions: any deviation from these conditions results in an increase of the diameter of the beam within the tube.

Let us add the remark that in the particular case of a long-periodical channel and for $\eta = 0$ the search produces as solution $\sigma(s)$ of Eq. (9), the vector of Floquet's modules $\rho(s)$ of the channel.

Let us assume, we are asked to determine the transmission characteristic of a tube for the acceleration of protons from an injection energy of 100 keV to a final energy of 1.65 MeV.

A schematic drawing of the acceleration and focusing system of the tube is shown in Fig. 1, which is analogous to a figure in Ref. 2. The protons are focused by 16 quadrupole lenses and by axially symmetrical electrostatic lenses formed by non-magnetic metallic cylinders. The aperture radius of every lens is $r_\ell = 2.5$ cm, the length of a lens is $\ell_\ell = 5.2$ cm, the pattern formula of the system is FODO with all straight sections of the same length and with the period $S_o = 15.$[illegible] cm. The accelerating cylinders are mechanically connected to the lenses and have an inner radius $r_c = 2.3$ cm and a length $\ell_c = 5.6$ cm. The section of the entrance into the tube has been given the co-ordinate $s = 0$, and the section of the exit the co-ordinate $s = L$.

As unit of length and unit of field gradient of the lenses have been chosen $\ell_o = 0.01$ m and $G_o = 3.13 \times 10^4$ T/m, respectively.

Based on the materials presented in Ref. 7 it is possible to calculate the gradient distribution along the tube axis of a single lens with the ratio $2r_\ell/\ell_\ell = 1$ (dashed curves in Fig. 1):

$$g(s) = \frac{g_m}{1 + \left|\frac{S_o(2n-1) - 4s}{4b}\right|^{\alpha}}, \qquad b \cong 1.5\,\frac{r_\ell}{\ell_o}, \qquad \alpha = 3.25 . \qquad (15)$$

Here is n the order number of a lens; n = 1.2 ... 16; $g_m = \pm\, G_m/G_o$ is the gradient of the centre of the lens.

If several lenses are combined and if the screening effect is small, the gradient distribution can be obtained from a superposition of the distributions of the single lenses (see Fig. 1). For a rough estimate it is even possible to replace it by a sinusoidal curve with the amplitude $a = 0.85\,|g_m|$.

The electrostatic potentials of the electrodes are indicated in Fig. 1 and the variation of the potential from the centre of one cylinder to another is given by (see Ref. 8):

$$\varphi(s) = \frac{U_{n+1} + U_n}{2\varphi_o} + \frac{U_{n+1} - U_n}{2\varphi_o}\,\mathrm{th}\left\{1.315\,\frac{\ell_o}{r_c}\left[s - (n-1)\,\frac{S_o}{2}\right]\right\} \qquad (16)$$

where n, $U_n$ are the order number and the potential of the left hand side cylinder of a pair; n = 1.2 ... 16.

In some cases it is practical to use quadrupole lenses made of permanent magnets (see Ref. 2). For the dimensions as indicated above one can easily obtain in the centre of the lenses a field gradient $G_m$ of about 10 T/m, and by using special magnetic materials even higher gradients can be obtained.

Below a system of identical lenses is considered with a gradient $G_m$ = 9.1 T/m.

Without acceleration and without current $\varphi' = 0$, $J = 0$ the focusing channel of the tube can be described by the system of Eqs. (12) with periodical coefficients. There is in this case a Floquet solution $\sigma(s) = \rho(s)$. The phase shift of incoherent oscillations over one element of periodicity $S_o$ of the channel is in this case $\mu_o = 0.482$ and the maximum of Floquet's module is $\rho_{max} = 6.38$. With the help of the theory developed in Ref. 3 one can calculate from these parameters the limit current $I_{lim}$ for $\varphi' = 0$:

$$I_{lim} = \frac{\mu_o}{2} \frac{r_n}{\ell_o}^2 \frac{\beta^3 \gamma^3}{S_o \rho^2_{max}} I_o. \tag{17}$$

Taking $r_n = 1.5$ cm we obtain from Eq. (17) $I_{lim} = 84$ mA, while the application of the described search procedure for $\varphi' = 0$ and $V_n = 0$ produces the value 82 mA.

For the determination of the limit transmission characteristic in presence of acceleration with $\varphi' \neq 0$ the search method has been applied for the following values of the parameter $\eta = J/E_y(o)$:

$$0; \quad 1.25 \times 10^{-7}; \quad 1.98 \times 10^{-7}; \quad 3.75 \times 10^{-7}; \quad 9.35 \times 10^{-7}.$$

Figure 2a shows as the result the maximum current $I_{max}$ transmitted by the given tube as function of the phase space area $V_n$ of the beam and for an amplitude of the incoherent oscillations of $r_n = 1.5$ cm. The calculated points agree well with the relationship found in Ref. 3:

$$I_{max} = I_{lim} \left[ 1 - \left( \frac{V_n}{V_o} \right)^2 \right]; \tag{18}$$

if one takes $I_{lim} = 350$ mA, $V_o = 0.93$ cm mrad.

It can be seen from Fig. 2a that acceleration of currents close to $I_{lim} = 350$ mA is coupled with high phase space densities $j_\varphi = I_{max}/V_n$ (see Ref. 3). Obviously, in our case it is difficult to obtain $I_{max} > 300$ mA. This would mean that the necessary phase space density would be higher than 1 A/cm mrad, a value which is close to the limit of the best existing proton sources (see Ref. 9).

As already mentioned, together with the limit current and the phase space volume the matched injection conditions have been calculated, which are necessary for reaching these values. If a working point has been chosen on the limit curve $I_{max} = f(V_n)$ of Fig. 2a, the injection conditions can be found from Fig. 2b, where the initial dimension x, y and the slopes x', y' of the beam envelopes at the entrance are plotted in Cartesian co-ordinates according to Eq. (1). The working region which would require a phase space density larger than 1 A/cm mrad are shaded in both Figs. 2a and 2b.

By comparing the values $I_{lim}$ from Eqs. (17) and (18) equal to 84 mA and 350 mA, respectively, one recognizes the increase of the transmission characteristic of the tube, due to the presence of acceleration $\varphi' \neq 0$.

It should be noted that the electrostatic forces appearing from $\varphi'' \neq 0$ increase the limit current very little: an acceleration in a homogeneous field with $\varphi' = \text{const}$, $\varphi'' = 0$ would lead to $I_{lim} = 325$ mA instead of $I_{lim} = 350$ mA from Eq. (18).

The acceleration of beams larger than 300 mA in a tube of the described design would require stronger focusing, which is possible only by abandoning permanent magnets and by using electro-magnets for the quadrupole lenses.

In Fig. 3a the transmission characteristic has been plotted for such a case with a field gradient of $G_m = 25$ T/m in the centre of the lens. This curve can also be described by Eq. (18) if one assumes $I_{lim} = 1.0$ A, $V_o = 1.97$ cm mrad. For every point of this curve one can find from Fig. 3b the corresponding matched injection conditions. Phase space densities $j_\varphi > 1$ A/cm mrad (the shaded areas in Figs. 3a and 3b) correspond to a limit current higher than 0.8 A.

The limit currents transmitted through the tube in absence of acceleration ($\varphi' = 0$), have been calculated with the help of Eq. (17) for the parameters $\mu_o = 1.42$ and $\rho_{max} = 4.97$ by applying the search method. The values which have been found are 409 mA and 438 mA, respectively.

The authors are gratefully indebted to B.V. Chirikov for the discussion of the present paper and his interest in it.

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Figure captions

Fig. 1 Scheme of the accelerating tube.

Fig. 2 The basic characteristics of an accelerating tube with quadrupole lenses formed by permanent magnets with a field gradient of $G_m = 9.1$ T/m in the centre of the lenses.

a) dependence of the maximum accelerated current $I_{max}$ (in mA) on the phase space volume of the beam $V_n$ (in cm mrad).

b) the matched values of the dimension x, y (in cm) and the slopes x', y' (dimensionless) of the beam envelope at the entrance into the tube.

Fig. 3 The basic characteristics of an accelerating tube with electro-magnet quadrupole lenses with a field gradient of $G_m = 25$ T/m in the centre of the lenses,

a) the dependence of the maximum accelerated current $I_{max}$ (in A) on the phase space volume of the beam $V_n$ (in cm mrad).

b) the matched values of the dimension x, y (in cm) and the slopes x', y' (dimensionless) of the beam envelope at the entrance into the tube.

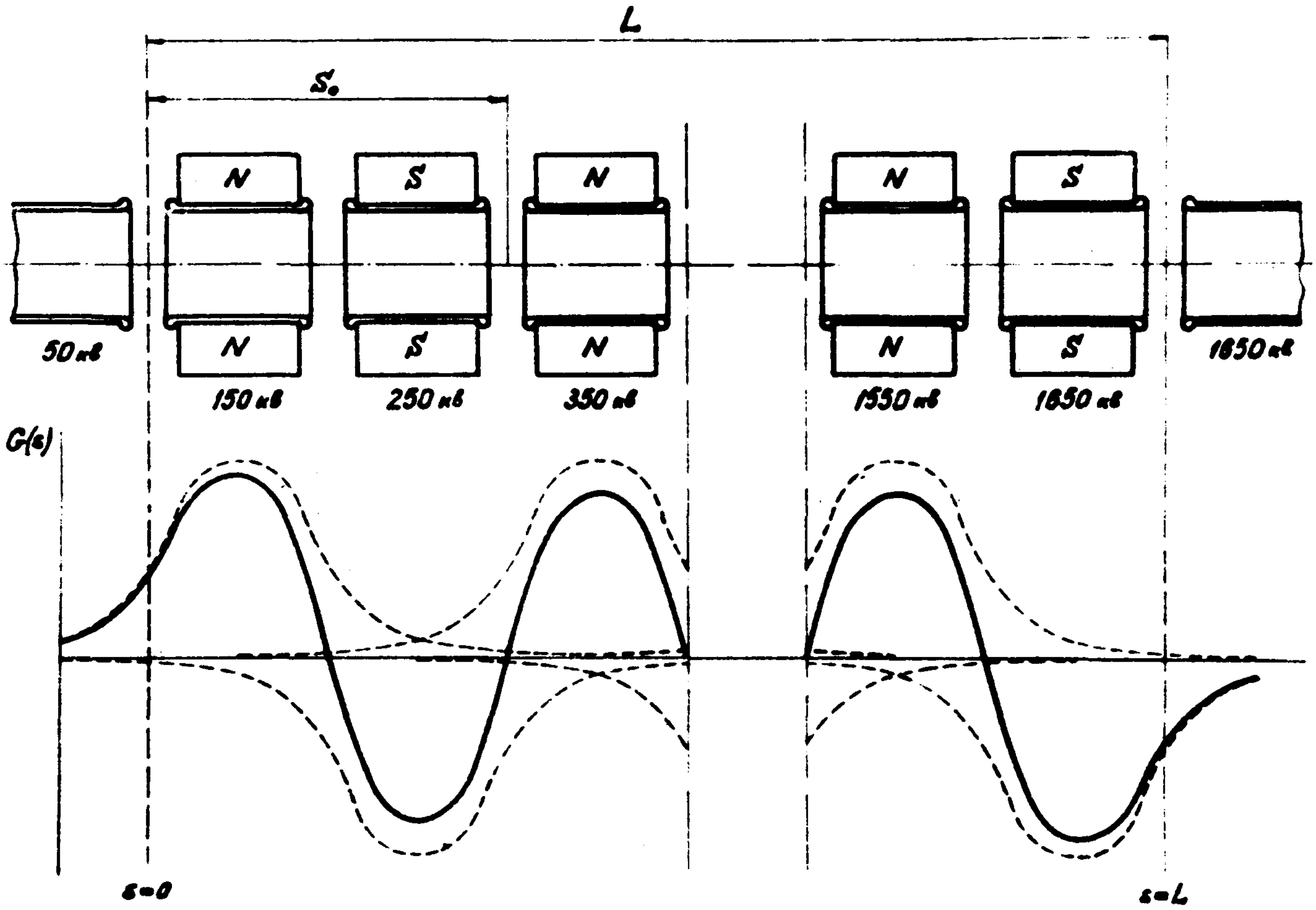
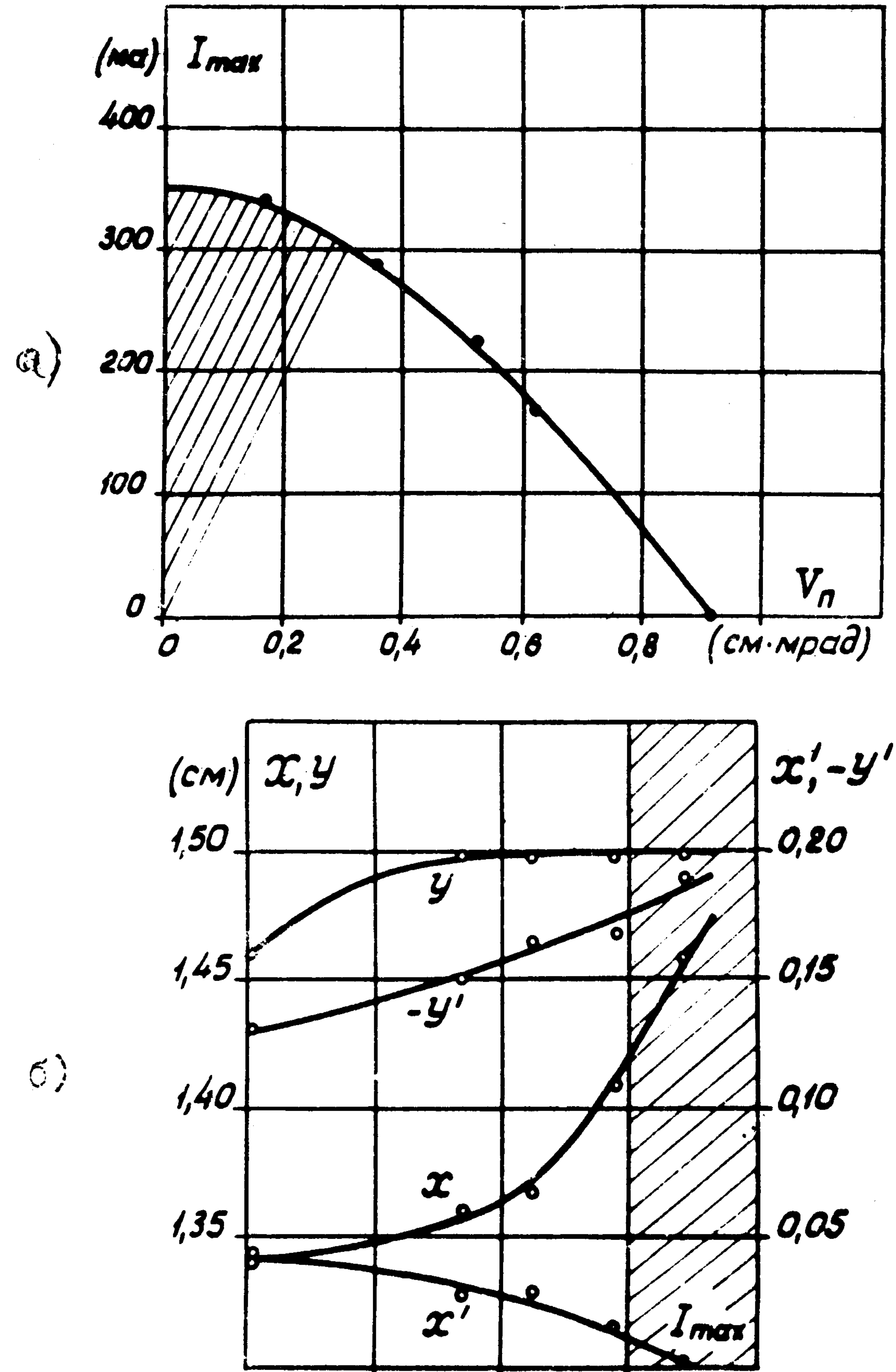



Fig. 1

SIS/R/15444


a)
(ма) $I_{max}$
400
300
200
100
0
$V_n$
0
0,2
0,4
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(см·мрад)
б)
(см) $x, y$
$x', -y'$
1,50
1,45
1,40
1,35
0,20
0,15
0,10
0,05
$y$
$-y'$
$x$
$x'$
$I_{max}$
0
100
200
300
(ма)

Fig. 2

a)

б)

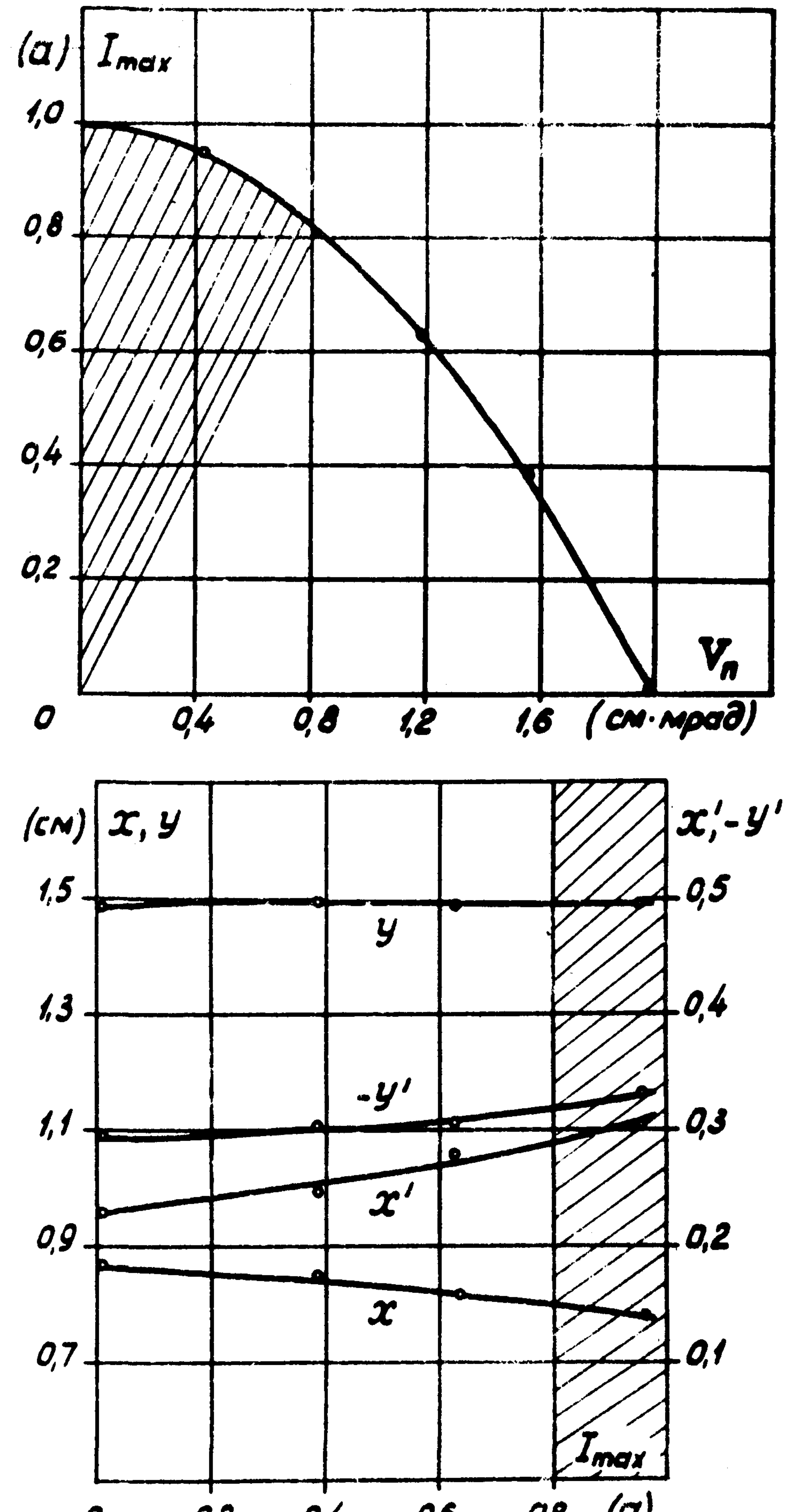
(a) I_max
1,0
0,8
0,6
0,4
0,2
0
0,4
0,8
1,2
1,6
V_n
(см·мрад)
(см) x, y
x', -y'
1,5
1,3
1,1
0,9
0,7
0,5
0,4
0,3
0,2
0,1
y
-y'
x'
x
I_max
0
0,2
0,4
0,6
0,8
(a)

Fig. 3