Collinear QCD Models[1]

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In ancient times, 't Hooft studied the mesons in QCD_{1+1} [1] to illustrate the power of the large N limit [2] in the light-cone formalism. Recently some generalisations of the 't Hooft model have been studied, which retain a remnant of transverse degrees of freedom, based on a dimensional reduction of QCD to $1+1$ dimensions [3, 4, 5, 6, 7, 8, 9]. In this collinear approximation, quarks and gluons are artificially restricted to move in one space dimension, but retain their polarization degree of freedom. In this lecture, a problem which in principle involves a large number of partons will be addressed in the context of the collinear model at large N. For light-cone quantisation, large numbers of partons are synonymous with small Bjorken-x. The example treated here[3] is the quark distribution function in a heavy meson, which is supposed to exhibit a version of Regge behaviour at small-x. The central idea involves high light-cone energy boundary conditions on wavefunctions — ladder relations — which typically connect Fock space sectors of differing numbers of partons. The same ideas carry over to $3+1$ dimensions [10].

We start from $SU(N)$ gauge theory in $3+1$-dimensions with one flavour of quarks. If we pick an arbitrary fixed space direction x^3 and restrict ourselves to zero momentum in the transverse directions

$$\partial_{x_\perp} A_\mu = \partial_{x_\perp} \Psi = 0 \tag{1}$$

for the gauge and quark fields, one finds an effectively two-dimensional gauge theory of adjoint scalars and fundamental Dirac spinors with action

$$
\begin{aligned}
S \quad = \quad & \int dx^0 dx^3 \;\; -\frac{1}{4g^2}\mathrm{Tr}\{F_{ab}F^{ab}\} + \mathrm{Tr}\{-\frac{1}{2}\bar{D}_a\phi_\rho \bar{D}^a\phi^\rho \\
& -\frac{tg^2}{4}[\phi_\rho,\phi_\sigma][\phi^\rho,\phi^\sigma] + \frac{1}{2}m_0^2\phi_\rho\phi^\rho\} + \frac{i}{\sqrt{2}}(\bar{u}\gamma^a D_a u + \bar{v}\gamma^a D_a v) \\
& +\frac{m_F}{\sqrt{2}}(\bar{u}v + \bar{v}u) - \frac{sg}{\sqrt{2}}(\bar{u}(\phi_1 + i\gamma^5\phi_2)u - \bar{v}(\phi_1 - i\gamma^5\phi_2)v) \; ,
\end{aligned}
\tag{2}
$$

where a and $b \in \{0,3\}$, $\rho \in \{1,2\}$, $\gamma^0 = \sigma^1$, $\gamma^3 = i\sigma^2$, $\gamma^5 = i\sigma^1\sigma^2$, $\phi_\rho = A_\rho/g$, $\bar{D}_a = \partial_a + i[A_a,.]$, $D_a = \partial_a + iA_a$, $\int dx^1 dx^2 = \mathcal{L}^2$, $g^2 = \tilde{g}^2/\mathcal{L}^2$, and $\tilde{g}$ is the four-dimensional coupling. The two-component spinors u and v are related to Ψ by

$$\Psi = \frac{1}{2^{1/4}\mathcal{L}}\begin{pmatrix} u_{R+} \\ u_{L+} \\ u_{L-} \\ u_{R-} \end{pmatrix} \;,\; u = \begin{pmatrix} u_{L+} \\ u_{R+} \end{pmatrix} \;,\; v = \begin{pmatrix} u_{L-} \\ u_{R-} \end{pmatrix} \;. \tag{3}$$

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[1] Invited talk at 'New Non-Perturbative Methods and Quantisation on the Light-Cone', Les Houches, France, 24 Feb - 7 Mar 1997. (To appear in the proceedings.)

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[3] Another example was treated in the lecture — asymptotic spectrum of glueballs — for which there was not enough room in these proceedings.

The suffices L (R) and $+$ $(-)$ in (3) represent Left (Right) movers and $+ve$ $(-ve)$ helicity, which is a conserved quantity. Thus u, v, ϕ_1, ϕ_2 represent the transverse polarisations of the $3 + 1$ dimensional quarks and gluons. Since the dimensional reduction procedure treats space asymmetrically, dimensionless parameters s and t, and a bare gluon mass m_0 can occur due to loss of transverse local gauge transformations. For the present application the precise choice of s and t will not be qualitatively important, so we set $s = 1$ and $t = 0$ for simplicity.

In the light-cone gauge $A_- = (A_0 - A_3)/\sqrt{2} = 0$, the fields A_+ and $u_{L\pm}$ are non-propagating in light-front time $x^+ = (x^0 + x^3)/\sqrt{2}$ so may be eliminated by their constraint equations

$$\partial_- u_{L\pm} = \mathrm{i} F_{\mp} \quad , \quad (\partial_-)^2 A_+ = g^2 J^+ \tag{4}$$

$$F_{\pm} = \frac{m_F}{\sqrt{2}} u_{R\pm} \pm g B_{\mp} u_{R\mp} \tag{5}$$

$$J^+ = \mathrm{i}[B_-, \partial_- B_+] + \mathrm{i}[B_+, \partial_- B_-] + u_{R+} u_{R+}^\dagger + u_{R-} u_{R-}^\dagger \tag{6}$$

and $B_\pm = (\phi_1 \pm \mathrm{i}\phi_2)/\sqrt{2}$. The exchange of non-propagating particles associated with the constrained fields results non-local interactions in the light-cone hamiltonian

$$P^- = \int dx^- \ F_+^\dagger \frac{1}{\mathrm{i}\partial_-} F_+ + F_-^\dagger \frac{1}{\mathrm{i}\partial_-} F_- - \frac{g^2}{2} \mathrm{Tr} \left\{ J^+ \frac{1}{(\partial_-)^2} J^+ \right\} + \frac{m_0^2}{2} \mathrm{Tr} \ \phi^2 \tag{7}$$

The zero momentum limit of the constraints (4) forces a condition on the quark-gluon combined system

$$\int_{-\infty}^{+\infty} dx^- F_\pm = 0 \quad , \quad \int_{-\infty}^{+\infty} dx^- J^+ = 0 \ , \tag{8}$$

assuming u_L and $\partial_- A_+$ vanish at $x^- = \pm\infty$. This is required for finiteness of the non-local interactions in (7). The relation involving J^+, which amongst other things restricts one to gauge singlets, will not be discussed further here.

Introducing the harmonic oscillator modes of the physical fields[4]

$$u_{R\pm i} = \frac{1}{\sqrt{2\pi}} \int_0^\infty dk \left(b_{\pm i}(k) \mathrm{e}^{-\mathrm{i}kx^-} + d_{\mp i}^\dagger(k) \mathrm{e}^{\mathrm{i}kx^-} \right) \tag{9}$$

$$B_{\pm ij} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{dk}{\sqrt{2k}} \left(a_{\mp ij}(k) \mathrm{e}^{-\mathrm{i}kx^-} + a_{\pm ji}^\dagger(k) \mathrm{e}^{\mathrm{i}kx^-} \right) \tag{10}$$

in the quantum theory we can expand any hadron state $|\Psi(P^+)>$ of total momentum P^+ in terms of a Fock basis [11]. The operators $a_\pm^\dagger$ create gluons with helicity ± 1, while $b_\pm^\dagger$ and $d_\pm^\dagger$ correspond to quarks and antiquarks (respectively) with helicities $\pm\frac{1}{2}$. At large N a gauge-singlet meson is a superposition

$$
\begin{aligned}
|\Psi(P^+)> \ = \ & \sum_{n=2}^\infty \int_0^{P^+} dk_1 \ldots dk_n \sum_{\alpha_i = \pm} \delta(k_1 + \cdots + k_n - P^+) \times \\
& \frac{f_{\alpha_1 \ldots \alpha_n}(k_1, k_2, \ldots, k_n)}{\sqrt{N^{n-1}}} \times \\
& d_{\alpha_1 i}^\dagger(k_1) a_{\alpha_2 ij}^\dagger(k_2) a_{\alpha_3 jk}^\dagger(k_3) \ldots a_{\alpha_{n-1} lm}^\dagger(k_{n-1}) b_{\alpha_n m}^\dagger(k_n)|0>
\end{aligned}
\tag{11}
$$

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[4] The superscript on k^+ has been dropped for clarity; $i, j \in \{1, \ldots, N\}$ are gauge indices and $\dagger$ is now understood as the quantum complex conjugate, so does not transpose them.

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Introducing the fourier transform $\tilde{F}(w)$, one finds that in the quantum theory Eq.(8) can be meaningfully applied as an annihilator of physical states for the cases

$$\lim_{w \to 0^+} \tilde{F}_{\pm i}(w) \cdot |\Psi(P^+)> \;=\; 0 \tag{12}$$

$$\lim_{w \to 0^-} \tilde{F}_{\pm i}^\dagger(w) \cdot |\Psi(P^+)> \;=\; 0 \;. \tag{13}$$

The first relation yields a condition on the Fockspace wavefunctions f involving vanishing quark momentum $k = w > 0$, the second on vanishing anti-quark momentum $k = -w > 0$:

$$
\begin{aligned}
&f_{\mp \pm \alpha_1 \cdots \alpha_n}(k, k_1, \ldots, k_{n+1}) \\
&= \;\pm \lambda \left[\frac{f_{\pm \alpha_1 \cdots \alpha_n}(k + k_1, k_2, \ldots, k_{n+1})}{\sqrt{k_1}} \right. \\
&\quad \left. + \int_0^\infty \frac{dp\,dq}{\sqrt{q}} \, \delta(p + q - k) f_{\pm \mp \pm \alpha_1 \cdots \alpha_n}(p, q, k_1, \ldots, k_{n+1}) \right] \;,
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&f_{\mp \mp \alpha_1 \cdots \alpha_n}(k, k_1, \ldots, k_{n+1}) \\
&= \;\pm \lambda \int_0^\infty \frac{dp\,dq}{\sqrt{q}} \, \delta(p + q - k) f_{\pm \mp \mp \alpha_1 \cdots \alpha_n}(p, q, k_1, \ldots, k_{n+1}) \;,
\end{aligned}
\tag{15}
$$

with $\lambda = \sqrt{g^2 N / 2\pi m_F^2}$ and a similar set of relations for quarks from (12); in (14)(15) the limit $k \to 0^+$ is understood. If we adopt the following momentum-space operator ordering in P^- (7)

$$\int_{-\infty}^0 \frac{dw}{w} \tilde{F}_{\pm i}^\dagger \tilde{F}_{\pm i} - \int_0^\infty \frac{dw}{w} \tilde{F}_{\pm i} \tilde{F}_{\pm i}^\dagger \;, \tag{16}$$

we then apparently have manifest finiteness as $w \to 0$ for physical states. Normal ordering the oscillator modes in P^- would spoil finiteness. Since we do not normal order the form (16), infinite quark self energies (self-inertias) are generated but no vacuum energies are generated.

However the above argument is flawed by the fact that infinities may also arise due to integration over the parton momenta in the wavefunction $|\Psi>$, since (14)(15) are to be interpreted at *fixed* k_i as $k \to 0^+$. This is evident from the light-cone Schrodinger equation, obtained by projecting $2P^+ P^- |\Psi> = \mathcal{M}^2 |\Psi>$ onto a specific n-parton Fock state

$$\left(\left[\frac{\mathcal{M}^2}{2P^+} \right] - \sum_{i=1}^n \frac{m_i^2}{2p_i} \right) f_{\alpha_1 \ldots \alpha_n}(p_1, \ldots, p_n) = \hat{V}\left[f_{\alpha_1 \ldots \alpha_n}(p_1, \ldots, p_n) \right] \tag{17}$$

where $\hat{V}$ is the interaction kernel (including self-inertias), $\mathcal{M}$ the boundstate mass, and m_i is m_F (quark) or m_B (gluon). The ladder relations (14)(15) are necessary for finiteness of the internal integrations in $\hat{V}$ at fixed external momenta p_i. However further renormalisation of $\hat{V}$ is necessary since the ladder relations do not ensure finiteness when one or more external momenta p_i vanish in (17).[5] In fact an explicit two-loop calculation of the fermion self-energy in light-cone Yukawa$_{1+1}$ [13] shows that divergences do not cancel for the same cutoff on all small momenta. In general the renormalisation that cures these divergences will depend on the precise cut-off(s) employed. It has been suggested to renormalise the fermion kinetic mass finitely to restore parity invariance in light-cone calculations [14], and this should coincide with ensuring finiteness of $\mathcal{M}$.[6]

Eqs.(14)(15) show that the meson wavefunction components do not vanish as the quark momentum vanishes. It will be demonstrated that this leads directly to a rising quark distribution function

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[5]This point, and also the integral terms in (14)(15), were missed in ref.[7].

[6]After this lecture was typed, a preprint appeared [15] which verifies this for certain examples with a Yukawa interaction only.

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at small $x = k/P^+$. The probability to find an anti-quark — the answer is the same for a quark — with momentum fraction $x = k/P^+$ of the meson is

$$
\begin{aligned}
Q(x) &= \sum_{n=2}^{\infty} \sum_{\alpha_i} \int_0^{P^+} dk_1 \ldots dk_n \, \delta(k_1 + \cdots + k_n - P^+) \times \\
&\quad \delta(k_1 - k) \, |f_{\alpha_1 \ldots \alpha_n}(k_1, k_2, \ldots, k_n)|^2 \\
&= \sum_{\alpha} < d_{\alpha i}^{\dagger} d_{\alpha i}(x) >
\end{aligned}
\tag{18}
$$

For the polarized version $\Delta Q(x)$ one inserts $\text{sgn}(\alpha)$. In order to make use of (14)(15) to evaluate $Q(x \to 0)$, it is helpful to eliminate the integral terms, which generate renormalisation of the other (non-integral) terms. Although the integrals are over a set of measure zero, they are non-zero due to the singular behaviour of the integrand. This singular behaviour can be found from the (correctly renormalised) Schrodinger equation (17). Let us consider the helicity $+1$ meson with valence component f_{++}. The general idea is to use an expansion in λ and $\log 1/x$ to evaluate (18). The leading orders we shall calculate are independent of any additional fermion kinetic mass renormalisation in (17). The leading log approximation amounts to considering the integration region $k_{n-2} >> k_{n-2} >> \cdots >> k_2 >> k_1 = k$ in (18). In this region we may use the ladder relations iteratively to express every n-parton non-valence contribution in terms of f_{++}. For example, truncating to no more than one gluon we obtain $Q(x \to 0) = \lambda^2 < 1/y >_{++}$ from the f_{-++} component, where $< 1/y >_{++}$ is the average inverse momentum fraction in the f_{++} component.

Truncating to no more than two gluons, we can compute some of the subleading λ and $\log 1/x$ corrections. For this we need to use the $n = 4$ boundstate equation (17) (for which there is neither a fermion self-inertia nor a finite kinetic mass renormalisation) to evaluate the integral term in (14). The following results neglect the A_+ exchange process between quark and gluon in (17), whose effects cannot be explicitly resummed. From resumming instantaneous fermion (u_L) processes the correct ladder relations in this 2-gluon approximation become

$$
f_{-++}(x \to 0, x_1, x_2) = \lambda^* \frac{f_{++}(x_1, x_2)}{\sqrt{x_1}}
\tag{19}
$$

$$
f_{+-++}(x \to 0, x_1, x_2, x_3) = -\lambda \frac{f_{-++}(x_1, x_2, x_3)}{\sqrt{x_1}}
\tag{20}
$$

where

$$
\lambda^* = \frac{\lambda}{1 + \lambda^2 \left[\frac{\log(m_F^2/m_B^2)}{1 - (m_B^2/m_F^2)} \right]}
\tag{21}
$$

Then to leading log the contributions from one and two gluon components of the wavefunction give

$$
Q(x \to 0) \approx (\lambda^*)^2 (1 + \lambda^2 \log 1/x) < 1/y >_{++} .
\tag{22}
$$

An example of a next-to-leading log contribution comes from integrating the two-gluon contribution over the region $\int_0^k dk_2 \int_k dk_3$ using (17) at $n = 4$

$$
\lambda^2 (\lambda^*)^2 \log \left(1 + \frac{m_F^2}{m_B^2} \right) < 1/y >_{++}
\tag{23}
$$

The analytic ladder results are compared with a non-perturbative DLCQ solution of (17) truncated to the same number of gluons in Figs. 1 and 2. The DLCQ calculations are formal at finite x since no additional fermion kinetic mass renormalisation has been carried out to ensure finite $\mathcal{M}$ as $K \to \infty$. In practice however, for heavy quarks the effects of this omission are very tiny at
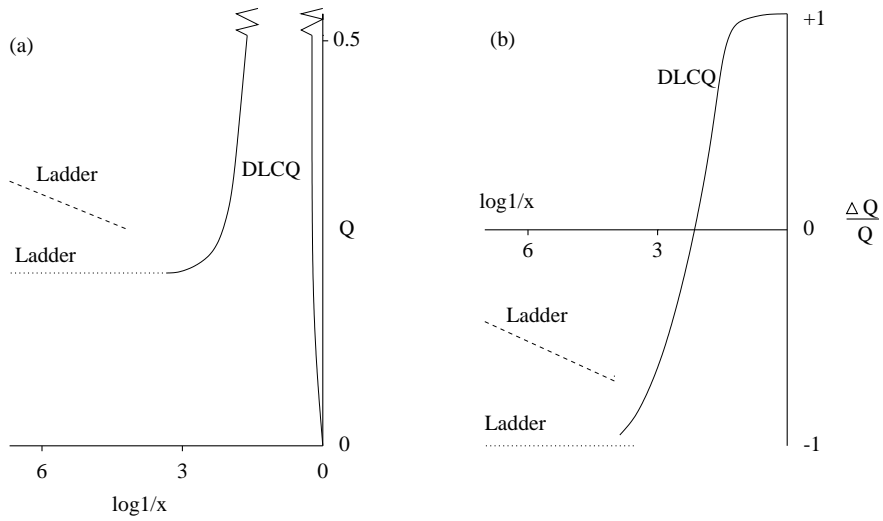
4

(a)

Ladder

DLCQ

Ladder

Q

0.5

0

6   3   0

log1/x

(b)

+1

DLCQ

log1/x

$\frac{\triangle Q}{Q}$

6   3   0

Ladder

Ladder

-1

Figure 1: (a) Unpolarized distribution function for helicity +1 meson at $\lambda^2 = 0.1$: (Dotted) 1-gluon ladder prediction ($< 1/y >_{++}$ is indistinguishable from 2 for heavy quarks); (solid) DLCQ up to 1 gluon, K=24; (dashed) arbitrary-gluon tree-level ladder prediction.(b) helicity asymmetry $\Delta Q/Q$.

(a)

DLCQ

0.5

Ladder

DLCQ
(non-valence)

Q

0

4   2   0

log1/x

(b)

+1

DLCQ

log1/x

$\frac{\triangle Q}{Q}$
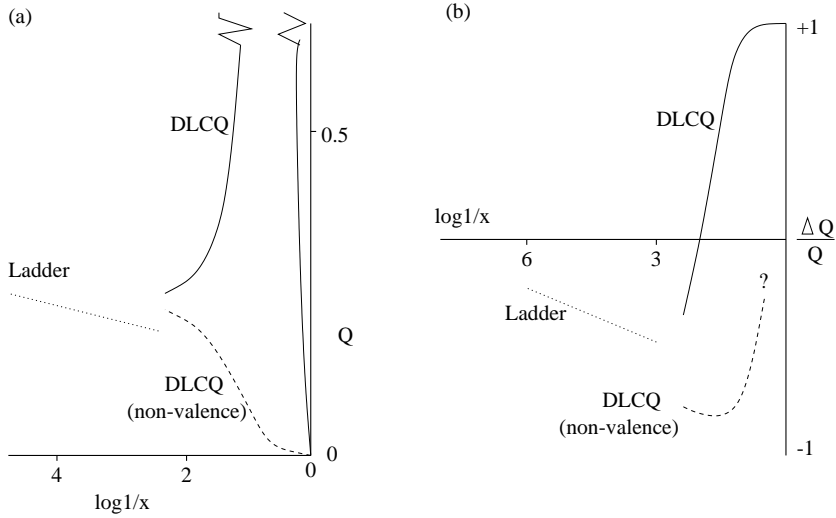
6   3

?

Ladder

DLCQ
(non-valence)

-1

Figure 2: (a) Unpolarized distribution function for helicity +1 meson at $\lambda^2 = 0.1$, $m_B^2/m_F^2 = 0.1$: (Dotted) ladder prediction for up to 2 gluons and next-to-leading log; (solid) DLCQ up to 2 gluons, K=15; (dashed) non-valence part of the DLCQ calculation. (b) Helicity asymmetry $\Delta Q/Q$.

finite K, and the plots shown should be a good representation of the exact result at finite x (the same comment applies to plots in ref.[7]). At $x \sim 1/K$ the DLCQ results should in any case match onto the analytic predictions. There are many sources in the approximations which might account for the remaining discrepancy in the normalisation and slope seen in Fig.2 at small x, although the agreement is much better if the DLCQ calculation is repeated without the quark-gluon A_+ exchange.

More generally if we consider the ladder relations to leading order in λ, which means neglecting the integrals in (14)(15), an arbitrary number of gluons can be eliminated to yield an exponential sum of leading $\log 1/x$'s (Fig.1)

$$Q(x \to 0) \approx \lambda^2 x^{-\lambda^2} < 1/y >_{++} \tag{24}$$

The integral terms should only renormalise λ in the previous expression. All the results point to a rising small-x unpolarized distribution. At large N the polarization asymmetry changes sign and then vanishes at small x in the leading log approximation, $\Delta Q / Q \sim x^{2\lambda^2}$

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