

## Anisotropic Correlated Electron Model Associated With the Temperley-Lieb Algebra

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### ABSTRACT

We present an anisotropic correlated electron model on a periodic lattice, constructed from an R-matrix associated with the Temperley-Lieb algebra. By modification of the coupling of the first and last sites we obtain a model with quantum algebra invariance.

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Since the discovery of high  $T_c$  superconductivity there has been a great interest in the area of integrable highly correlated electron systems. Notably for many years the Hubbard and the supersymmetric  $t - J$  models, which are both exactly solvable in one dimension, stood as the prototypes for such types of models. Subsequently other correlated electron models have been formulated [1–3].

In this letter we present a new  $q$ -deformed integrable electronic model following the approach of [4], which consists of using the Temperley-Lieb (TL) algebra to obtain solutions of the Yang-Baxter equation. In [3] this was achieved utilizing a 4-dimensional module of Lie superalgebra  $gl(2/1)$ , here we proceed along the same lines by investigating the corresponding module of  $U_q(gl(2/1))$  in order to obtain a representation of the TL algebra.

The procedure adopted in [3] involved a symmetry breaking transformation so that the resulting Hamiltonian, defined on a one-dimensional periodic lattice, was not invariant with respect to  $gl(2/1)$  but rather its even subalgebra  $gl(2) \otimes u(1)$ . For the present situation, the usual imposition of periodic boundary conditions has the effect of also breaking the  $U_q(gl(2)) \otimes u(1)$  symmetry due to the non-cocommutativity of the quantum algebra generators. However we will show, following the methodology of [5–8] that a quantum algebra invariant closed system can be defined by the introduction of an operator coupling the first and last sites into the expression for the Hamiltonian.

Let  $\{|x\rangle\}_{x=1}^4$  be an orthonormal basis for a four-dimensional  $U_q(gl(2/1))$  module  $V$ . The quantum superalgebra  $U_q(gl(2/1))$  obtained by deforming  $gl(2/1)$  has simple generators  $\{E_i^i\}_{i=1}^3 U\{E_{i+1}^i, E_i^{i+1}\}_{i=1}^2$  which act on this module according to

$$\begin{aligned}
E_i^i|j\rangle &= -(\delta_j^i + \delta_4^j)|j\rangle ; i = 1, 2 , j = 1, 2, 3, 4 \\
E_3^3|j\rangle &= \left(\frac{1}{2} + \delta_4^j - \delta_1^j\right)|j\rangle ; j = 1, 2, 3, 4 \\
E_2^1|j\rangle &= \delta_3^j|2\rangle ; j = 1, 2, 3, 4 \\
E_1^2|j\rangle &= \delta_2^j|3\rangle ; j = 1, 2, 3, 4 \\
E_3^2|1\rangle &= E_3^2|3\rangle = E_2^3|2\rangle = E_2^3|4\rangle = 0 \\
E_3^2|2\rangle &= \left[\frac{1}{2}\right]_q^{1/2}|1\rangle , E_3^2|4\rangle = \left[\frac{1}{2}\right]_q^{1/2}|3\rangle \\
E_2^3|1\rangle &= -\left[\frac{1}{2}\right]_q^{1/2}|2\rangle , E_2^3|3\rangle = \left[\frac{1}{2}\right]_q^{1/2}|4\rangle
\end{aligned} \tag{1}$$

where

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad x \in C.$$

The quantum superalgebra  $U_q(gl(2/1))$  carries the following parity

$$[E_j^i] = ([i] + [j]) \pmod{2}, \quad (2)$$

where  $[1] = [2] = 0$ ,  $[3] = 1$ , and by consistency the parity of the basis vectors is chosen to be

$$[[1]] = [[4]] = 0 \quad [[2]] = [[3]] = 1 \quad (3)$$

Associated to  $U_q(gl(2/1))$  there is also a co-product structure

$$\Delta : U_q(gl(2/1)) \rightarrow U_q(gl(2/1)) \otimes U_q(gl(2/1))$$

given by

$$\begin{aligned} \Delta(E_i^i) &= I \otimes E_i^i + E_i^i \otimes I \quad i = 1, 2, 3 \\ \Delta(E_j^i) &= E_j^i \otimes q^{(E_1^1 - E_2^2)/2} + q^{-(E_1^1 - E_2^2)/2} \otimes E_j^i \quad i, j = 1, 2 \\ \Delta(E_\ell^k) &= E_\ell^k \otimes q^{(E_2^2 + E_3^3)/2} + q^{-(E_2^2 + E_3^3)/2} \otimes E_\ell^k \quad k, \ell = 2, 3 \end{aligned} \quad (4)$$

Everywhere we shall use the graded-tensor product law, defined by.

$$(a \otimes b)(c \otimes d) = (-1)^{[b][c]} (ac \otimes bd)$$

Following a strategy analogous to the one employed in ref. [3] to construct an hermitian Hamiltonian, we now consider the operator

$$T = |\psi\rangle\langle\psi|$$

where  $|\psi\rangle$  is an unnormalized vector of  $V \otimes V$  defined by

$$\begin{aligned} |\psi\rangle &= \left( q^{-1/2}|4\rangle \otimes |1\rangle + q^{+1/2}|1\rangle \otimes |4\rangle \right) \\ &+ \left( q^{-1/2}|3\rangle \otimes |2\rangle - q^{1/2}|2\rangle \otimes |3\rangle \right) \end{aligned} \quad (5)$$

and

$$\begin{aligned} \langle\psi| &= \left( q^{-1/2}\langle 4| \otimes \langle 1| + q^{+1/2}\langle 1| \otimes \langle 4| \right) \\ &+ \left( -q^{-1/2}\langle 3| \otimes \langle 2| + q^{+1/2}\langle 2| \otimes \langle 3| \right) \end{aligned} \quad (6)$$

A straightforward calculation shows that

$$T^2 = [2(q + q^{-1})]T \quad (7)$$

and, using the fact that  $|\psi\rangle$  spans a 1-dimensional submodule of  $U_q(gl(2/1))$  (see [3, 4])

$$\begin{aligned} (T \otimes I)(I \otimes T)(T \otimes I) &= T \otimes I \\ (I \otimes T)(T \otimes I)(I \otimes T) &= I \otimes T, \end{aligned} \quad (8)$$

such that  $T$  provides a representation of the  $TL$  algebra. This can be used to obtain an R-matrix by the transformation [4]

$$\check{R}(u) = PR(u) = I + \frac{\sinh(u)}{\sinh(\eta - u)} T, \quad (9)$$

where  $\cosh(\eta) = (q + q^{-1})$  and  $P$  is the  $Z_2$ -graded permutation operator defined by  $P(|x\rangle \otimes |y\rangle) = (-1)^{[x][y]}|y\rangle \otimes |x\rangle$ ,  $\forall 1 \leq x, y \leq 4$ .

It is easy to check that it satisfies the Yang-Baxter equation

$$(I \otimes \check{R}(u)(\check{R}(u+v) \otimes I)(I \otimes \check{R}(v)) = (\check{R}(v) \otimes I)(I \otimes \check{R}(u+v))(\check{R}(u) \otimes I) \quad (10)$$

A local Hamiltonian can be defined by [9]

$$H_{i,i+1} = \sinh(\eta) \left. \frac{d}{du} \check{R}(u)_{i,i+1} \right|_{u=1} = T_{i,i+1}, \quad (11)$$

where on the N-fold tensor product space we denoted

$$\check{R}(u)_{i,i+1} = I^{\otimes(i-1)} \otimes \check{R}(u) \otimes I^{\otimes(N-i-1)}.$$

Finally in view of the grading the basis vectors of the module  $V$  can be identified with the electronic states as follows

$$|1\rangle \equiv |+-\rangle = c_+^+ c_-^+ |0\rangle, \quad |2\rangle \equiv |- \rangle = c_-^+ |0\rangle, \quad |3\rangle \equiv |+\rangle = c_+^+ |0\rangle, \quad |4\rangle \equiv |0\rangle$$

allowing  $H_{i,i+1}$  to be expressed in terms of the canonical fermion operators as

$$\begin{aligned} H_{i,i+1} &= qn_{i,+}n_{i,-}(1 - n_{i+1,+})(1 - n_{i+1,-}) + q^{-1}(1 - n_{i,+})(1 - n_{i,-})n_{i+1,+}n_{i+1,-} \\ &+ q^{-1}n_{i,+}(1 - n_{i,-})n_{i+1,-}(1 - n_{i+1,+}) + qn_{i,-}(1 - n_{i,+})n_{i+1,+}(1 - n_{i+1,-}) \\ &- S_i^+ S_{i+1}^- - S_i^- S_{i+1}^+ + c_{i,+}^+ c_{i,-}^+ c_{i+1,-} c_{i+1,+} + c_{i+1,+}^+ c_{i+1,-}^+ c_i^- c_i^+ \\ &+ qc_{i,+}^+ c_{i+1,+} n_{i,-}(1 - n_{i+1,-}) + h.c. - c_{i,-}^+ c_{i+1,-} n_{i,+}(1 - n_{i+1,+}) + h.c. \\ &+ c_{i,-} c_{i+1,-}^+ n_{i+1,+}(1 - n_{i,+}) + h.c. - q^{-1}c_{i,+} c_{i+1,+}^+ n_{i+1,-}(1 - n_{i,-}) + h.c. \end{aligned} \quad (12)$$

where the  $c_{i\pm}^{(+)}$  are spin up or down annihilation (creation) operators, the  $S'_i$ 's spin matrices and the  $n'_i$ 's occupation numbers of electrons at lattice site  $i$ .

Thus we obtain a Hamiltonian describing electron pair hopping, correlated hopping and generalized spin interactions. We notice that in the limit  $q \rightarrow 1$  we recover the isotropic Hamiltonian discussed in [3].

By means of the quantum inverse scattering method [9, 10] it is possible to show that the model is integrable. Basically, by this procedure, the Hamiltonian (12) is related to the transfer matrix of a graded vertex model [11] (see also [12]), constructed from the spectral parameter dependent R-matrix. The associated Yang-Baxter algebra implies the commutativity of the transfer matrix for different spectral parameters, which reflects the integrability of the model.

The global Hamiltonian takes the form

$$H = \sum_{i=1}^{N-1} H_{i,i+1} + H_{N1} . \quad (13)$$

It is not invariant with respect to  $U_q(gl(2)) \otimes u(1)$  since  $H_{N1} \neq H_{1N}$  reflecting the non-cocommutativity of the co-product. However, by modifying the above Hamiltonian we can obtain a quantum algebra invariant model as follows [c.f. 5- 8].

Let  $\sigma$  denote the braid generator defined by

$$\begin{aligned} \sigma &= \lim_{u \rightarrow \infty} \check{R}(u) \\ &= I - e^{-\eta} T \end{aligned} \quad (14)$$

which satisfies the braid relations

$$(\sigma \otimes I)(I \otimes \sigma)(\sigma \otimes I) = (I \otimes \sigma)(\sigma \otimes I)(I \otimes \sigma) . \quad (15)$$

Setting  $G = \sigma_{12}\sigma_{23} \cdots \sigma_{N-1N}$ , then it follows from the TL relations that

$$GH_{i,i+1}G^{-1} = H_{i+1,i+2} \quad , \quad i = 1, \cdots, N-2$$

We now define

$$H_o = GH_{N-1,N}G^{-1} \quad (16)$$

which can be shown to satisfy

$$GH_oG^{-1} = H_{12}$$

and set our new Hamiltonian to be

$$H = \sum_{i=1}^{N-1} H_{i,i+1} + H_o \quad (17)$$

satisfying  $[H, G] = 0$  and additionally invariance with respect to the quantum algebra  $U_q(\mathfrak{gl}(2)) \otimes u(1)$ .

The details about the Bethe ansatz solution of the present model will be developed elsewhere.

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