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# Brane Dynamics and N=1 Supersymmetric Gauge Theory

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We discuss some aspects of the relation between space-time properties of branes in string theory, and the gauge theory on their worldvolume, for models invariant under four supercharges in three and four dimensions. We show that a simple set of rules governing brane dynamics reproduces many features of gauge theory. We study theories with  $U(N_c)$ ,  $SO(N_c)$  and  $Sp(N_c)$  gauge groups and matter in the fundamental and two index tensor representations, and use the brane description to establish Seiberg's electric-magnetic duality for these models.

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## 1. Introduction

In the past three years there has been remarkable progress in field and string theory. In four dimensional field theory with  $N \geq 1$  supersymmetry, a combination of symmetries and holomorphicity of certain terms in the low energy quantum effective action was found to provide constraints that in many cases are sufficient to completely determine the strongly coupled low energy dynamics. These constraints become more powerful as the number of supersymmetries increases. Theories with fewer supersymmetries typically have richer dynamics, but the handle one has on their dynamics is more limited.

The organizing principle that seems to emerge is electric-magnetic, strong-weak coupling duality.  $N = 4$  supersymmetric gauge theories are now believed to exhibit an exact Montonen-Olive duality [1].  $N = 2$  supersymmetric theories, as discussed by Seiberg and Witten [2], can be solved (in the BPS sector) using duality and supersymmetry.  $N = 1$  supersymmetric gauge theories have the richest dynamics of the three, are the closest to phenomenology, and in many cases can be studied at low energies by a generalization of Montonen-Olive duality due to Seiberg [3,4]. However, while there are many examples of  $N = 1$  supersymmetric gauge theories that can be analyzed using duality, the situation is not as satisfactory as that for theories with  $N > 1$  supersymmetry, since there are many cases for which the standard techniques appear to be inadequate.  $N = 1$  theories that are understood, are usually analyzed on a case by case basis with no obvious unified picture. There is no general a priori understanding of whether any given model possesses a useful dual.

One of the most interesting recent developments is the realization that many of the field theory phenomena mentioned above can be understood by embedding gauge theories in string theory and using properties of the latter to study the former. A very useful idea has been to use the interplay between the geometry and dynamics of branes, and the quantum field theory on their worldvolume [5-8]. In particular, in [9] it was shown that applying string duality to a configuration of branes in type IIB string theory provides an explanation of a certain “mirror symmetry” in 2+1 dimensional  $N = 4$  supersymmetric gauge theory discovered in [10] (see also [11,12]).

In a recent paper [13] it has been proposed that configurations of branes in type II string theory preserving four supercharges provide a natural arena for the study of four dimensional  $N = 1$  supersymmetric gauge dynamics. Seiberg’s duality relates theories that, in the brane construction, can be connected by a continuous path in the moduli space

of vacua, along which the gauge symmetry is completely broken and the infrared dynamics is weak. The relation between the electric and magnetic theories in this framework is reminiscent of the well known correspondence between two dimensional sigma models on Calabi-Yau hypersurfaces in weighted projective spaces and Landau-Ginzburg models with  $N = (2, 2)$  SUSY [14-16]; the continuous path is analogous to the interpolation between the CY and LG descriptions of the theory, both of which are embedded in the larger framework of the (non-conformal) gauged linear sigma model [17].

The purpose of this paper is to further study the dynamics of branes realizing four dimensional  $N = 1$  supersymmetric field theories on their worldvolume. The strategy is to use the interplay between brane dynamics and gauge theory to shed light on both. One can use simple gauge theory phenomena to learn about properties of branes, and then use these properties to study more complicated gauge theory phenomena using branes. We will use these ideas to complete and extend the explanation of Seiberg's duality from string theory proposed in [13], and present some related results in three and four dimensional gauge theories. As will become clear below, our discussion will touch on only a small subset of the many possible questions. A better understanding of the brane-gauge theory dictionary is likely to improve both our understanding of the role of branes in string theory and of gauge theory dynamics.

Another approach to studying the dynamics of  $N = 1$  supersymmetric four dimensional gauge theory and in particular Seiberg's duality that has been discussed recently is F theory [18-20]. It provides a complimentary picture of gauge dynamics by a different geometrization of the gauge theory phenomena. Our approach and F theory are related by T-duality. It is clearly important to pursue both approaches further to gain insight into strongly coupled gauge theory and string theory.

The plan of this paper is as follows. In sections 2 – 4 we introduce the set of branes and orientifolds which we use in subsequent sections to construct different gauge theories. We check that these branes preserve  $N = 1$  SUSY in four dimensions, and describe some of their classical and quantum properties. These properties can be deduced by constructing gauge theories out of brane configurations, and requiring that brane dynamics reproduce known properties of the gauge theory. This approach to gauge/brane dynamics provides a nice geometric realization of the moduli space of vacua in gauge theory, and of the space of deformations. In the course of the discussion we often find it convenient to compactify one of the four spacetime coordinates on a circle of radius  $R$  and study the physics as a function

of  $R$ . This leads to interesting connections to recent discussions of three dimensional  $N = 2$  supersymmetric gauge theories.

In sections 5, 6 we discuss four dimensional  $N = 1$  supersymmetric QCD and its compactification on a circle. We start in section 5 by constructing a configuration of branes describing at low energies  $N = 1$  supersymmetric Yang-Mills theory with gauge group  $U(N_c)$  and  $N_f$  flavors of quarks in the fundamental representation (we will refer to this as the “electric” theory). We describe the classical space of deformations of the model (the moduli space of vacua and mass deformations) and discuss the global symmetries and their realization in the brane configuration. We also describe a different brane configuration corresponding to a “magnetic” SQCD theory with gauge group  $U(N_c)$ ,  $N_f$  flavors of quarks and a gauge singlet “magnetic meson” field which couples to the quarks via a cubic superpotential. We show that the Higgs moduli space of the electric theory with  $N_c$  colors and  $N_f$  flavors and that of the magnetic theory with  $N_f - N_c$  colors and  $N_f$  flavors provide different descriptions of a single smooth space – the moduli space of supersymmetric brane configurations. This is the string theory origin of the relation between the two theories, discovered in gauge theory by Seiberg [21].

In section 6 we turn to quantum effects. Using the results of section 4 we reproduce the known quantum global symmetries, moduli spaces and deformations of SQCD with different numbers of flavors  $N_f$ , in three and four dimensions. We discuss quantum effects in the “magnetic” SQCD model of section 5, and complete the proof of Seiberg’s duality from string theory.

In section 7 we consider  $U(N_c)$  gauge theory with one or two adjoint superfields with polynomial superpotentials, as well as a number of fundamentals. We recover the field theory results of [22], and study some generalizations of that work.

In sections 8, 9 we discuss the case of symplectic and orthogonal groups, with matter in the fundamental, symmetric and antisymmetric tensor representations. To study such theories we introduce, in addition to branes, orientifold hyperplanes. We reproduce many of the results obtained in gauge theory in [21-27] as well as some generalizations.

In section 10 we summarize the results and comment on them. An appendix outlines the derivation in gauge theory of one of the results we deduce in the text from brane theory.

## 2. The cast of characters

We will be mostly working in type IIA string theory in ten dimensional Minkowski space. The theory is non-chiral; denoting by  $Q_L, Q_R$  the space-time supercharges generated

by left and right moving worldsheet degrees of freedom [28], we have:

$$\begin{aligned}\Gamma^0 \cdots \Gamma^9 Q_L &= + Q_L \\ \Gamma^0 \cdots \Gamma^9 Q_R &= - Q_R\end{aligned}\tag{2.1}$$

Type IIA string theory is actually an eleven dimensional theory. The eleventh dimension,  $x^{10}$ , is a circle of radius  $R_{10}$  proportional to the string coupling  $\lambda$ . In the M-theory limit,  $\lambda \rightarrow \infty$ , eleven dimensional Lorentz invariance is restored, and the two spinors  $Q_R, Q_L$  in (2.1) combine into a single spinor, the **32** of  $SO(10, 1)$ .

We will study three kinds of objects, each of which separately preserves half of the SUSY of the theory:

- (a) Neveu-Schwarz (NS), or solitonic, fivebranes, with tension proportional to  $1/\lambda^2$ . These branes couple magnetically to the NS  $B_{\mu\nu}$  field and are thus magnetic duals of fundamental IIA strings. In M-theory the NS fivebrane is interpreted as a “transverse” fivebrane, living at a point in the eleventh dimension  $x^{10}$ . At weak string coupling, the configuration of an NS fivebrane in flat Minkowski spacetime has been studied by Callan, Harvey and Strominger [29] who constructed the appropriate CFT for this situation. This CFT is not well understood, especially close to the core of the brane, where the dilaton formally blows up and the CFT description is not reliable, even when the string coupling far from the fivebrane is weak. The worldvolume field theory has (2,0) SUSY in six dimensions; the massless spectrum includes [29] a self-dual Kalb-Ramond field and five scalars parametrizing fluctuations of the fivebrane in the five transverse dimensions of M-theory.
- (b) Dirichlet (D)  $p$ -branes, with tension proportional to  $1/\lambda$ . In type IIA string theory there are branes with  $p = 0, 2, 4, 6, 8$ , which couple to appropriate  $p + 1$  form gauge fields in the Ramond sector. We will be mainly interested in four and six branes, although other branes will also play a role, providing non-perturbative corrections to the classical dynamics. The D fourbrane corresponds in M-theory to a “longitudinal” fivebrane wrapped around  $x^{10}$ . The D sixbrane can be thought of as a Kaluza-Klein monopole. At weak string coupling, D branes are governed by a CFT which has been constructed by Green [30], Polchinski [31], and others [32], and is rather well understood. The worldvolume theory on an infinite Dirichlet  $p$ -brane is a  $p + 1$  dimensional field theory invariant under sixteen supercharges; the massless spectrum includes a  $p + 1$  dimensional  $U(1)$  gauge field and  $9 - p$  scalars parametrizing fluctuations of the  $p$ -brane in the transverse dimensions. For  $N$  parallel  $p$ -branes, both the gauge field

and  $9 - p$  scalars are promoted to  $N \times N$  matrices [33] giving rise to  $U(N)$  gauge theory on the worldvolume with  $9 - p$  adjoint scalar fields, and the fermions needed for supersymmetry.

- (c) Orientifolds (O) are objects that at least at weak string coupling are not dynamical [32]; otherwise, they are similar to D branes. An orientifold  $p$ -plane breaks the same half of the SUSY as a parallel Dirichlet  $p$ -brane. It carries charge under the same Ramond  $p$  form gauge fields as such a D brane. The charge of an orientifold  $p$ -plane is equal and opposite to that of  $2^{p-5}$  *physical* Dirichlet  $p$ -branes. In the presence of an orientifold plane all D branes (which are outside the orientifold) acquire mirror images; thus the number of such D branes is double the number of physical branes.

Clearly, there are many ways to construct configurations of branes preserving four of the supercharges (1/8 of the original supersymmetry). For our purposes it will be sufficient to consider a class of configurations built out of four kinds of differently oriented branes, and two kinds of orientifold planes:

- (1) NS fivebrane with worldvolume  $(x^0, x^1, x^2, x^3, x^4, x^5)$ , which lives at a point in the  $(x^6, x^7, x^8, x^9)$  directions. The NS fivebrane preserves supercharges of the form  $\epsilon_L Q_L + \epsilon_R Q_R$ , with

$$\begin{aligned}\epsilon_L &= \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5 \epsilon_L \\ \epsilon_R &= \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5 \epsilon_R.\end{aligned}\tag{2.2}$$

- (2) A differently oriented NS fivebrane, which we will refer to as the NS' fivebrane. Its worldvolume is  $(x^0, x^1, x^2, x^3, x^8, x^9)$ ; it lives at a point in the  $(x^4, x^5, x^6, x^7)$  directions, and preserves the supercharges

$$\begin{aligned}\epsilon_L &= \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^8 \Gamma^9 \epsilon_L \\ \epsilon_R &= \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^8 \Gamma^9 \epsilon_R.\end{aligned}\tag{2.3}$$

- (3) D sixbrane with worldvolume  $(x^0, x^1, x^2, x^3, x^7, x^8, x^9)$ , which lives at a point in the  $(x^4, x^5, x^6)$  directions. The D sixbrane preserves supercharges satisfying

$$\epsilon_L = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^7 \Gamma^8 \Gamma^9 \epsilon_R.\tag{2.4}$$

- (4) D fourbrane with worldvolume  $(x^0, x^1, x^2, x^3, x^6)$ , which lives at a point in the  $(x^4, x^5, x^7, x^8, x^9)$  directions, and preserves supercharges satisfying

$$\epsilon_L = \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^6 \epsilon_R.\tag{2.5}$$

- (5) Orientifold sixplane, at a point in the  $(x^4, x^5, x^6)$  directions. As discussed above, it behaves similarly to the D sixbrane of item (3).
- (6) Orientifold fourplane, at a point in the  $(x^4, x^5, x^7, x^8, x^9)$  directions, which behaves similarly to the D fourbrane of item (4).

It is easy to see that a configuration including all the objects (1) – (6) preserves four supercharges. Indeed, the second equations in (2.2), (2.3) are satisfied by four of the sixteen independent spinors  $\epsilon_R$ .  $\epsilon_L$  is determined by (say) (2.4), and one can check that the first equations in (2.2), (2.3) and (2.5) are identities.

It should be mentioned for completeness that more general configurations preserving  $N = 1$  supersymmetry can be constructed by adding NS fivebranes, D sixbranes and O6 planes which are rotated with respect to the original ones in the  $(x^4, x^5, x^8, x^9)$  directions (see [34,35] for recent discussions). We will not discuss such configurations here, except for a few scattered comments.

### 3. Brane configurations and some of their classical properties

The field theory on an infinite fourbrane is a five dimensional gauge theory invariant under sixteen supercharges. We will study, following [9], the theory on fourbranes stretched between fivebranes. In this case, the boundary conditions eliminate some of the physical fluctuations of the infinite fourbrane, and reduce the amount of SUSY of the theory.

Consider, for example, a single D fourbrane stretched along the  $x^6$  direction between two NS fivebranes [9]. Since the ends of the fourbrane are constrained to lie on the fivebranes, the modes corresponding to fluctuations of the fourbrane in the  $(x^6, \dots, x^9)$  directions are massive. The configuration preserves eight supercharges (those satisfying (2.2), (2.5)), and describes at long distances a four dimensional  $N = 2$  supersymmetric  $U(1)$  gauge theory<sup>1</sup>. The position of the D fourbrane on the NS fivebranes in the  $(x^4, x^5)$  plane corresponds to the expectation value of the scalar in the  $U(1)$  vector multiplet. As

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<sup>1</sup> In a recent paper [36], E. Witten pointed out that this  $U(1)$  is in fact frozen (*i.e.* its gauge coupling vanishes) due to a divergence which has an infrared interpretation on the fivebrane and an ultraviolet one on the fourbrane. This does not modify the present discussion (or the constructions in the rest of the paper) since we can cancel the divergence in question by adding additional fourbranes attached to the NS fivebranes “at infinity” in  $(x^4, x^5)$ . Since all states charged under the  $U(1)$  gauge fields on both the original and the new fourbranes are arbitrarily heavy, we can ignore them.

is clear from the brane description, physics is independent of this expectation value – the scalar does not carry  $U(1)$  charge. The distance between the two NS fivebranes in the  $x^6$  direction,  $L_6$ , determines the gauge coupling of the  $N = 2$  supersymmetric gauge theory on the brane,  $1/g^2 \propto L_6/\lambda$ .

Consider now adding a D sixbrane at a point in the  $x^6$  direction that is between the locations of the two NS fivebranes. The additional brane does not break any further SUSY [9]. The fourbrane worldvolume dynamics corresponds now to  $N = 2$  supersymmetric  $U(1)$  gauge theory with a single charged hypermultiplet, which arises from 4 – 6 strings. The relative position of the D fourbrane and D sixbrane in the  $(x^4, x^5)$  directions determines the mass of the hypermultiplet.

The relative position of the two NS branes in the  $(x^7, x^8, x^9)$  directions plays the role of a Fayet-Iliopoulos (FI) D-term in the  $U(1)$  gauge theory. From the brane configuration it is clear that only when the two NS fivebranes are at the same value of  $(x^7, x^8, x^9)$ , can the D fourbrane stretch between them without breaking SUSY. This is a reflection of the fact that in the presence of a non-vanishing FI D-term, the gauge theory has no supersymmetric vacuum with unbroken gauge symmetry. For non zero D-terms there is an isolated vacuum, in which the theory is in a Higgs phase. The interpretation of this in brane theory is the following [9]: although for non zero FI D-terms a D fourbrane cannot connect two NS fivebranes, it can break into two pieces, each of which connects the D sixbrane to one of the two NS fivebranes. Since  $(x^7, x^8, x^9)$  are part of the worldvolume of the D sixbrane, as the two NS branes are separated in these directions, the two pieces of the original fourbrane could separate in the  $(x^7, x^8, x^9)$  directions while staying on the D sixbrane.

Thus we learn that the Higgs mechanism is described in brane theory as the breaking of D fourbranes stretched between NS branes, on D sixbranes. The fact that in the  $U(1)$  gauge theory considered above the vacuum one finds for non zero D-terms is isolated, implies that there are no massless modes associated with fourbranes stretched between D sixbranes and NS fivebranes. This is consistent with the fact that such a D fourbrane cannot fluctuate in the  $(x^4, \dots, x^9)$  directions due to the boundary conditions on the two ends.  $N = 2$  SUSY implies that there is also no massless worldvolume gauge field on it.

An interesting qualitatively new effect arises when we generalize the previous discussion to the case of  $N_c > 1$  fourbranes stretched between two NS fivebranes (still in the presence of one D sixbrane between the NS branes). This configuration describes an  $N = 2$



supersymmetric  $U(N_c)$  gauge theory with a hypermultiplet in the fundamental representation of the gauge group. If we now displace one of the NS fivebranes relative to the other in the  $(x^7, x^8, x^9)$  directions, turning on a FI D-term, naively the previous discussion applies, and the  $N_c$  fourbranes can break on the single D sixbrane to yield an isolated supersymmetric vacuum for all  $N_c$ . A gauge theory analysis reveals that this conclusion is false – for all  $N_c > 1$  the F and D term conditions for a supersymmetric vacuum have no solution. This and many related puzzles would be resolved if the following “s-rule” held [9]:

*A configuration in which an NS fivebrane and a D sixbrane are connected by more than one D fourbrane is not supersymmetric.*

A geometric explanation of the s-rule has been proposed in [20]. In brane dynamics it may be related to the fact that two fourbranes connecting a given NS fivebrane to a given D sixbrane are necessarily on top of each other – a rather singular situation.

Later in the paper we will be interested in moving NS fivebranes through D sixbranes. At first sight this appears to be a singular procedure, since when the two kinds of branes are at the same value of  $x^6$ , they actually meet in spacetime. In [9] it was proposed that when an NS fivebrane passes through a D sixbrane in the  $x^6$  direction, a third brane is created, a D fourbrane connecting the two original branes. It was pointed out that this creation of a brane is necessary to preserve magnetic charge. More importantly for us, in [9] evidence was presented that this process also implies that the low energy dynamics is insensitive to the relative position of D sixbranes and the NS fivebrane. Thus, despite appearances, the process of moving an NS fivebrane through a D sixbranes is smooth. This is going to play a role in the construction of the next sections.

In the next sections we will be interested in configurations which include NS' fivebranes; hence, it is useful to repeat the above discussion with NS' fivebranes replacing NS fivebranes. We will next mention a few of the main points.

Consider a configuration of  $N_c$  fourbranes stretched between two NS' fivebranes along the  $x^6$  direction, with  $N_f$  sixbranes located between the two NS' fivebranes. Unlike the previous case, this configuration preserves four supercharges, and describes at long distances a four dimensional  $N = 1$  supersymmetric  $U(N_c)$  gauge theory with one adjoint chiral multiplet,  $X$ , and  $N_f$  fundamental multiplets,  $Q, \tilde{Q}$  (we suppress flavor indices). The difference with the previous case is that the superpotential  $W = \tilde{Q}XQ$  required by  $N = 2$  SUSY in  $4d$  is absent here. The resulting model is in fact one of the simplest examples of  $N = 1$  supersymmetric gauge theory where the theory is known to be in a non abelian

Coulomb phase for all  $N_f \geq 1$ , but the long distance dynamics is not understood (see e.g. the third reference in [22] for some comments on this model).

Fluctuations of the  $N_c$  fourbranes inside the NS' fivebrane in the  $(x^8, x^9)$  plane parametrize the Coulomb branch of the model. Displacements of the  $N_f$  D sixbranes relative to the NS' fivebranes in the  $(x^4, x^5)$  directions give masses to the fundamental multiplets  $Q, \tilde{Q}$ . The relative position of the two NS' fivebranes in the  $(x^4, x^5)$  directions is not an independent parameter, as it can be compensated by changing the positions of the D sixbranes in the  $(x^4, x^5)$  plane (*i.e.* the masses of the fundamentals), and an overall rotation of the configuration.

The relative displacement of the two NS' fivebranes in the  $x^7$  direction plays the role of a FI D-term; note that unlike the  $N = 2$  SUSY case discussed before, here the D-term is a single real number, in agreement with field theory expectations. Separating the two NS' fivebranes in the  $x^7$  direction (*i.e.* turning on a FI D-term) should not break SUSY for all  $N_c, N_f > 0$ . In fact, a gauge theory analysis leads to the conclusion that complete Higgsing is possible for all  $N_f \geq 1$ ; the (complex) dimension of the Higgs branch is:  $2N_f N_c + N_c^2 - N_c^2 = 2N_f N_c$ . The first two terms on the left hand side are the numbers of components in the fundamental and adjoint chiral multiplets, and the negative term accounts for degrees of freedom eaten up by the Higgs mechanism.

The brane configuration provides a simple picture of the moduli space of vacua. As we have learned before, complete Higgsing corresponds to breaking all  $N_c$  fourbranes on various D sixbranes. The resulting fourbranes can then further break on other D sixbranes; a generic point in the Higgs branch corresponds to splitting each of the  $N_c$  fourbranes into  $N_f + 1$  pieces, the first connecting the left NS' fivebrane to the first D sixbrane, the second connecting the first two D sixbranes etc. To calculate the dimension of the moduli space one notes that:

- (a) A fourbrane stretched between two D sixbranes has two complex massless degrees of freedom, corresponding to fluctuations of the brane in the  $(x^7, x^8, x^9)$  directions, and  $A_6$ , the compact component of the fourbrane worldvolume gauge field.
- (b) A fourbrane stretched between an NS' fivebrane and a D sixbrane has one complex massless degree of freedom, corresponding to fluctuations of the brane in the  $(x^8, x^9)$  directions.
- (c) One does not expect an analog of the s-rule for fourbranes stretched between an NS' fivebrane and a D sixbrane, *e.g.* because two such fourbranes can be separated in the  $(x^8, x^9)$  directions, which are common to both kinds of branes.

Using these rules we find that the dimension of moduli space of brane configurations with completely broken  $U(N_c)$  gauge symmetry is:  $2N_c(N_f - 1) + 2N_c = 2N_cN_f$ , in agreement with the gauge theory analysis. Of course, one can read the above arguments in the opposite direction; the agreement of the gauge theory analysis with the brane configuration counting is strong evidence for the validity of assumptions (a) – (c) about brane dynamics.

#### 4. Some quantum properties of branes

One of the interesting features of brane configurations preserving  $N = 1$  SUSY in  $4d$  (or  $N = 2$  in  $3d$ ) is the fact that the classical picture of branes connected at right angles and not exerting any forces on each other is corrected quantum mechanically<sup>2</sup>. In this section we will describe the quantum bending of branes [36], and formulate some phenomenological rules regarding the quantum forces between different fourbranes. These forces will be deduced by comparison to gauge theory. One of the main advantages of the brane description is its universality; interactions deduced in a particular situation can be used in others to learn about strongly coupled gauge dynamics.

##### 4.1. Bending of NS fivebranes

Consider a configuration where a D fourbrane ends on an NS fivebrane from the left<sup>3</sup>. The fourbrane is actually an M-theory fivebrane spanning the complex  $s = x^6 + ix^{10}$  plane (in addition to  $(x^0, x^1, x^2, x^3)$  which we will suppress here), and is located at fixed  $v = x^4 + ix^5$ , say  $v = a$ . The NS fivebrane fills the  $v$  plane, and is classically located at a fixed value of  $s$ .

Quantum mechanically [36], the NS fivebrane bends away from the D fourbrane. Far from the point at which classically the fourbrane meets the fivebrane,  $v = a$ , the location of the latter is described by:

$$s_5 = R_{10} \ln(v - a) \tag{4.1}$$

This bending can be thought of as a quantum effect in the type IIA string description. Equation (4.1) implies that the classical symmetry of continuous rotation in the  $v$  plane,  $(v - a) \rightarrow e^{i\alpha}(v - a)$ , is broken quantum mechanically. This is the geometrical M-theory realization of the gauge theory anomaly in a chiral global  $U(1)$  current.

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<sup>2</sup> In contrast, in theories with eight or more supercharges, branes do not exert forces on each other quantum mechanically as well, since there are no superpotentials. There are of course other interesting quantum effects [9], [36].

<sup>3</sup> Here and below, positions of various branes refer to the  $x^6$  direction, unless stated otherwise.

#### 4.2. Three dimensional $N=2$ supersymmetric gauge theory

To deduce the quantum forces between branes we will compare some recent discussions of three dimensional  $N = 2$  supersymmetric gauge theories [37-39] with the brane picture. We start by describing the relevant gauge theory results.

Consider a four dimensional  $N = 1$  supersymmetric  $U(N_c)$  gauge theory coupled to  $N_f = N_c$  flavors of quarks  $Q^i, \tilde{Q}_{\tilde{i}}$  ( $i, \tilde{i} = 1, \dots, N_f$ ), in the fundamental representation of  $U(N_c)$  (the “electric” theory). Compactify one of the dimensions on a circle of radius  $R$ ; this gives rise at energies much lower than  $1/R$  to a three dimensional  $N = 2$  supersymmetric gauge theory. Classically, this model has an  $N_c (= N_f)$  dimensional<sup>4</sup> Coulomb branch and a  $2N_c N_f - N_c^2 = N_c^2$  dimensional Higgs branch which meet at the origin, as well as mixed Higgs – Coulomb branches in which part of the gauge symmetry is unbroken. One can also add “real masses” to the quarks which split the singularities in which the Coulomb and Higgs branches meet. We will discuss their role in the brane picture later.

Quantum mechanically, the low energy behavior of this theory (in the absence of real masses) is reproduced [39] by a sigma model for  $N_c^2 + 2$  chiral superfields  $V_{\pm}, M_{\tilde{i}}^i$ , with the superpotential

$$W = V_+ V_- (\det M - \eta) \tag{4.2}$$

where  $\eta$  is related to the three and four dimensional gauge couplings:  $\eta \sim e^{-1/Rg_3^2} \sim e^{-1/g_4^2}$ . In the four dimensional limit  $R \rightarrow \infty$ ,  $\eta$  turns into the four dimensional QCD scale  $\eta \rightarrow \Lambda_4^{2N_c}$ , while in the three dimensional limit  $R \rightarrow 0$  (with  $g_3$  fixed),  $\eta \rightarrow 0$ .  $M$  should be thought of as representing the meson field  $M_{\tilde{i}}^i = Q^i \tilde{Q}_{\tilde{i}}$  and  $V_{\pm}$  parametrize the Coulomb branch.

Varying (4.2) w.r.t. the fields  $V_{\pm}, M$  gives rise to the equations of motion

$$V_{\pm} (\det M - \eta) = 0; \quad V_+ V_- (\det M) (M^{-1})_{\tilde{i}}^{\tilde{i}} = 0 \tag{4.3}$$

Consider first the three dimensional case ( $R = \eta = 0$ ). There are three branches of moduli space:

- 1)  $V_+ = V_- = 0$ ;  $M$  arbitrary.
- 2)  $V_+ V_- = 0$ ;  $M$  has rank at most  $N_c - 1$ .
- 3)  $V_+, V_-$  arbitrary;  $M$  has rank at most  $N_c - 2$ .

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<sup>4</sup> Here and below we always count complex dimensions of moduli spaces.

The first branch can be thought of as a Higgs branch, while the last two are mixed Higgs – Coulomb branches. The three branches meet on a complex hyperplane on which the rank of  $M$  is  $N_c - 2$  and  $V_+ = V_- = 0$ . Most of the classical  $N_c$  dimensional Coulomb branch is lifted; its only remnants are  $V_\pm$ .

The understanding of the theory with  $N_f = N_c$  allows us to study models with any  $N_f \leq N_c$  by adding masses to some of the flavors and integrating them out. Adding a complex quark mass term  $W = -mM$  to (4.2), the following structure emerges. If the rank of  $m$  is one, one finds after integrating out the massive flavor a moduli space of vacua with  $V_+V_- \det M = 1$  ( $M$  is the  $(N_f - 1) \times (N_f - 1)$  matrix of classically massless mesons). If the rank of  $m$  is larger than one, one finds a superpotential with a runaway behavior. For example, if we add two non-vanishing masses:

$$W = V_+V_- \det M - m_1^1 M_1^1 - m_2^2 M_2^2 \quad (4.4)$$

we find, after integrating out the massive mesons  $M_1^i, M_j^1, M_2^i, M_j^2$ :

$$W = -\frac{m_1^1 m_2^2}{V_+V_- \det M} \quad (4.5)$$

where, again,  $M$  represents the  $(N_f - 2)^2$  classically massless mesons. Clearly, the superpotential (4.5) does not have a minimum at finite values of the fields; there is no stable vacuum.

When the radius of the circle is not strictly zero ( $\eta \neq 0$ ), the analysis of (4.3) changes somewhat. There are now only two branches:

- 1)  $V_+ = V_- = 0$ ;  $M$  arbitrary.
- 2)  $V_+V_- = 0$ ;  $\det M = \eta$ .

In particular, there is no analog of the third branch of the three dimensional problem. The two branches meet on a complex hyperplane on which  $\det M = \eta$  and  $V_+ = V_- = 0$ . The structure for all  $\eta \neq 0$  agrees with the four dimensional analysis [3,4].

If we add to (4.2) a complex mass term  $W = -mM$  with a mass matrix  $m$  whose rank is smaller than  $N_f$ , the vacuum is destabilized (including the case of a mass matrix of rank one where previously there was a stable vacuum). If the rank of  $m$  is  $N_f$ , so that the low energy theory is pure  $U(N_c)$  SYM, there are  $N_c (= N_f)$  isolated vacua which run off to infinity as the radius of the circle  $R$  goes to zero (there is also a decoupled moduli space for the  $U(1)$  piece of the gauge group).

Another theory which will prove useful is the compactified  $N = 1$  SYM theory with gauge group  $U(N_c)$  and  $N_f = N_c$  flavors of quarks described above, coupled to a gauge singlet superfield  $N_i^{\tilde{i}}$  via a Yukawa interaction:

$$W = N_i^{\tilde{i}} Q^i \tilde{Q}_{\tilde{i}} = N_i^{\tilde{i}} M_{\tilde{i}}^i \quad (4.6)$$

Classically, this “magnetic” theory has an  $N_c^2 + N_c$  dimensional moduli space of vacua, corresponding to giving the singlet meson  $N$  and the adjoint scalar (together with the dual of the gauge field) arbitrary expectation values, keeping  $Q = \tilde{Q} = 0$ . At generic points on this moduli space the gauge group is broken to  $U(1)^{N_c}$ .

To study the quantum mechanical situation, we add the classical superpotential (4.6) to (4.2). Varying the resulting superpotential with respect to the fields  $M$ ,  $N$ ,  $V_{\pm}$  gives rise to the equations:

$$V_{\pm}\eta = 0; \quad M = 0; \quad N = 0 \quad (4.7)$$

Thus, in the three dimensional case  $\eta = 0$  we find a two complex dimensional Coulomb phase parametrized by  $V_+$ ,  $V_-$ , while for non-zero  $R$  there is an isolated vacuum at the origin. The classical  $N_c^2$  dimensional moduli space parametrized by  $N$  is completely lifted. For non-zero  $R$  the Coulomb branch is lifted as well, while for  $R = 0$  a two dimensional subspace remains.

To understand the gauge theory results described above in brane theory, we next turn to the construction of these gauge theories using branes.

#### 4.3. The brane construction of classical 3d $N=2$ SQCD

Consider a configuration of  $N_c$  D fourbranes stretched between an NS fivebrane and an NS' fivebrane along the  $x^6$  direction, with the geometry described in section 2. This configuration preserves four supercharges, and corresponds at distances much larger than the separation between the NS and NS' branes,  $L_6$ , to a four dimensional  $N = 1$  supersymmetric theory on the worldvolume of the fourbranes.

In analogy to section 3, it is clear that the boundary conditions on the two ends of each fourbrane suppress long wavelength fluctuations of the fourbranes in the  $(x^4, \dots, x^9)$  directions. The only massless worldvolume degrees of freedom are the  $U(N_c)$  gauge bosons and their superpartners. To add matter, we insert  $N_f$  D sixbranes at values of  $x^6$  that are between those corresponding to the positions of the NS and NS' branes. 4 – 6 strings describe  $N_f$  chiral multiplets in the fundamental representation of  $U(N_c)$ ,  $Q^i, \tilde{Q}_i$ .

To compare to the electric gauge theory described in the previous subsection we compactify one of the fourbrane worldvolume dimensions, say  $x^3$ , on a circle of radius  $R$ . It is convenient to perform a T-duality transformation which transforms type IIA to type IIB and turns the D fourbrane wrapped around  $x^3$  into a D threebrane at a point on the dual circle of radius  $R_3 = 1/R$ . The NS and NS' fivebranes transform to themselves under this T-duality, while the D sixbrane turns into a D fivebrane at a point in  $(x^3, x^4, x^5, x^6)$ . The positions of the different D fivebranes in  $x^3$  play the role of the real masses of the quarks. As  $R_3 \rightarrow \infty$  we recover the three dimensional  $N = 2$  SQCD theory with  $\eta = 0$  discussed in the previous subsection, while the four dimensional limit corresponds to  $R_3 \rightarrow 0$ .

The magnetic theory with the singlet meson  $N$  (4.6) is obtained from a different but related configuration. We place an NS' fivebrane to the left of an NS fivebrane and add  $N_f$  D fivebranes to the left of the NS' fivebrane. Each of the D fivebranes is connected to the NS' fivebrane by a D threebrane, and there are  $N_c$  threebranes stretched between the NS' and NS fivebranes<sup>5</sup>. The quarks arise as 3 – 3 strings stretched between the two kinds of threebranes, while the meson  $N$  arises from 3 – 3 strings connecting threebranes stretched between the NS' fivebrane and different D fivebranes. The classical moduli space obtained by giving independent expectation values to all components of  $N$  is realized in the brane picture by breaking all  $N_f$  D threebranes connecting the NS' fivebrane to the D fivebranes in all possible ways.

#### 4.4. Three dimensional gauge theory and brane interactions

The quantum gauge theory effects described above are due in the brane picture to two basic quantum properties of threebranes:

- (1) There is a long range interaction between a D threebrane stretched between an NS' fivebrane and an NS fivebrane, and any other threebrane ending on the same NS' fivebrane. The interaction is repulsive if the two threebranes end on the same side of the NS' fivebrane; it is attractive if one ends on the NS' fivebrane from the left and the other from the right.
- (2) The repulsion between different threebranes ending on the same side of an NS' fivebrane can be “screened” by D fivebranes located between the threebranes (in  $x^3$ ). This screening is effective if and only if the threebrane stretched between NS and NS'

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<sup>5</sup> We will discuss the four dimensional version of this theory in section 5.

fivebranes mentioned in (1) can break on the D fivebrane before reaching the other threebrane.

Clearly, by T-duality there are similar interactions between fourbranes ending on an NS' fivebrane in type IIA string theory. In the rest of this subsection we will explain how the two rules stated above encode different gauge theory phenomena described in section 4.2. Later we will apply these rules to more general situations in three and four dimensions and will see that they pass some additional consistency tests.

Consider first the electric theory of section 4.2. If the mass matrix  $m$  has rank  $N_c$ , the theory describes  $N = 1$  supersymmetric  $U(N_c)$  gauge theory with no massless matter, on  $R^3 \times S^1$ . As we saw, if the radius of the circle,  $R$ , vanishes, the theory has no stable vacuum due to a runaway superpotential on the classical Coulomb branch. When  $R > 0$  the classical moduli space is replaced by  $N_c$  isolated vacua which run off to infinity as  $R \rightarrow 0$ .

In the brane picture these results are consequences of the repulsion between threebranes stretched between NS and NS' fivebranes. Recall that the threebranes live on a dual circle with radius  $R_3 = 1/R$ . For finite  $R_3$  the branes arrange around the circle at equal spacings, maximizing the distances between them and leading to an isolated vacuum. It is clear that the  $N_c$  dimensional Coulomb phase is lifted, except for the decoupled  $U(1)$  Coulomb branch which appears as the freedom to rigidly rotate all  $N_c$  threebranes around the  $x^3$  circle. The fact that there are  $N_c$  vacua has to do with the dual of the three dimensional gauge field, and is not expected to be seen geometrically (in  $3d$ ). It is also clear that in the three dimensional limit  $R_3 \rightarrow \infty$  the vacua run off to infinity, due to the long range repulsion.

The repulsive potential between a pair of threebranes can be thought of in this case<sup>6</sup> as due to Euclidean D strings stretched between the NS and NS' fivebranes and between the two D threebranes [9], [37] (which are instantons in the low energy three dimensional gauge theory). Since there are two fermionic zero modes in the presence of these instantons, they lead to a superpotential on the classical Coulomb branch. For D threebranes stretched between NS fivebranes (discussed in [9]) the same instantons contribute to the metric on the Coulomb branch, since there are then four zero modes in the presence of the instanton.

In situations when the rank of the mass matrix is smaller than  $N_c$ , there are massless quarks in the gauge theory; in the brane description, there are D fivebranes that can

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<sup>6</sup> Note, however, that this explanation does not apply to some of the other attractive and repulsive interactions between threebranes implied by the rules above.



“screen” the interactions between the threebranes. This screening can be seen directly by studying the Euclidean D strings stretched between D threebranes. If the worldsheet of such a D string intersects a D fivebrane, two additional zero modes appear and the contribution to the superpotential vanishes.

Consider for example the case where the number of massless flavors is  $N_f = N_c - 2$  in the three dimensional limit  $R = 0$ . We saw in section 4.2 that this theory develops a runaway quantum superpotential (4.5). In the brane picture, we have  $N_c$  threebranes stretched between NS and NS' fivebranes, and  $N_c - 2$  D fivebranes located at the same value of  $x^3$  (we are restricting to the case of vanishing real masses for now) between the NS and NS' fivebranes.

Due to the repulsion between unscreened threebranes stretched between NS and NS' fivebranes,  $N_c - 2$  of the  $N_c$  threebranes must break on different D fivebranes. The s-rule implies that once this has occurred, no additional threebranes attached to the NS fivebrane can break on these D fivebranes. We are left with two unbroken threebranes, one on each side of the D fivebranes (in  $x^3$ ). These threebranes repel each other, as well as the pieces of the broken threebranes stretched between the NS' fivebrane and the D fivebrane closest to it. There is no screening in this situation since all  $N_c - 2$  D fivebranes are connected to the NS fivebrane; hence the two threebranes stretched between the NS and NS' fivebranes cannot break on these D fivebranes. The system is unstable, and some or all of the threebranes mentioned above must run away to infinity.

This is in agreement with the gauge theory analysis of the superpotential (4.5). One can think of  $V_{\pm}$  as the positions in  $x^3$  of the two threebranes stretched between NS and NS' fivebranes mentioned above (as usual, together with the dual of the three dimensional gauge field). The potential obtained from (4.5) indeed suggests a repulsion between the different threebranes.

It is clear that the arguments above continue to hold when the radius of the circle on which the threebranes live is finite. While the two threebranes stretched between the NS and NS' fivebranes can no longer run away to infinity in the  $x^3$  direction, those connecting the NS' fivebrane to a D fivebrane (representing components of  $M$ ) can, and there is still no stable vacuum. This is in agreement with gauge theory; adding the term  $W = \eta V_+ V_-$  to (4.5) and integrating out  $V_{\pm}$  leads to a superpotential of the form  $W \sim (\det M)^{-1/2}$ .

The above discussion can be repeated with the same conclusions for all values  $1 \leq N_f \leq N_c - 2$ .

For  $N_f = N_c - 1$  the gauge theory answer is different; there is still no vacuum in the four dimensional case  $\eta \neq 0$ , while in three dimensions there is a quantum moduli space of vacua with  $V_+ V_- \det M = 1$ . In brane theory there are now  $N_c - 1$  D fivebranes, and the interaction between the D threebranes stretched between NS and NS' fivebranes can be screened. Indeed, consider a situation where  $N_c - 2$  of the  $N_c$  threebranes stretched between NS and NS' fivebranes break on D fivebranes. This leaves two threebranes and one D fivebrane that is not connected to the NS fivebrane. If  $R_3 = \infty$  (*i.e.*  $\eta = 0$ ), the single D fivebrane can screen the repulsion between the two threebranes. If the threebrane is at  $x^3 = 0$ , then using the rules in the beginning of this subsection we deduce that any configuration where one of the threebranes is at  $x^3 > 0$  while the other is at  $x^3 < 0$  is stable. The locations in  $x^3$  of the two threebranes give the two moduli  $V_{\pm}$ . Thus, the brane picture predicts correctly the existence of the quantum moduli space and its dimension. The precise shape of the moduli space (the relation between  $\det M$  and  $V_+ V_-$ ) is more difficult to see geometrically; nevertheless, it is clear that due to the repulsion there is no vacuum when either  $V_+$  or  $V_-$  vanish.

If the radius of the fourth dimension  $R$  is not zero, the brane picture predicts a qualitative change in the physics. Since  $R_3$  is now finite, the two threebranes stretched between NS and NS' fivebranes are no longer screened by the D fivebrane – they interact through the other side of the circle. Thus one of them has to break on the remaining D fivebrane, and one remains unbroken because of the s-rule. The repulsion between that threebrane and the threebranes stretched between the NS' fivebrane and a D fivebrane which is no longer screened leads to vacuum destabilization, in agreement with the gauge theory analysis.

For  $N_f = N_c$  (and vanishing real masses) the brane theory analysis is similar to the previous cases, and the conclusions are again in agreement with gauge theory. For  $R_3 = \infty$  one finds three phases corresponding to a pure Higgs phase in which there are no threebranes stretched between NS and NS' fivebranes, and two mixed Higgs – Coulomb phases in which there are one or two threebranes stretched between the NS and NS' fivebranes; the locations of the threebranes in  $x^3$  are parametrized by  $V_{\pm}$ . When there are two unbroken threebranes, they must be separated in  $x^3$  by the D fivebranes, which provide the necessary screening.

For finite  $R_3$ , the structure is similar, except for the absence of the branch with two unbroken threebranes, which is lifted by the same mechanism to that described in the case  $N_f = N_c - 1$  above.

#### 4.5. The magnetic brane configuration

In section 4.3 we have presented a brane configuration realizing a  $U(N_c)$  gauge theory with  $N_f$  flavors of quarks coupled to a singlet meson  $N_c^i$  via the superpotential (4.6). The quantum moduli space of this gauge theory with  $N_f = N_c$  was analyzed at the end of section 4.2. In this subsection we will revisit this theory from the point of view of brane dynamics.

The brane configuration describing this theory contains  $N_c$  threebranes connecting the NS' to an NS fivebrane on its right (we will refer to these as threebranes of the first kind), and  $N_c$  threebranes (which we will refer to as threebranes of the second kind) connecting it to  $N_c$  D fivebranes on its left.

Classically, this configuration has an  $N_c^2 + N_c$  dimensional moduli space corresponding to arbitrary breaking of the threebranes of the second kind on D fivebranes, and to independent motions of the threebranes of the first kind in the  $x^3$  direction. The rules of section 4.4 imply that quantum mechanically threebranes of the first kind have repulsive interactions among themselves, and attractive interactions with threebranes of the second kind. These together with the s-rule imply that the theory has a unique vacuum at the origin where all  $N_c$  D threebranes are aligned and can be thought of as stretching between the NS fivebrane and the  $N_c$  D fivebranes.

Looking back at equation (4.7) we discover a puzzle. For  $\eta \neq 0$  (*i.e.* finite  $R_3$ ) the gauge theory analysis is in agreement with the brane picture, yielding a unique vacuum at the origin of moduli space  $V_{\pm} = M = N = 0$ . For  $R_3 = \infty$  (the three dimensional limit) the gauge theory analysis yields a two dimensional moduli space parametrized by  $V_{\pm}$ . If true, it would imply that two of the threebranes of the first kind can remain unattached to threebranes of the second kind, without feeling any attraction to them or repulsion from each other. This is puzzling since the relevant physics in this problem is the interplay between the attraction between branes of different kinds and repulsion of branes of the same kind, and the existence of such an unlifted branch of moduli space should not be sensitive to whether  $R_3$  is finite or infinite.

The above puzzle leads us to propose that in fact the magnetic brane configuration described above gives rise quantum mechanically to the following sigma model:

$$W = V_+ V_- (\det M - \eta) + NM + V_+ W_- + V_- W_+ \quad (4.8)$$

This is the same as the quantum superpotential of the gauge theory (4.6) except for the fact that the fields  $V_{\pm}$  are now massive even for  $\eta = 0$ . To give them a mass we had to

introduce dynamical fields  $W_{\pm}$ , whose geometrical role in the brane configuration is not clear. Of course, the description (4.8) is a quantum one. Classically, the fields  $V_{\pm}$  are not well defined; as described in detail above, they parametrize the potentially massless modes on the Coulomb branch after most of it has been lifted by quantum effects.

## 5. Classical electric and magnetic N=1 SQCD

After the detour to three dimensions, we return in this section to four dimensional  $N = 1$  supersymmetric gauge theory. The main purpose of this and the next section is to derive Seiberg's duality [21] in  $U(N_c)$  gauge theory with matter in the fundamental representation, by embedding the problem in string theory. To achieve that we will in sections 5.1, 5.2 study brane configurations realizing the electric and magnetic theories; in section 5.3 we will show that, in the presence of a small FI D-term which breaks the gauge symmetry completely, the moduli spaces of the two theories in fact provide two different descriptions of a single smooth space – the moduli space of vacua of the underlying brane theory. This explains the equivalence between the two theories in the Higgs phase. In section 6 we will complete the demonstration of duality by studying the quantum effects, which become large when we turn off the D-term and approach the origin of moduli space.

### 5.1. The electric theory

Consider the configuration of  $N_c$  D fourbranes stretched between an NS fivebrane and an NS' fivebrane along the  $x^6$  direction, with  $N_f$  D sixbranes at values of  $x^6$  that are between those corresponding to the positions of the NS and NS' branes discussed in section 3. It is instructive to relate the supersymmetric deformations of the gauge theory to parameters defining the brane configuration, using the dictionary established in the previous sections:

#### 1) Higgs moduli space:

In the gauge theory, the structure of moduli space is as follows. For  $N_f < N_c$ , the  $U(N_c)$  gauge symmetry can be maximally broken to  $U(N_c - N_f)$ . The complex dimension of the moduli space of vacua is:  $2N_c N_f - (N_c^2 - (N_c - N_f)^2) = N_f^2$ . For  $N_f \geq N_c$  the gauge symmetry can be completely broken, and the complex dimension of the moduli space is  $2N_c N_f - N_c^2$ .

In the brane description, Higgsing corresponds to splitting fourbranes on sixbranes [9]. Consider, *e.g.*, the case  $N_f \geq N_c$  (the case  $N_f < N_c$  is similar). A generic point in

moduli space is described as follows. The first D fourbrane is broken into  $N_f + 1$  segments connecting the NS fivebrane to the first (*i.e.* leftmost) D sixbrane, the first D sixbrane to the second, etc., with the last segment connecting the rightmost D sixbrane to the NS' fivebrane. The second D fourbrane can now only be broken into  $N_f$  segments, because of the s-rule: the first segment must stretch between the NS fivebrane and the *second* D sixbrane, with the rest of the breaking pattern as before.

Recalling (see section 3) that a D fourbrane stretched between two D sixbranes has two complex massless degrees of freedom, while a D fourbrane stretched between a D sixbrane and an NS' fivebrane has one complex massless d.o.f., we conclude that the dimension of moduli space is:

$$\sum_{l=1}^{N_c} [2(N_f - l) + 1] = 2N_f N_c - N_c^2 \quad (5.1)$$

in agreement with the gauge theory result.

2) *Mass deformations:*

In gauge theory we can turn on a mass matrix for the (s)quarks, by adding a superpotential

$$W = m_i^j Q^i \tilde{Q}_j \quad (5.2)$$

with  $m$  an arbitrary  $N_f \times N_f$  matrix of complex numbers. In the brane configuration, masses correspond to relative displacement of the D sixbranes and the D fourbranes (or equivalently the NS' fivebrane) in the  $(x^4, x^5)$  directions. The configuration can be thought of as realizing a superpotential of the form (5.2), with the mass matrix  $m$  satisfying the constraint

$$[m, m^\dagger] = 0 \quad (5.3)$$

Thus, we can diagonalize  $m, m^\dagger$  simultaneously; the locations of the D sixbranes are the eigenvalues of  $m$ .

Hence, the brane configuration describes only a subset of the possible deformations of the gauge theory. This is a rather common situation in string theory, and we will encounter additional examples of this phenomenon below. Note also that the condition (5.3) appears as a consistency condition in  $N = 2$  supersymmetric gauge theories. Our theory is clearly not  $N=2$  supersymmetric; nevertheless, it is not surprising that the condition (5.3) arises, since one can think of  $m$  as the expectation value of a superfield in the adjoint of the  $U(N_f)$  gauge group on the D sixbranes. The theory on the infinite sixbranes is invariant

under sixteen supercharges in the bulk of the worldvolume, and while it is broken by the presence of the other branes, it inherits (5.3) from the theory with more supersymmetry.

3) *FI D-term:*

In the gauge theory it is possible to turn on a D-term for  $U(1) \subset U(N_c)$ . For  $N_f < N_c$  this breaks SUSY. For  $N_f \geq N_c$  there are supersymmetric vacua in which the gauge symmetry is broken, and the system is in a Higgs phase. In the brane description, the role of the FI D-term is played by the relative displacement of the NS and NS' fivebranes in the  $x^7$  direction. Clearly, when the two are at different values of  $x^7$ , a fourbrane stretched between them breaks SUSY. To preserve SUSY, all such fourbranes must break on D sixbranes, which as we saw above is only possible for  $N_f \geq N_c$  because of the s-rule. Once all fourbranes have been split, there is no obstruction to moving the NS and NS' fivebranes to different locations in  $x^7$ . Note that the Higgs phase depends smoothly on the D-term. In the brane construction the reason is that once all  $N_c$  D fourbranes have been broken on D sixbranes in a generic way, nothing special happens when the relative displacement of the two fivebranes in  $x^7$  vanishes.

4) *Global symmetries:*

Supersymmetric QCD with gauge group  $SU(N_c)$  and  $N_f$  quarks has the global symmetry  $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_x \times U(1)_a$ . The two  $SU(N_f)$  factors rotate the quarks  $Q^i, \tilde{Q}_i$ , respectively;  $U(1)_B$  is a vectorlike symmetry, which assigns charge +1 (-1) to  $Q$  ( $\tilde{Q}$ ).  $U(1)_x$  and  $U(1)_a$  are  $R$  symmetries under which the gaugino is assigned charge one, and the quarks  $Q, \tilde{Q}$  have charge 0 or 1. Only one combination of the two  $R$  symmetries is anomaly free, but this quantum effect is not expected to be visible in the classical brane construction (we will derive it from brane dynamics in section 6) .

In our case, the baryon number symmetry is gauged. The brane configuration has a manifest  $SU(N_f)$  symmetry, which is a gauge symmetry on the D sixbranes and a global symmetry on the D fourbranes. The other  $SU(N_f)$  is presumably broken at short distances and emerges at long distances as an accidental symmetry. The symmetries  $U(1)_x, U(1)_a$  are exact symmetries in the brane picture. They correspond to rotations in the  $(x^4, x^5)$  and  $(x^8, x^9)$  planes,  $U(1)_{45}, U(1)_{89}$ . From the discussion of the mass deformations and Higgs moduli space above, it is clear that the mass parameters (5.2) are charged under  $U(1)_{45}$ , while the quarks  $Q, \tilde{Q}$  are charged under  $U(1)_{89}$ . These rotations are part of the Lorentz group in ten dimensions and, therefore, the four dimensional supercharges are charged under both  $U(1)_{45}$  and  $U(1)_{89}$ ; hence, the two  $U(1)$  symmetries are R-symmetries.

Summarizing, the charge assignments under  $U(1)_{45} \times U(1)_{89}$  are:  $Q$  and  $\tilde{Q}$  have charges  $(0, 1)$ , the mass parameters  $m$  in (5.2) have charges  $(2, 0)$ , and the superspace coordinates  $\theta_\alpha$  have charges  $(1, 1)$ . With these assignments, the mass term (5.2) is invariant under both global symmetries<sup>7</sup>.

## 5.2. The magnetic theory

The magnetic brane configuration is similar to that discussed in section 4.5. It contains  $N_c$  fourbranes connecting the NS' fivebrane to an NS fivebrane on its right (we will refer to these as fourbranes of the first kind), and  $N_f$  fourbranes (of the second kind) connecting it to  $N_f$  D sixbranes on its left. We will consider the case  $N_f \geq N_c$  in what follows.

This configuration describes SQCD with gauge group  $U(N_c)$  (with the gauge bosons coming as before from 4 – 4 strings connecting fourbranes of the first kind),  $N_f$  flavors of quarks  $q_i, \tilde{q}^i$  (4 – 4 strings connecting the  $N_c$  fourbranes of the first kind with the  $N_f$  fourbranes of the second kind), and a complex meson  $M_j^i$  ( $i, j = 1, \dots, N_f$ ) arising from 4 – 4 strings connecting fourbranes of the second kind. The standard open string coupling gives rise to a superpotential

$$W_{\text{mag}} = M_j^i q_i \tilde{q}^j \quad (5.4)$$

The analysis of moduli space and deformations of this model is similar to the electric theory, with a few differences due to the existence of the superpotential (5.4). Consider first mass deformations. In gauge theory we can add a mass term to the magnetic quarks, by modifying the superpotential to:

$$W_{\text{mag}} = M_j^i q_i \tilde{q}^j + \delta M_j^i q_i \tilde{q}^j \quad (5.5)$$

The mass parameters  $\delta M$  can be absorbed into the expectation value of the magnetic meson  $M_j^i$ , and should thus be thought of as parametrizing a moduli space of vacua. The  $N_f^2$  resulting parameters are described in the brane language by splitting the  $N_f$  D fourbranes of the second kind on the D sixbranes in the most general way consistent with the geometry. It is easy to check that there are generically a total of  $N_f^2$  massless modes in this brane configuration, corresponding to the  $N_f^2$  components of  $M$ :  $N_f$  of them describe fluctuations in the  $(x^8, x^9)$  plane of fourbranes stretched between the NS' fivebrane and the rightmost

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<sup>7</sup> These global symmetries may be used to argue that the three adjoint superfields corresponding to fluctuations of the fourbranes in  $(x^4, \dots, x^9)$  are infinitely massive; this will be discussed in section 7.

D sixbrane, and  $\sum_{l=1}^{N_f-1} 2l = N_f(N_f - 1)$  parametrize fluctuations in  $(x^6, x^7, x^8, x^9)$  of the fourbranes connecting different sixbranes.

To give the magnetic meson field a mass in gauge theory, we add a linear term in  $M$  to the magnetic superpotential:

$$W_{\text{mag}} = M_j^i (q_i \tilde{q}^j - m_i^j) \quad (5.6)$$

Integrating out the massive field  $M$  we find that in the presence of the mass parameters  $m_i^j$  the gauge group is broken; thus the “mass parameters”  $m$  play the role of Higgs expectation values. In the brane description, these deformations correspond to a process where fourbranes of the first kind are aligned with those of the second kind and reconnected to stretch between the NS fivebrane and a D sixbrane. If  $m$  has rank  $n (\leq N_c)$ ,  $n$  such fourbranes are reconnected. The D sixbranes on which the reconnected fourbranes end can then be moved in the  $(x^4, x^5)$  directions, taking the fourbranes with them and breaking the  $U(N_c)$  gauge group to  $U(N_c - n)$ . The brane description realizes only a subset of the allowed “mass” matrices  $m$ , namely those which satisfy (5.3) (the reason is similar to the one described there). We will soon see that this analogy is not coincidental.

Another deformation of the magnetic gauge theory and of the corresponding brane configuration, which will play a role in the sequel, is switching on a FI D-term for the  $U(1)$  subgroup of  $U(N_c)$ . In the brane construction this corresponds to a relative displacement of the NS and NS' fivebranes in the  $x^7$  direction. To preserve SUSY, all  $N_c$  fourbranes of the first kind have to be reconnected to  $N_c$  of the  $N_f$  fourbranes of the second kind, leading to a situation where  $N_c$  fourbranes stretch between the NS fivebrane and  $N_c$  different sixbranes and  $N_f - N_c$  fourbranes stretch between the NS' fivebrane and the remaining sixbranes. Once this occurs, the two fivebranes can be separated in  $x^7$ .

Unlike the electric theory, here there is classically a jump in the dimension of the moduli space of the theory as we vary the D-term. For non-vanishing D-term there are only  $N_f - N_c$  fourbranes that give rise to moduli (the other  $N_c$  are frozen because of the s-rule), and the moduli space is easily checked to be  $N_f^2 - N_c^2$  dimensional. When the D-term vanishes, the previously frozen fourbranes can be reconnected to yield the original configuration, with unbroken  $U(N_c)$ , and we gain access to the full  $N_f^2$  dimensional moduli space discussed above. Quantum mechanically, this classical jump in the structure of the moduli space disappears, as will be discussed in section 6.

The magnetic brane configuration is invariant under the same global symmetries as the electric theory described above. It is not difficult to show that the charge assignments



under the  $U(1)_{45} \times U(1)_{89}$  symmetry are as follows: the magnetic quarks  $q, \tilde{q}$  have charges  $(1, 0)$ , the mass parameters  $m$  have charges  $(2, 0)$ , the magnetic meson  $M$  has charges  $(0, 2)$ , and the superspace coordinates  $\theta_\alpha$  have charges  $(1, 1)$ .

### 5.3. Seiberg's duality in the classical brane picture

We have now constructed using branes two  $N = 1$  supersymmetric gauge theories, the electric and magnetic theories discussed in the previous two subsections. Seiberg has shown [21] that the electric gauge theory with gauge group  $U(N_c)$  and the magnetic theory with gauge group  $U(N_f - N_c)$  are equivalent in the extreme infrared<sup>8</sup> (*i.e.* they flow to the same infrared fixed point). Seiberg's duality is a quantum symmetry, but it has classical consequences in situations where the gauge symmetry is completely broken and there is no strong infrared dynamics. In such situations Seiberg's duality reduces to a classical equivalence of Higgs branches and their deformations.

In this subsection we show using brane theory that the Higgs branches of the electric and magnetic theories with gauge groups  $U(N_c)$  and  $U(N_f - N_c)$ , respectively, provide different parametrizations of a single space, the moduli space of vacua of the appropriate brane configuration. This explains the classical part of Seiberg's duality. As one approaches the root of the Higgs branch, non-trivial quantum gauge dynamics appears, and we have to face the resulting strong coupling problem. This will be addressed in section 6.

Start, for example, with the electric theory with gauge group  $U(N_c)$ . As we learned above, higgsing corresponds to breaking D fourbranes on D sixbranes. It is convenient before entering the Higgs branch to move the NS fivebrane to the right through the  $N_f$  D sixbranes. This is a smooth process which does not influence the low energy physics on the fourbranes [9]. When the NS fivebrane passes through the  $N_f$  D sixbranes, it generates  $N_f$  D fourbranes connecting it to the D sixbranes; these new D fourbranes are rigid.

We now enter the Higgs phase by reconnecting the  $N_c$  original fourbranes to  $N_c$  of the  $N_f$  new fourbranes created previously; we then further reconnect the resulting fourbranes in the most general way consistent with the rules described in section 3. The resulting moduli space is  $2N_f N_c - N_c^2$  dimensional, as described in section 5.1. Note that, generically, there are now  $N_f - N_c$  D fourbranes attached to the NS fivebranes, and  $N_c$  D fourbranes

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<sup>8</sup> Seiberg actually considered the  $SU(N_c)$  and  $SU(N_f - N_c)$  theories, but the statement for  $U(N_c), U(N_f - N_c)$  follows from his results by gauging baryon number.

connected to the NS' fivebranes (the other ends of all these fourbranes lie on different D sixbranes).

Once we are in the Higgs phase, we can freely move the NS fivebrane relative to the NS' fivebrane, and in particular the two branes can pass each other in the  $x^6$  direction without ever meeting in space. This can be achieved by taking the NS fivebrane around the NS' fivebrane in the  $x^7$  direction, *i.e.* turning on a FI D-term in the worldvolume gauge theory. At a generic point in the Higgs branch of the electric theory, turning on such a D-term is a completely smooth procedure; this is particularly clear from the brane description, where in the absence of D fourbranes connecting the NS fivebrane to the NS' fivebrane, the relative displacement of the two in the  $x^7$  direction can be varied freely.

After exchanging the NS and NS' fivebranes, the brane configuration we find can be interpreted as describing the Higgs phase of *another* gauge theory. To find out what that theory is, we approach the root of the Higgs branch by aligning the  $N_f - N_c$  D fourbranes emanating from the NS fivebrane with the NS' fivebrane, and the  $N_c$  D fourbranes emanating from the NS' fivebrane with D fourbranes stretched between D sixbranes.

We then reconnect the D fourbranes to obtain a configuration consisting of  $N_f - N_c$  D fourbranes connecting the NS' fivebrane to an NS fivebrane which is to the left of it; the NS' fivebrane is further connected by  $N_f$  D fourbranes to the  $N_f$  D sixbranes which are to the left of it. This is the magnetic SQCD of section 5.2, with gauge group  $U(N_f - N_c)$ .

To summarize, we have shown that the moduli space of vacua of the electric SQCD theory with (completely broken) gauge group  $U(N_c)$  and  $N_f$  flavors of quarks  $Q^i$ ,  $\tilde{Q}_i$ , and the moduli space of vacua of the magnetic SQCD model with (broken) gauge group  $U(N_f - N_c)$ , can be thought of as providing different descriptions of a single moduli space of supersymmetric brane configurations. One can smoothly interpolate between them by varying the scale  $\Lambda$ , keeping the FI D-term fixed but non-zero. Since the only role of  $\Lambda$  in the low energy theory is to normalize the operators [22], theories with different values of  $\Lambda$  are equivalent. The electric and magnetic theories will thus share all features, such as the structure of the chiral ring (which can be thought of as the ring of functions on moduli space), that are independent of the interpolation parameter  $\Lambda$ .

The above smooth interpolation relies on the fact that the gauge symmetry is completely broken, due to the presence of the FI D-term. As mentioned above, it is not surprising that duality appears classically in this situation since there is no strong infrared gauge dynamics.

The next step is to analyze what happens as the gauge symmetry is restored when the D-term goes to zero and we approach the origin of moduli space. Classically, we find a disagreement. In the electric theory, we saw in section 5.1 that nothing special happens when the gauge symmetry is restored. New massless degrees of freedom appear, but there are no new branches of the moduli space that we gain access to.

In the magnetic theory the situation is different. When we set the FI D-term to zero, we saw in section 5.2 that a large moduli space of previously inaccessible vacua becomes available. While the electric theory has a  $2N_f N_c - N_c^2$  dimensional smooth moduli space, the classical magnetic theory experiences a jump in the dimension of its moduli space from  $2N_f N_c - N_c^2$  for non-vanishing FI D-term, to  $N_f^2$  when the D-term is zero. However, in the magnetic theory when the D-term vanishes the  $U(N_f - N_c)$  gauge symmetry is restored, and to understand what really happens we must study the quantum dynamics. We will discuss this in section 6, where we shall see that quantum mechanically the jump in the magnetic moduli space disappears, and the quantum moduli spaces of the electric and magnetic theories agree.

It is instructive to map the deformations of the classical electric theory to those of the classical magnetic one. Turning on masses (5.2) in the electric theory corresponds to moving the D sixbranes away from the D fourbranes (or equivalently from the NS' fivebrane) in the  $(x^4, x^5)$  directions. As discussed in section 5.2, in the magnetic description, the electric mass parameters correspond to Higgs expectation values (5.6).

Turning on expectation values to the electric quarks, which was described in the brane language in section 5.1, corresponds on the magnetic side to varying the expectation value of the magnetic meson  $M$ , (5.5). This gives masses to the magnetic quarks.

The transmutation of masses into Higgs expectation values and vice versa observed in the brane construction is one of the hallmarks of Seiberg's duality.

## 6. Quantum electric and magnetic N=1 SQCD

To complete the demonstration of Seiberg's duality in brane theory we have to address the strong coupling quantum effects which become important near the origin of moduli space. In this section we will do that, taking the opportunity to describe some additional aspects of the quantum physics of the electric and magnetic theories in three and four dimensions.

### 6.1. Quantum effects in the electric theory

#### 1) Global symmetries:

In section 5.1 we saw that the classical electric theory is invariant under two chiral  $R$  symmetries, corresponding in the brane language to rotations in the (4,5) and (8,9) planes. In gauge theory it is well known that quantum mechanically only one combination of the two  $R$  symmetries is preserved, due to chiral anomalies. This can be seen directly in brane theory.

At finite type IIA coupling  $\lambda$  we should interpret our brane configurations as describing fivebranes and sixbranes in M-theory. Defining (as in section 4.1):

$$\begin{aligned} s &= x^6 + ix^{10} \\ v &= x^4 + ix^5 \\ w &= x^8 + ix^9 \end{aligned} \tag{6.1}$$

the D fourbrane corresponds “classically” to an M-theory fivebrane at  $v = w = 0$  and it is extended in  $s$ , the NS fivebrane is at  $s = w = 0$  and is extended in  $v$ , while the NS' fivebrane is at  $v = 0$ ,  $s = S$  and is extended in  $w$ .

Quantum mechanically [36] the NS and NS' fivebranes are deformed by the fourbranes ending on them according to (4.1). Consider the classical electric configuration corresponding to  $N_f$  D sixbranes connected by D fourbranes to an NS fivebrane which is to their right, with the NS fivebrane further connected by  $N_c$  fourbranes to an NS' fivebrane which is to its right. Equation (4.1) implies that this classical configuration is deformed quantum mechanically. Far from the origin of the  $(v, w)$  plane (the location of the fourbranes), the location of the NS fivebrane in the  $s$  plane is described by

$$s_5 = R_{10}(N_f - N_c) \ln v \tag{6.2}$$

while the location of the NS' fivebrane in the  $s$  plane is described by

$$s_{5'} = R_{10}N_c \ln w \tag{6.3}$$

Thus, while the classical brane configuration is invariant both under rotations in the  $(x^4, x^5)$  plane,  $v \rightarrow e^{i\alpha}v$ , and in the  $(x^8, x^9)$  plane,  $w \rightarrow e^{i\beta}w$ , the quantum mechanical

configuration breaks both symmetries. Nevertheless, one combination of the two  $U(1)$  symmetries is preserved, if accompanied by an appropriate translation in  $x^{10}$ . The unbroken  $R$  symmetry is the one under which

$$s_5 - s_{5'} = R_{10} \ln (w^{-N_c} v^{N_f - N_c}) \quad (6.4)$$

is invariant. It is not difficult to check that if (by definition) the  $R$  charge of  $\theta$  under this symmetry is one, that of  $Q$ ,  $\tilde{Q}$  is  $B_f = 1 - N_c/N_f$ , in agreement with the gauge theory answer [3,4].

2) *Moduli space of vacua for  $N_f > N_c$ :*

In section 4 we described the quantum moduli space of  $N = 1$  SQCD with  $N_f \leq N_c$  in four dimensions and its compactification on a circle, comparing the gauge theory picture of [39] to that obtained by using the brane interactions of section 4.4. Here we will comment on the case  $N_f > N_c$  and the role of real masses in the compactified theory.

As is clear from [39] and from the discussion of section 4, the physics of the four dimensional theory compactified on a circle of finite radius  $R$  is similar to that of the infinite  $R$  one. As  $R \rightarrow 0$  one often encounters different phenomena. It will be convenient, as in section 4, to discuss the compactified theory, as it helps to geometrize more of the gauge theory phenomena. We will also perform the T-duality transformation of sections 4.3, 4.4 and discuss threebranes at points on a circle of radius  $R_3 = 1/R$  rather than fourbranes wrapped around a circle of radius  $R$ .

Consider the electric theory on  $R^3 \times S^1$  with  $N_f > N_c$ , and vanishing real masses. We start with the case  $R_3 = \infty$  in which the theory describes a three dimensional  $N = 2$  supersymmetric QCD. The moduli space of vacua is similar to that of the theory with  $N_f = N_c$  (see section 4.2). There are three branches of moduli space:

- a) Higgs branch:  $V_+ = V_- = 0$ . The meson field  $M$  gets an expectation value, generically breaking the  $U(N_c)$  gauge group completely. The Higgs branch is  $2N_f N_c - N_c^2$  dimensional, and corresponds in the brane picture to breaking all  $N_c$  threebranes that are initially stretched between the NS and NS' fivebranes on D fivebranes.
- b) Mixed Coulomb – Higgs branch, where  $V_+ V_- = 0$ , but either  $V_+$  or  $V_-$  is massless. This branch is realized by breaking all but one of the threebranes on the D fivebranes, and allowing the remaining threebrane to fluctuate in the  $x^3$  direction, either above ( $V_+$ ) or below ( $V_-$ ) the location of the D fivebranes. All repulsive interactions between the single threebrane stretched between the NS and NS' fivebranes and the broken

threebranes are screened, and hence this branch of moduli space is not lifted. Its dimension is  $(2N_f(N_c - 1) - (N_c - 1)^2) + 1$  with the term in parenthesis coming from the mesons  $M$  and the  $+1$  from  $V_{\pm}$ .

- c) Another mixed Coulomb – Higgs branch is obtained by breaking all but two of the threebranes on D fivebranes, and placing one of the two above and the other below the D fivebranes in  $x^3$ . In this branch  $V_+$  and  $V_-$  are massless while the meson moduli space, arising from the  $N_c - 2$  broken threebranes, is  $2N_f(N_c - 2) - (N_c - 2)^2$  dimensional. Again, all repulsive interactions are screened.

The repulsion between threebranes stretched between NS and NS' fivebranes does not allow us to leave more than two such threebranes unbroken. Therefore, the branches discussed above exhaust the possibilities for unlifted parts of moduli space. For finite  $R_3$  the structure is very similar, except that the third branch of the three dimensional moduli space discussed above is lifted, because the interaction between the two unbroken threebranes is no longer screened (as in the case  $N_f = N_c$  discussed in section 4.4).

In the four dimensional limit  $R_3 \rightarrow 0$  the Coulomb branches degenerate (the threebranes have no room to move in the  $x^3$  direction), and we are left with the  $2N_c N_f - N_c^2$  dimensional Higgs branch familiar from the classical theory. An important conclusion is that in the quantum theory, just like in the classical one, there is no discontinuous jump in the structure of the moduli space at the root of the Higgs branch.

### 3) Adding real masses for the quarks:

In addition to complex masses (5.2), which correspond to the positions of D fivebranes in the  $(x^4, x^5)$  plane, in the compactified theory one can add “real masses” for the quarks by displacing the D fivebranes in the  $x^3$  direction (see [38], [39] for the gauge theory description of these mass parameters). When all  $N_f$  real masses are different (*i.e.* the D fivebranes are all at different values of  $x^3$ ,  $a_i$ , with  $i = 1, \dots, N_f$ ), the discussion of the moduli space has to be modified somewhat.

For  $1 \leq N_f < N_c - 1$  there is still no vacuum both for finite and infinite  $R_3$ , since there is no way to place the threebranes such that all the repulsive interactions are screened.

For  $N_f = N_c - 1$  and infinite  $R_3$  there is an  $N_c$  dimensional Coulomb branch<sup>9</sup> which is described in brane theory by placing a D fivebrane between every two threebranes, thus screening the repulsive interactions between the threebranes. The Coulomb branch is then parametrized by the locations of the  $N_c$  threebranes (and the dual of the  $3d$  worldvolume

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<sup>9</sup> The existence of such a branch was observed in gauge theory in [38,39].

gauge field),  $V_i$  ( $i = 1, \dots, N_c = N_f + 1$ ), satisfying  $a_{i-1} < |V_i| < a_i$  (where  $a_0 = -\infty$ ,  $a_{N_f+1} = \infty$ ). The  $U(N_c)$  gauge group is broken to  $U(1)^{N_c}$  everywhere on this Coulomb branch. The boundaries of the Coulomb branch,  $|V_i| = a_{i-1}, a_i$ , are singular surfaces with no Higgs branches emanating from them<sup>10</sup> since whenever a threebrane breaks on a fivebrane in this situation, unscreened repulsive forces push parts of it to infinity.

For finite  $R_3$  the phase discussed above disappears, since as we have seen a number of times before, there is now an unscreened interaction between the first and last threebranes.

It is easy to repeat the discussion above for the case that some of the real masses are equal. The brane picture predicts a moduli space corresponding to particular combinations of Higgs and Coulomb branches uniquely determined by the rules of section 4.4. We will not go through the details of that analysis here. It is also not difficult to generalize the discussion to  $N_f \geq N_c$ . Even when all the real masses are different, there are now possibilities for mixed Coulomb – Higgs branches. Their analysis is straightforward and will be left to the reader.

## 6.2. Quantum effects in the magnetic theory

To complete the demonstration of Seiberg’s duality using branes we now turn to a discussion of quantum effects in the magnetic SQCD model. We will restrict to the four dimensional case,  $R = \infty$ , and will comment on the compactified theory in section 10.

The classical magnetic configuration is invariant under  $U(1)_{45} \times U(1)_{89}$  just like the electric one. Quantum mechanically the NS fivebranes are deformed due to the presence of the “fourbranes”; equations (6.2), (6.3) which were found for the electric configuration, are valid for the magnetic one as well. This guarantees that the charge assignments of the various fields ( $q, \tilde{q}, M$ ) agree with those found in gauge theory, and with the electric configuration.

One may interpret, following [36], the form (6.2), (6.3) of the electric coupling  $s_5 - s_{5'}$  as describing a  $v, w$  dependent electric QCD scale  $\Lambda_e$ :

$$\Lambda_e^{3N_c - N_f} = \mu^{3N_c - N_f} e^{-(s_5 - s_{5'})} = \mu^{3N_c - N_f} (w^{-N_c} v^{N_f - N_c})^{-R_{10}} \quad (6.5)$$

where  $\mu$  is some fixed scale independent of  $v, w$ . The first equality in (6.5) is exact, while the second is correct asymptotically, when  $v, w$  are large. The fact that in the magnetic

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<sup>10</sup> This seems to disagree with [38].

configuration the NS and NS' fivebranes are reversed implies that the magnetic QCD scale  $\Lambda_m$  has a different dependence on  $v, w$ :

$$\Lambda_m^{3\bar{N}_c - N_f} = \mu^{3\bar{N}_c - N_f} e^{+(s_5 - s_{5'})} \quad (6.6)$$

where  $\bar{N}_c \equiv N_f - N_c$ . Equations (6.5), (6.6) lead to the scale matching relation:

$$\Lambda_e^{3N_c - N_f} \Lambda_m^{3\bar{N}_c - N_f} = \mu^{N_f} \quad (6.7)$$

which is familiar from the study of Seiberg's duality in gauge theory [4,22]. Strictly speaking, the argument above shows that while  $\Lambda_e$  and  $\Lambda_m$  depend on  $v$  and  $w$ , for large  $v, w$  the product (6.7) is constant. To study the constant and, in particular, its relation to the Yukawa coupling in the magnetic superpotential [4], one has to perform a more detailed analysis.

Classically, the moduli spaces of the electric and magnetic gauge theories were found in section 5 to be different. The electric moduli space is smooth as the FI D-term approaches zero, while in the magnetic theory there is a jump in the dimension of moduli space from  $2N_f N_c - N_c^2$  for non zero D-term, to  $N_f^2$  for vanishing D-term. The quantum analysis of the electric theory performed in the previous subsection implies that quantum effects do not lead to a qualitative change in the structure of the electric moduli space.

In the magnetic theory, quantum effects do lead to a qualitative change in the structure of the moduli space. The  $N_f - N_c$  fourbranes stretched between the NS and NS' fivebranes are attracted to the  $N_f$  fourbranes stretched between the NS' fivebrane and the sixbranes (see section 4.4). Hence,  $N_f - N_c$  of the  $N_f$  fourbranes of the second kind align with the fourbranes of the first kind, and reconnect, giving rise to  $N_f - N_c$  fourbranes stretched between the NS fivebrane and  $N_f - N_c$  different sixbrane (in agreement with the s-rule). The remaining  $N_c$  fourbranes of the second kind give rise to the usual  $2N_c N_f - N_c^2$  moduli.

In gauge theory, the same conclusions follow from the fact that the classical magnetic superpotential (5.4) is corrected quantum mechanically to [21]:

$$W_{\text{quantum}} = M_j^i q_i \tilde{q}^j + \Lambda^{\frac{3N_c - N_f}{N_c - N_f}} (\det M)^{\frac{1}{N_f - N_c}} \quad (6.8)$$

Physically, the origin of the second term in (6.8) is the fact that when  $M$  gets an expectation value, the magnetic quarks become massive due to the classical coupling (5.4). If the rank of  $M$  is larger than  $N_c$ , the vacuum is destabilized by non-perturbative SQCD effects. This



reduces the dimension of moduli space to  $N_f^2 - (N_f - N_c)^2 = 2N_f N_c - N_c^2$ . The analysis of the superpotential (6.8) leads to the same conclusions [21].

Thus, quantum effects eliminate the discontinuous jump in the chiral ring of the magnetic theory as the FI D-term is tuned to zero. Quantum mechanically one does not have access to the full  $N_f^2$  dimensional classical moduli space, but to a  $2N_f N_c - N_c^2$  dimensional subspace thereof – precisely the subspace that connects smoothly to the electric moduli space via our construction!

It is now clear that the equivalence of the chiral rings of the electric and magnetic theories constructed above extends to the root of the Higgs branch (for vanishing D-term), since in the quantum theory there is no discontinuous jump in the structure of either the electric or magnetic moduli space there.

## 7. Theories with two adjoints

### 7.1. The brane configuration and its interpretation

In this section we will consider a configuration of  $k$  coincident NS fivebranes connected by  $N_c$  D fourbranes to  $k'$  coincident NS' fivebranes, with  $N_f$  D sixbranes located between the NS and NS' branes. The discussion of the previous sections corresponds to the case  $k = k' = 1$ .

The low energy theory on the fourbranes is in this case  $N = 1$  SYM with gauge group  $U(N_c)$ ,  $N_f$  fundamental flavors  $Q^i$ ,  $\tilde{Q}_i$ , and two adjoint superfields  $X$ ,  $X'$ . The classical superpotential is

$$W = \frac{s_0}{k+1} \text{Tr} X^{k+1} + \frac{s'_0}{k'+1} \text{Tr} X'^{k'+1} + \tilde{Q}^i X' Q_i \quad (7.1)$$

$X$  and  $X'$  can be thought of as describing fluctuations of the fourbranes in the  $(x^8, x^9)$  and  $(x^4, x^5)$  directions, respectively. They are massless, but the superpotential (7.1) implies that there is still a potential for the corresponding fluctuations, allowing only infinitesimal deviations from the vacuum at  $X = X' = 0$ . The couplings  $s_0$ ,  $s'_0$  should be thought of as very large:  $s_0, s'_0 \rightarrow \infty$ . This can be deduced *e.g.* on the basis of the transformation properties of (7.1) under the global symmetries  $U(1)_{45}$  and  $U(1)_{89}$ .

Indeed, in addition to the charge assignments for  $Q$ ,  $\tilde{Q}$ ,  $m$  and  $\theta$  discussed in section 5.1, the charges of the adjoint fields under  $U(1)_{45} \times U(1)_{89}$  are  $(0, 2)$  for  $X$  and  $(2, 0)$  for

$X'$ . Note that the adjoint field  $X'$  transforms in the same way as the quark mass  $m$ ; this is consistent with the fact that turning on an expectation value for  $X'$  changes  $m$ , as is clear from the form of the superpotential (7.1).

We see that with the above charge assignments for the fields, the last term in (7.1) is invariant under the global symmetry, while the first two are not. This means that the coefficients  $s_0, s'_0$  must be infinite. This is consistent with the case  $k = k' = 1$  considered in sections 5, 6. There, the statement is that the masses of  $X$  and  $X'$  are not of order  $1/L_6$  as one would naively expect, but rather infinite. That this must be the case follows from the following argument. If the mass of  $X'$ ,  $m'$ , was finite, by integrating  $X'$  out using (7.1) we would have obtained a superpotential of the form

$$W = \frac{1}{m'} \tilde{Q}_i Q^j \tilde{Q}_j Q^i \quad (7.2)$$

for the quarks, which would have lifted some or all of the flat directions of SQCD. The fact that we have found the full moduli space of SQCD in the brane construction of section 5 is further evidence that the mass in that case, or more generally  $s'_0$  in (7.1), is infinite.

One can discuss the case of finite mass for  $X'$  by rotating the NS' fivebrane in the  $(v, w)$  plane (as in [35]). The mass  $m'$  varies from zero when the brane lies along the  $v$  axis, to infinity when it lies along the  $w$  axis. In between, there is a non vanishing superpotential of the form (7.2) for the quarks, which modifies the moduli space of vacua rather significantly. One can repeat the counting leading to (5.1) for this case and find that the dimension of the Higgs branch is  $2N_f N_c - 2N_c^2$ . The counting (5.1) is modified because a) fourbranes stretched between a D sixbrane and the rotated NS' fivebrane are now rigid, and b) there is an analog of the s-rule for fourbranes stretched between a D sixbrane and the rotated NS' fivebrane. Thus the structure for any non zero rotation angle is very similar to that of the  $N = 2$  supersymmetric case discussed in [9].

The occurrence of infinite coefficients in the superpotential seems unsatisfactory. It is quite possible that the description of the theory given here is an effective one, arising after integrating out some additional fields, and the underlying theory with these fields included does not exhibit any such singularities. We will leave an elucidation of these issues for future work.

A calculation similar to that of section 6.1 leads to the conclusion that only one combination of the two  $U(1)$   $R$  symmetries described above survives in the quantum theory.

The bending of the NS and NS' fivebranes is given for arbitrary  $k$  and  $k'$  (and again for large  $v, w$ ) by:

$$\begin{aligned} s_5 &= R_{10} \left( N_f - \frac{N_c}{k} \right) \ln v \\ s_{5'} &= R_{10} \frac{N_c}{k'} \ln w \end{aligned} \tag{7.3}$$

and the unbroken symmetry is the combination of  $v \rightarrow e^{i\alpha}v$ ,  $w \rightarrow e^{i\beta}w$  which preserves  $s_5 - s_{5'}$ .

It is not difficult to compute the  $R$ -charge of  $Q$ ,  $X$  and  $X'$  under this symmetry. One finds that for generic  $k, k'$  the answer one gets is not the one expected from gauge theory (the gauge theory answer for  $k' = 1$  and arbitrary  $k$  appears in [22]; for  $k' > 1$  the gauge theory predicts that there is no unbroken  $R$  symmetry except at specific values of  $N_f/N_c$ ).

The origin of the discrepancy is clear. The gauge theory analysis assumes finite  $s_0, s'_0$  which fixes already classically the  $R$  charge of  $X$ ,  $X'$  and  $Q$ . The brane construction corresponds to infinite  $s_0, s'_0$ , in which case the corresponding classical gauge theory is invariant under *three*  $U(1)$  symmetries, only two of which are visible in the brane picture as rotations in ten dimensions. Therefore, in gauge theory there are in this case two non-anomalous  $U(1)_R$  symmetries, only one combination of which is seen in the brane picture.

## 7.2. Deformations and moduli space

The simplest way to see that the configuration of branes constructed in the previous subsection indeed describes a gauge theory with the stated matter content is to match the deformations of the brane configuration with those of the gauge theory (7.1). We start with the parameters corresponding to the locations of the NS and NS' fivebranes.

Initially, the  $k$  NS fivebranes reside at the same point in the  $(x^8, x^9)$  plane. Displacing them to  $k$  different points  $a_j = x_j^8 + ix_j^9$ ,  $j = 1, \dots, k$  gives rise to many possible configurations, labeled by a set of non-negative integers  $(r_1, \dots, r_k)$ , with  $\sum_j r_j = N_c$ . The integers  $r_j$  specify the number of fourbranes stretched between the  $j$ 'th NS fivebrane, located at  $a_j$ , and the NS' fivebranes which we still take to be coincident. Since the  $N_c$  fourbranes end on the NS fivebranes, their locations in  $(x^8, x^9)$  follow those of the NS branes. On the worldvolume of the fourbranes one can think of the locations  $\{a_j\}$  as determining the expectation value of the adjoint field  $X$  describing fluctuations of the fourbranes in the

$(x^8, x^9)$  directions. It is thus clear how the parameters  $a_j$  appear in the worldvolume gauge theory; there is a superpotential for the adjoint field  $X$  of the form:

$$W = \sum_{j=0}^k \frac{s_j}{k+1-j} \text{Tr} X^{k+1-j} \quad (7.4)$$

For generic  $\{s_j\}$  the superpotential has  $k$  distinct minima  $\{a_j\}$  related to the parameters in the superpotential via the relation:

$$W'(x) = \sum_{j=0}^k s_j x^{k-j} \equiv s_0 \prod_{j=1}^k (x - a_j) \quad (7.5)$$

Vacua are labeled by sequences of integers  $(r_1, \dots, r_k)$ , where  $r_l$  is the number of eigenvalues of the matrix  $X$  residing in the  $l$ 'th minimum of the potential  $V = |W'(x)|^2$ . Thus, the set of  $\{r_j\}$  and  $\{a_j\}$  determines the expectation value of the adjoint field  $X$ , in agreement with the brane picture. When all  $\{a_j\}$  are distinct, the adjoint field is massive and the gauge group is broken:

$$U(N_c) \rightarrow U(r_1) \times U(r_2) \times \dots \times U(r_k) \quad (7.6)$$

The theory splits in the infrared into  $k$  decoupled copies of SQCD with gauge groups  $\{U(r_i)\}$  and  $N_f$  flavors of quarks. The brane description makes this structure manifest.

The above discussion can be repeated for the parameters corresponding to the locations of the  $k'$  NS' fivebranes in the  $(x^4, x^5)$  directions. These  $k'$  complex numbers can be thought of as parametrizing the extrema of a polynomial superpotential in  $X'$  of order  $k' + 1$ , in complete analogy to (7.4), (7.5). The only new element is that when we displace the  $k'$  NS' fivebranes in the  $(x^4, x^5)$  directions leaving the  $N_f$  D sixbranes fixed, we make the quarks  $Q, \tilde{Q}$  massive, with masses of order  $\langle X' \rangle$ . This is the origin of the Yukawa coupling in the superpotential (the last term on the r.h.s. of (7.1)).

One can also consider situations where both NS and NS' fivebranes are displaced in the  $(8, 9)$  and  $(4, 5)$  directions, respectively. There are then  $k \times k'$  minima of the potential corresponding to all possible combinations of the  $k$  minima of the  $X$  superpotential and the  $k'$  minima of the  $X'$  superpotential. Vacua are labeled by integers  $r_{l,m}$  ( $l = 1, \dots, k$ ,  $m = 1, \dots, k'$ ) specifying the number of eigenvalues of  $X$  and  $X'$  that are in the  $l$ 'th minimum of the potential for  $X$  and in the  $m$ 'th minimum of the potential for  $X'$  (with  $Q = \tilde{Q} = 0$ ). Clearly,  $\sum_{l,m} r_{l,m} = N_c$ . This is again in agreement with the brane picture,

where  $r_{l,m}$  is the number of fourbranes stretched between the  $l$ 'th NS fivebrane and the  $m$ 'th NS' fivebrane.

Note that in analyzing the vacuum structure for  $k, k' > 1$  we have assumed that we can diagonalize the matrices  $X$  and  $X'$  simultaneously. The theory seems to inherit this property from the  $N = 4$  supersymmetric theory on infinite fourbranes. There it can be understood in gauge theory as arising from a superpotential of the form  $W = \text{Tr}(X''[X, X'])$ . Here, the field  $X''$  is infinitely massive and the mechanism for enforcing  $[X, X'] = 0$  is less clear.

As another check of the brane theory – gauge theory correspondence, consider the Higgs moduli space of the theory. In the brane language, the counting is the following. Suppose

$$N_c = km + l \tag{7.7}$$

where  $m, l$  are non-negative integers, and  $0 \leq l < k$ . As we have seen before, we have to study all possible ways to decompose the  $N_c$  fourbranes into segments by splitting them on the D sixbranes, taking into account the s-rule. Since there are now  $k$  NS fivebranes, the s-rule allows  $k$  D fourbranes to stretch between them and any given D sixbrane. Taking this into account and repeating the analysis done for SQCD above, we find that the dimension of moduli space is

$$k \sum_{j=1}^m [2(N_f - j) + 1] + l [2(N_f - m - 1) + 1] = 2N_f N_c - km^2 - l(2m + 1) \tag{7.8}$$

In particular, it is independent of  $k'$ .

To understand the counting directly in gauge theory it is convenient [22] to slightly deform the singularity (7.1) as in (7.4) (for both  $X$  and  $X'$ ). For generic couplings and choices of vacua (*i.e.* the occupation numbers  $r_{l,m}$  defined previously) the theory describes  $k \times k'$  decoupled SQCD systems, however, because of the  $\tilde{Q}X'Q$  coupling in (7.1) at most  $k$  of them contain massless quarks. Thus, the moduli space we are looking for is that of  $k$  decoupled SQCD systems with gauge groups  $U(r_j)$ ,  $j = 1, \dots, k$  and  $N_f$  flavors of quarks. We know from subsection 5.1 that the dimension of this moduli space is:

$$\sum_{j=1}^k (2N_f r_j - r_j^2) = 2N_f N_c - \sum_{j=1}^k r_j^2 \tag{7.9}$$

We should choose the degeneracies  $r_j$  to maximize (7.9). It is easy to see that the largest dimension is obtained by taking  $l$  of the  $r_j$  to be equal to  $m + 1$ , and the remaining  $k - l$  to be equal to  $m$  ( $l$  and  $m$  are defined by (7.7)). Substituting in (7.9), one finds that the dimension is equal to that deduced from the brane counting (7.8).

### 7.3. The magnetic theory and duality for $k > 1$ , $k' = 1$

The magnetic brane configuration consists in this case of an NS' fivebrane connected by  $\bar{N}_c = kN_f - N_c$  fourbranes to a cluster of  $k$  coincident NS fivebranes on its right, and by  $k$  fourbranes to each of  $N_f$  D sixbranes on its left. The low energy theory on the fourbranes stretched between fivebranes is a  $U(\bar{N}_c)$  gauge theory with an adjoint superfield  $Y$  coming from 4 – 4 strings connecting fourbranes of the first kind,  $N_f$  fundamental flavors of quarks  $q_i, \tilde{q}^i$  ( $i = 1, \dots, N_f$ ) coming from 4 – 4 strings connecting fourbranes of the first kind to those of the second kind, and  $k$  magnetic meson fields  $M_j$  ( $j = 1, \dots, k$ ), each of which is an  $N_f \times N_f$  matrix, coming from 4 – 4 strings connecting fourbranes of the second kind<sup>11</sup>.

This theory has been shown in [22] to be dual to the electric theory described above. Its superpotential is

$$W_m = \frac{\bar{s}_0}{k+1} \text{Tr} Y^{k+1} + \sum_{j=1}^k M_j \tilde{q} Y^{k-j} q \quad (7.10)$$

where  $\bar{s}_0 = -s_0$ , and we have set a scale to one. The analysis of deformations is similar to that described for  $k = 1$  in section 5.2 and will not be repeated for this case. The classical moduli space is  $kN_f^2$  dimensional. In the gauge theory it corresponds to setting  $q = \tilde{q} = Y = 0$  and turning on arbitrary expectation values for the singlets  $M_j$ . In the brane configuration, since there are  $k$  fourbranes connecting the NS' fivebrane to each of the  $N_f$  sixbranes, one simply gets  $k$  copies of the moduli space of the magnetic theory described in section 5.2.

It is easy to generalize the discussion of duality in sections 5, 6 to  $k > 1$ . By turning on a relative separation between the  $k$  NS fivebranes and the NS' fivebrane, which again corresponds to a FI D-term for  $U(1) \subset U(N_c)$ , one enters the Higgs phase in both the electric and magnetic theories, with the gauge group completely broken. The electric and magnetic Higgs branches, both of whose dimensions are given by (7.8), are then seen to parametrize the same space; as in section 5, they are related by a smooth change in the scale of the theory, and are therefore identified. Duality relates the theory with  $N_c$  colors to one with  $kN_f - N_c$  colors essentially because the electric configuration contains  $k$  fivebranes which are not parallel to the D sixbranes.

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<sup>11</sup> There seem to be  $kN_f \times kN_f$  strings of this sort at the origin of moduli space, corresponding to  $(kN_f)^2$  massless fields, but this is misleading. At a generic point in moduli space we will see momentarily that there are  $kN_f^2$  massless meson fields  $M_j$ .

As one approaches the origin of the Higgs branch by switching off the D-term, classical physics is smooth in the electric theory, while in the magnetic one the dimension of moduli space jumps from (7.8) to  $kN_f^2$ . A straightforward extension of the analysis of section 6 reveals that quantum mechanically, due to the attraction between fourbranes of the first and second kinds one does not have access to the full  $kN_f^2$  dimensional classical moduli space, but only to the subspace whose dimension is given by (7.8) that connects smoothly to the electric description. This establishes the equivalence between the two theories.

#### 7.4. The magnetic theory and duality for $k = 1, k' > 1$

The magnetic brane configuration consists in this case of  $k'$  NS' fivebranes connected by  $\bar{N}_c = N_f - N_c$  fourbranes to an NS fivebrane on their right, and by a single fourbrane to each of  $N_f$  D sixbranes on their left. The low energy theory on the fourbranes stretched between fivebranes has a  $U(\bar{N}_c)$  gauge group and an adjoint superfield  $Y'$  coming from 4 – 4 strings connecting fourbranes of the first kind, and  $N_f$  fundamental flavors of quarks  $q_i, \tilde{q}^i$  ( $i = 1, \dots, N_f$ ) coming from 4 – 4 strings connecting fourbranes of the first kind to those of the second kind.

The adjoint field and the quarks are further coupled to a single magnetic meson field  $M$ , which is an  $N_f \times N_f$  matrix, coming from 4 – 4 strings connecting fourbranes of the second kind. The superpotential is:

$$W_m = \frac{\bar{s}'_0}{k' + 1} \text{Tr} Y'^{k'+1} + \tilde{q} Y' q + M \tilde{q} q \quad (7.11)$$

The discussion of moduli space of vacua and deformations of this theory is similar to previous cases and will be skipped. Its relation to the electric theory of section 7.1 is an essentially known result in gauge theory. It follows from the duality of [22] by turning on a particular perturbation<sup>12</sup>; for completeness we review the gauge theory derivation in appendix A. There is one slight subtlety in the comparison to gauge theory, which we discuss in the Appendix, having to do with the question of whether  $s'_0$  is finite or not. For finite  $s'_0$ , gauge theory predicts that all magnetic mesons in the dual  $U(N_f - N_c)$  gauge theory are massive. As  $s'_0 \rightarrow \infty$  we show in the Appendix that one magnetic meson comes down to zero mass, in agreement with the brane picture.

As explained previously, we can study theories with finite  $s'_0$ , by rotating the stack of NS' fivebranes in the  $(v, w)$  plane. The picture that arises is in complete agreement with

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<sup>12</sup> For  $k' = 2$  this perturbation has been analyzed in gauge theory in [40].

the gauge theory discussion. If the NS' fivebranes are not parallel to the D sixbranes, the fourbranes of the second kind that stretch between them are rigid, and hence the magnetic meson  $M$  becomes massive. In that case, the duality one finds is just that familiar from gauge theory.

### 7.5. The magnetic theory and duality for $k, k' > 1$

In the general case, the magnetic brane configuration consists of  $k'$  NS' fivebranes connected by  $\bar{N}_c = kN_f - N_c$  fourbranes to  $k$  NS fivebranes on their right, and by  $k$  fourbranes to each of  $N_f$  D sixbranes on their left. The low energy theory on the fourbranes stretched between fivebranes has a  $U(\bar{N}_c)$  gauge group and two adjoint superfields  $Y, Y'$  coming from 4 – 4 strings connecting fourbranes of the first kind,  $N_f$  fundamental flavors of quarks  $q_i, \tilde{q}^i$  ( $i = 1, \dots, N_f$ ) coming from 4 – 4 strings connecting fourbranes of the first kind to those of the second kind, and  $k$  magnetic meson fields  $M_j$  ( $j = 1, \dots, k$ ), each of which is an  $N_f \times N_f$  matrix, coming from 4 – 4 strings connecting fourbranes of the second kind. The superpotential is:

$$W_m = \frac{\bar{s}_0}{k+1} \text{Tr} Y^{k+1} + \frac{\bar{s}'_0}{k'+1} \text{Tr} Y'^{k'+1} + \tilde{q} Y' q + \sum_{j=1}^k M_j \tilde{q} Y^{k-j} q \quad (7.12)$$

In addition, there is a constraint that enforces  $[Y, Y'] = 0$ , which as in the electric theory is not fully understood.

This duality with two adjoint fields is not known in gauge theory. It would clearly be interesting to complete its analysis; this is left for future work.

## 8. Theories with $SO$ and $Sp$ groups: I

In this and the next section we briefly discuss the generalization of the results of sections 5 – 7 to other classical groups. We start in this section by discussing a configuration of branes near an orientifold fourplane (which we will denote by O4); in the next section we will describe configurations including an orientifold sixplane (O6). Both constructions give rise to orthogonal and symplectic groups, depending on the choice of the orientation projection; in general they differ in the field content. Theories with  $SO$  and  $Sp$  gauge groups have been also studied recently in [19,20,41].



### 8.1. $SO(N_c)$

The electric theory corresponds to the following brane configuration. There is an orientifold plane O4 with a single NS' fivebrane<sup>13</sup> which is stuck at the orientifold (it cannot move in the  $(x^4, x^5, x^7)$  directions because it does not have an O4-mirror partner). To the left of the NS' fivebrane in the  $x^6$  direction there are  $k$  NS fivebranes and their  $k$  mirror images, as well as a single NS fivebrane which does not have a mirror and, therefore, is stuck at the orientifold. The NS fivebranes are connected to the NS' fivebrane by a total of

$$N_c = r_0 + 2 \sum_{j=1}^k r_j \quad (8.1)$$

D fourbranes;  $r_0$  fourbranes end on the single NS fivebrane which is stuck at the orientifold.  $r_j$  fourbranes end on the  $j$ 'th NS fivebrane,  $j = 1, \dots, k$ ; their  $r_j$  mirrors end on the O4-mirror partners of the NS fivebranes. We also place  $N_f$  D sixbranes and their  $N_f$  O4-mirror partners between the NS and NS' fivebranes.

The worldvolume dynamics on the D fourbranes describes at long distances a four dimensional  $N = 1$  supersymmetric gauge theory with gauge group  $SO(N_c)$ ,  $2N_f$  quarks  $Q^i$  in the vector representation, and a field  $X$  in the adjoint representation. The superpotential is

$$W_e = \sum_{j=0}^k \frac{s_{2j}}{2(k+1-j)} \text{Tr} X^{2(k+1-j)} \quad (8.2)$$

As in section 7, all the couplings  $\{s_{2j}\}$  should be thought of as tending uniformly to infinity. Note that:

- 1) The addition of the O4 plane does not change the number of unbroken supercharges. Therefore, our configuration describes at low energies a four dimensional  $N = 1$  supersymmetric gauge theory.
- 2) There are  $\frac{1}{2}N_c(N_c - 1)$  open string sectors connecting different D fourbranes and their mirrors. These correspond to the  $\dim SO(N_c) = \frac{1}{2}N_c(N_c - 1)$  vector multiplets on the worldvolume of the D fourbranes.
- 3) The 4 - 6 strings connecting the  $N_f$  D sixbranes and their  $N_f$  mirrors to the  $N_c$  D fourbranes describe  $2N_f$  chiral multiplets in the fundamental representation of

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<sup>13</sup> As in section 7, one can study the generalization to  $k' > 1$  NS' fivebranes; we will not consider it here.

$SO(N_c)$ . The  $Sp(N_f)$  gauge symmetry on the D sixbranes corresponds to an  $Sp(N_f)$  global symmetry of the low energy theory on the worldvolume of the D fourbranes<sup>14</sup>.

- 4) As before, a fourbrane stretched between the NS and NS' fivebranes can break on D sixbranes into pieces with a relative splitting in the  $(x^6, x^7, x^8, x^9)$  directions. This corresponds to turning on Higgs expectation values for the quarks. The dimension of the Higgs moduli space can be obtained, as in section 5.1, by counting the number of all possible breakings. For  $2k + 1 = 1$  and  $N_c = 2r$ , the result is

$$\sum_{l=1}^r \{2[2N_f - (2l - 1)] + 1\} = 2N_f(2r) - r(2r - 1) \quad (8.3)$$

For  $N_c = 2r + 1$  the result is

$$\sum_{l=1}^r \{2[2N_f - (2l - 1)] + 1\} + 2(N_f - r) = 2N_f(2r + 1) - r(2r + 1) \quad (8.4)$$

In (8.3), (8.4) we use the fact that there are two massless complex scalars parametrizing fluctuations of a D fourbrane stretched between two D sixbranes as well as between a D sixbrane and its mirror. Moreover, the s-rule allows a D fourbrane and its mirror to connect the NS fivebrane only with a single D sixbrane and its mirror. The extra term in (8.4) is due to the D fourbrane which does not have a mirror partner when  $N_c$  is odd. Equations (8.3), (8.4) are in agreement with the gauge theory analysis.

- 5) As before, relative positions of the NS and NS' fivebranes in the  $x^7$  direction play the role of FI D-terms in the gauge theory on the fourbranes.
- 6) For generic values of the couplings  $\{s_{2j}\}$  in (8.2), the bosonic potential  $V \sim |W'|^2$  has  $2k + 1$  minima: one at the origin, and  $k$  paired minima at  $\{\pm a_j\}$ :

$$W'(x) = s_0 x \prod_{j=1}^k (x^2 - a_j^2) \quad (8.5)$$

If  $r_0$  eigenvalues of  $X$  sit at zero, and  $r_j$  eigenvalues sit at  $\pm a_j$ , the gauge symmetry is spontaneously broken:

$$SO(N_c) \rightarrow SO(r_0) \times U(r_1) \times \cdots \times U(r_k) \quad (8.6)$$

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<sup>14</sup> The maximal flavor symmetry of  $SO(N_c)$  with  $2N_f$  quarks is  $SU(2N_f)$ ; here we only see an  $Sp(N_f)$  subgroup. This is similar to the  $U(N_c)$  case, where the flavor symmetry seen by the brane configuration is the diagonal  $SU(N_f)$  subgroup of the  $SU(N_f) \times SU(N_f)$  maximal global symmetry, as discussed in section 5.

The only massless matter near the minimum at the origin is  $2N_f$  fundamentals of an  $SO(r_0)$  gauge group; the non zero minima  $\{a_j\}$  correspond to SQCD with gauge group  $U(r_j)$  and  $N_f$  quarks in the fundamental representation. Thus, for generic  $\{a_j\}$ , the model describes at low energies  $1 + k$  decoupled supersymmetric QCD theories with gauge groups  $SO(r_0)$  and  $U(r_j)$ , respectively (8.6). In the brane description,  $\{a_j\}$  correspond to locations in the  $(x^8, x^9)$  plane of the  $k$  NS fivebranes that are free to leave the orientifold plane. The parameters  $\{r_j\}$  correspond to the number of fourbranes attached to the different NS fivebranes (see (8.1)). Separations of the  $k$  fivebranes in  $(x^6, x^7)$  correspond to changing the couplings and FI D-terms in the  $U(r_j)$  factors in (8.6). The rest of the analysis follows closely that of section 7.

- 7) The distance in the  $(x^4, x^5)$  directions between the NS' fivebrane and the  $N_f$  D sixbranes determines the masses of the  $2N_f$  chiral multiplets. Due to the O4-mirror reflection in the  $(x^4, x^5)$  directions, the same mass is given to pairs of quarks: a quark and its mirror image.
- 8) *R-charge*: the inclusion of an orientifold four plane does not change the discussion of the classical  $R$ -symmetries presented in section 5. However, to consider the nonanomalous  $R$  symmetry of quarks, we need to discuss charges of branes and orientifolds. An orientifold fourplane has  $-1/2$  times the charge of a physical D fourbrane (see section 2). Therefore, if we normalize the charge of a D fourbrane and its mirror partner to be  $+1$  (so that the total charge of a physical D fourbrane is  $+2$ ), the charge of an O4 is  $-1$ . When one places NS and/or NS' fivebranes on top of the orientifold, the O4 charge flips sign each time it crosses an NS or an NS' brane in the  $x^6$  direction [41]. Therefore, it looks on the fivebrane as a source of charge  $-2$ , and hence it contributes to the bending discussed in section 6.1. Repeating the analysis there one finds that the location of the NS fivebrane in the  $s$  plane is described by

$$s_5 = R_{10}(2N_f - N_c + 2) \ln v \tag{8.7}$$

while the location of the NS' fivebrane is described by

$$s_{5'} = R_{10}(N_c - 2) \ln w \tag{8.8}$$

The unbroken  $R$ -symmetry is the one under which

$$s_5 - s_{5'} = R_{10} \ln (w^{-N_c+2} v^{2N_f-N_c+2}) \tag{8.9}$$

is invariant. It is not difficult to check that the  $R$  charge of  $Q$  under this symmetry is  $B_f = 1 - (N_c - 2)/2N_f$ , in agreement with the gauge theory answer<sup>15</sup> [3,4].

This concludes the discussion of the electric theory and its description in terms of brane dynamics and gauge theory.

To find the magnetic theory we follow the discussion of section 5.3. Consider first the case  $2k + 1 = 1$ . The single NS fivebrane is stuck on the orientifold, and can no longer avoid crossing the NS' fivebrane when it approaches it in  $x^6$ . This is potentially problematic since to demonstrate Seiberg's duality on the classical level we want, as in section 5.3, to embed the electric and magnetic theories in a single smooth moduli space of vacua.

We propose that the lesson one should draw from the analysis of section 5 is the following. The process of moving an NS fivebrane through an NS' fivebrane is smooth if and only if the “linking number” on each fivebrane is conserved, without reconnecting branes. This linking number [9] is defined in the presence of O4 planes of charge  $Q(O4)$  by:

$$l_{NS} = \frac{1}{2}(R_{D6} - L_{D6}) + (L_{D4} - R_{D4}) + Q(O4)(L_{O4} - R_{O4}) \quad (8.10)$$

where  $L_{D6}$  ( $L_{O4}$ ) [ $L_{D4}$ ] is the number of D sixbranes (O4 planes) [D fourbranes] to the left of the NS fivebrane and, similarly,  $R_{D6}$  ( $R_{O4}$ ) [ $R_{D4}$ ] is the number of D sixbranes (O4 planes) [D fourbranes] to its right. When the NS fivebrane is to the left of the NS' fivebrane it sees an O4 plane of charge +1 on its left and an O4 plane of charge  $-1$  on its right, therefore, the contribution of the O4 plane to the linking number is +2. Similarly, the contribution of the O4 plane to the linking number of the NS' fivebrane is  $-2$ .

To allow a smooth transition one should neutralize this difference, by putting two of the  $N_c$  D fourbranes on top of the orientifold plane. In addition, one must break the remaining  $N_c - 2$  fourbranes on D sixbranes, and enter the Higgs branch. In this situation, there is no obstruction to smoothly vary the separation between the two fivebranes in  $x^6$ , and one can continue the discussion as in section 5.

The magnetic theory, obtained by smoothly passing the NS fivebrane through the NS' fivebrane and approaching the root of the Higgs branch, is an  $SO(2N_f - N_c + 4)$  gauge

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<sup>15</sup> In the context of  $N = 2$  supersymmetric brane configurations, equation (8.7) can be used to generalize the brane calculation of [36] of the  $\beta$  function of  $N = 2$  SQCD to the other classical groups.

theory with  $2N_f$  quarks and a magnetic meson field  $M$  representing the electric bilinears  $M = QQ$ , coupled to the magnetic quarks via a Yukawa interaction.

For general  $k$  the analysis is similar. One can either repeat the previous discussion  $2k + 1$  times<sup>16</sup>, or remove  $k$  pairs of NS fivebranes from the orientifold plane (breaking the gauge symmetry as in (8.6)), and repeat the discussion of section 5.

After all the  $2k + 1$  fivebranes cross to the other side of the NS' fivebrane we obtain the magnetic brane configuration, in which the  $2k + 1$  NS fivebranes are connected to the NS' fivebrane by  $(2k + 1)2N_f - N_c + 4$  fourbranes, and the NS' fivebrane is further connected by  $(2k + 1) \times 2N_f$  D fourbranes to the  $N_f$  D sixbranes and their  $N_f$  mirrors. This is the magnetic description [26] of the original theory:

- 1) The gauge group on the fourbranes worldvolume is  $G_m = SO(\bar{N}_c)$ , where

$$\bar{N}_c = (2k + 1)2N_f - N_c + 4 \quad (8.11)$$

- 2) The fourbranes connecting the D sixbranes to the NS' fivebrane give rise to magnetic meson fields  $(M_j)_{fg}$ ,  $j = 0, \dots, 2k$ ,  $f, g = 1, \dots, 2N_f$ , coming from 4 - 4 string configurations invariant under the orientifold projection.  $M_j$  is in the  $N_f(2N_f - 1)$  ( $N_f(2N_f + 1)$ ) dimensional antisymmetric (adjoint) representation of the  $Sp(N_f)$  flavor symmetry for odd (even)  $j$ .
- 3) As in the electric theory, the locations of the NS fivebranes in the magnetic theory are encoded in a polynomial magnetic superpotential for an adjoint superfield  $Y$ , corresponding to O4-invariant 4 - 4 strings describing fluctuations of the fourbranes in the  $(x^8, x^9)$  directions.
- 4) The couplings of the magnetic mesons  $M_j$  to the magnetic quarks  $q$  and to the magnetic adjoint  $Y$  are as expected. The full magnetic superpotential is:

$$W_m = \sum_{j=0}^k \frac{\bar{s}_{2j}}{2(k+1-j)} \text{Tr} Y^{2(k+1-j)} + \sum_{j=0}^{2k} M_j q Y^{2k-j} q \quad (8.12)$$

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<sup>16</sup> Noting that each time an NS fivebrane crosses the NS' fivebrane the magnetic charge between the remaining NS fivebranes and the NS' fivebrane flips sign.

## 8.2. $Sp(N_c)$

To obtain an  $Sp(N_c)$  gauge group instead of an  $SO(N_c)$  one, all we have to do is to change the sign of the orientifold charge<sup>17</sup> [19,20,41]. Symmetric O4-projections are interchanged with antisymmetric O4-projections and, therefore, in particular, we are forced to place a total even number of D fourbranes:  $N_c$  D fourbranes and their  $N_c$  O4-mirrors. In the presence of  $N_f$  D sixbranes and their  $N_f$  mirrors<sup>18</sup>, and with  $2k + 1$  NS fivebranes, the electric theory is an  $N = 1$  supersymmetric  $Sp(N_c)$  gauge theory with  $2N_f$  chiral multiplets in the fundamental representation, and a field  $X$  in the adjoint representation of  $Sp(N_c)$  with a superpotential (8.2). To obtain the magnetic theory, we follow similar steps to subsection 8.1, and obtain the dual theory with superpotential (8.12). Here we shall present only some of the main differences between the  $Sp$  and  $SO$  projections:

- 1) The dimension of the Higgs moduli space is obtained by counting all possible breakings of D fourbranes on D sixbranes. For  $2k + 1 = 1$  the dimension is:

$$\sum_{l=1}^{N_c} \{2[2N_f - 2l] + 1\} = 2(2N_f)N_c - N_c(2N_c + 1) \quad (8.13)$$

The differences between equations (8.13) and (8.3) are that here the total number of D fourbranes and their O4-mirrors is  $2N_c$ , thus  $r$  in (8.3) is being replaced by  $N_c$ , and the  $2l - 1$  in (8.3) is replaced by  $2l$  because the antisymmetric projection eliminates one combination of the motions of fourbranes stretched between sixbranes.

- 2) For generic values of the  $\{a_j\}$  in (8.5), the bosonic potential  $V \sim |W'|^2$  has  $2k + 1$  distinct minima: one at the origin, and  $k$  pairs at  $\pm a_j$ . If  $2r_0$  eigenvalues of  $X$  sit at the origin, and  $r_j$  sit at  $\pm a_j$ , the gauge symmetry is spontaneously broken:

$$Sp(N_c) \rightarrow Sp(r_0) \times U(r_1) \times \cdots \times U(r_k) \quad (8.14)$$

---

<sup>17</sup> A simple way to see that D fourbranes near an O4 plane can give rise to both orthogonal and symplectic gauge groups is to consider two fivebranes attached to an O4 plane stretched around a circle of finite radius in the  $x^6$  direction.  $2N_c$  fourbranes are stretched between the two NS fivebranes on one side, and  $2N_f$  are stretched on the other. This configuration preserves  $N = 2$  SUSY; if on the  $2N_c$  fourbranes of the first kind the gauge symmetry is  $SO(2N_c)$ ,  $N = 2$  SUSY requires that the global symmetry due to the presence of  $N_f$  fundamental hypermultiples is  $SP(N_f)$ . But this global symmetry is the gauge symmetry on the fourbranes of the second kind. This implies that the charge of the O4 plane flips sign when one passes through an NS (or NS') fivebrane. A similar sign ambiguity appears for the O6 plane, to be discussed in section 9; it can be seen by a simple generalization of the above construction.

<sup>18</sup> An odd number of sixbranes leads to a theory with a global anomaly.

This is clearly seen in the brane description.

- 3) *R-symmetry*: following the discussion in sections 6.1, 8.1, we find that quantum mechanically, the location of the NS fivebrane in the  $s$  plane is described by

$$s_5 = 2R_{10}(N_f - N_c - 1) \ln v \quad (8.15)$$

while the location of the NS' fivebrane in the  $s$  plane is described by

$$s_{5'} = 2R_{10}(N_c + 1) \ln w \quad (8.16)$$

The unbroken  $R$ -symmetry is the one under which

$$s_5 - s_{5'} = -2R_{10} \ln (w^{N_c+1} v^{N_c+1-N_f}) \quad (8.17)$$

is invariant. The  $R$  charge of  $Q$  is  $B_f = 1 - (N_c + 1)/N_f$ , in agreement with the gauge theory answer [26].

- 4) The single NS brane, which is stuck on the orientifold, can only cross the NS' brane smoothly if we neutralize the charge difference between both sides of the NS brane and the NS' brane. In the  $Sp$  case, the charge of the part of the orientifold between the NS and NS' branes is  $+1$ , while that outside is  $-1$ . To compensate the difference one can do one of two things: a) modify the electric brane configuration, by replacing two physical D sixbranes by two pairs of semi-infinite fourbranes, one stretching to the right of the NS' fivebrane, the other to the left of the NS fivebrane. The charge everywhere along the orientifold is now  $+1$ , and the discussion proceeds as above; b) create a pair of a D fourbrane and a D anti-fourbrane together with their mirror partners. Now, the two anti-fourbranes can neutralize the charge difference along the orientifold. After the NS branes cross the NS' brane, the anti-fourbranes annihilate with D fourbranes, leaving a supersymmetric configuration. One may regard such anti-fourbranes as virtual.

The magnetic theory one finds in this case is an  $Sp(\bar{N}_c)$  gauge theory with

$$\bar{N}_c = (2k + 1)N_f - N_c - 2 \quad (8.18)$$

in agreement with [26]. The matter content and interactions arising from the brane construction agree with the field theory studies.

## 9. Theories with $Sp$ and $SO$ groups: II

$SO$  and  $Sp$  gauge groups with somewhat different matter content can be constructed using an orientifold sixplane<sup>19</sup> (O6). Consider an electric brane configuration, containing an O6 plane with an NS' fivebrane embedded in it. To the left of the O6 plane in the  $x^6$  direction, there are  $k$  NS fivebranes connected to the NS' fivebrane by  $N_c$  D fourbranes. We also place  $N_f$  D sixbranes between the O6 plane and the stack of NS fivebranes.

All branes except the NS' fivebrane have ‘‘O6-mirrors’’. In particular, the  $k$  NS fivebranes have  $k$  O6-mirror images which can leave the orientifold plane in pairs. The NS' fivebrane cannot be removed from the O6 (and as in section 8, we will not consider configurations with more than one NS' fivebrane).

The fourbrane worldvolume dynamics describes at long distances an  $Sp(N_c)$  supersymmetric gauge theory with  $2N_f$  chiral multiplets in the fundamental representation and an antisymmetric tensor  $X$  in the  $N_c(2N_c - 1)$  dimensional representation of  $Sp(N_c)$ , with a superpotential

$$W_e = \sum_{j=0}^k \frac{s_j}{(k+1-j)} \text{Tr} X^{(k+1-j)} \quad (9.1)$$

As in section 8, one notes that:

- 1) The orientifold O6 does not break any additional supercharges. Therefore, the configuration has four unbroken supercharges, and it describes at low energies a four dimensional  $N = 1$  supersymmetric gauge theory.
- 2) There are  $N_c^2$  4 – 4 string sectors corresponding to gauge multiplets of the  $U(N_c)$  subgroup of  $Sp(N_c)$ . In addition, there are  $N_c(N_c + 1)$  sectors, connecting a fourbrane on one side of the orientifold with a fourbrane on the other side of the orientifold, which are invariant under the  $Z_2$  orientifold projection. Altogether, there are  $N_c(2N_c + 1)$  O6-invariant 4 – 4 string configurations, corresponding to the  $\dim Sp(N_c) = N_c(2N_c + 1)$  gauge multiplets on the worldvolume of the D fourbranes; the gauge group is  $Sp(N_c)$ .
- 3) The 4 – 6 strings stretched between the  $N_f$  D sixbranes and the  $N_c$  D fourbranes describe  $2N_f$  chiral multiplets in the fundamental representation of  $Sp(N_c)$ . The  $SO(2N_f)$  gauge symmetry on the D sixbranes corresponds to an  $SO(2N_f)$  global symmetry of the low energy theory on the worldvolume of the D fourbranes.

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<sup>19</sup> It is interesting that pure SYM theories with gauge groups  $Sp(N_c)$  and  $SO(N_c)$  may be obtained from geometrically different brane configurations, *e.g.* ones with O4 or O6 planes. This may be interpreted as an infrared duality in the space of brane theories.



- 4) As before, it is possible for a fourbrane stretched between the NS and NS' fivebranes to break on D sixbranes into pieces with a relative splitting in the  $(x^6, x^7, x^8, x^9)$  directions. This corresponds to turning on Higgs expectation values for the quarks. To obtain the whole Higgs moduli space, we need to locate the  $N_f$  D sixbranes and their O6-mirrors at the orientifold plane. Now, the counting of all possible breakings gives for  $k = 1$ :

$$\sum_{l=1}^{N_c} [4(N_f - l) + 1] = 4N_f N_c - N_c(2N_c + 1) \quad (9.2)$$

The result (9.2) agrees with the counting in gauge theory.

- 5) One can again describe the deformations of the superpotential (9.1),  $\{s_j\}$ , by displacing NS fivebranes in the  $(x^8, x^9)$  plane. For generic values of the  $\{s_j\}$  the gauge symmetry is spontaneously broken:

$$Sp(N_c) \rightarrow Sp(r_1) \times Sp(r_2) \times \cdots \times Sp(r_k) \quad (9.3)$$

and the theory corresponding to the  $j$ 'th minimum is SQCD with gauge group  $Sp(r_j)$  and  $2N_f$  fundamentals.

The magnetic theory is obtained by going into the Higgs phase so that no fourbranes connect the NS fivebrane to the NS' fivebrane, and then moving the NS fivebrane through the NS' fivebrane in the  $x^6$  direction. The only new element in the discussion is the fact that as the NS fivebrane passes through the NS' fivebrane it also meets the O6 plane. This process is smooth: as we saw in section 2 the O6 plane behaves like a D sixbrane; its charge is  $-2$  times the charge of a physical sixbrane.

Thus it is reasonable to expect that when the NS fivebrane passes the O6 plane with an NS' fivebrane embedded in it, two of the fourbranes connecting it to the D sixbranes reconnect to the NS' fivebrane and stay on the orientifold. This is consistent with conservation of the “linking number” of [9] which reads in the presence of orientifolds:

$$l_{NS} = \frac{1}{2}(R_{D6} - L_{D6}) + (L_{O6} - R_{O6}) + (L_{D4} - R_{D4}) \quad (9.4)$$

where  $L_{D6}$  ( $L_{O6}$ ) [ $L_{D4}$ ] is the number of D sixbranes (orientifolds O6) [D fourbranes] to the left of the NS fivebrane and, similarly,  $R_{D6}$  ( $R_{O6}$ ) [ $R_{D4}$ ] is the number of D sixbranes (orientifolds O6) [D fourbranes] to its right.

An alternative description of the process of moving an NS fivebrane through an orientifold which leads to the same conclusion is to say that since the O6 plane has the opposite

charge to two D sixbranes, when an NS brane crosses an O6 plane, two D anti-fourbranes are generated connecting the NS fivebrane with the NS' fivebrane embedded in the orientifold six plane. After the NS brane passes the orientifold, these anti-fourbranes annihilate two fourbranes stretched between the NS and NS' branes.

After all the  $k$  fivebranes cross to the other side of the orientifold we obtain the following final brane configuration. The  $k$  NS fivebranes are connected to the NS' fivebrane by  $k(N_f - 2) - N_c$  fourbranes. The NS' fivebrane is further connected by  $k \times N_f$  D fourbranes to the  $N_f$  D sixbranes. This is the magnetic description [25] of the original theory:

- 1) The gauge group on the worldvolume of the fourbranes is  $G_m = Sp(\bar{N}_c)$ , where

$$\bar{N}_c = k(N_f - 2) - N_c \quad (9.5)$$

- 2) There are  $k \times N_f(2N_f - 1)$  magnetic mesons  $(M_j)_{fg}$ ,  $j = 1, \dots, k$ ,  $f, g = 1, \dots, 2N_f$ , coming as usual from open strings connecting fourbranes of the second kind.
- 3) The couplings of the magnetic mesons  $M$  to the magnetic quarks  $q$  and to the magnetic antisymmetric tensor  $Y$  are via the magnetic superpotential:

$$W_m = \frac{\bar{s}_0}{k+1} \text{Tr} Y^{k+1} + \sum_{j=1}^k M_j q Y^{k-j} q \quad (9.6)$$

Finally, if one flips the charge of the O6 plane one finds an electric theory corresponding to an  $SO(2N_c)$  gauge theory with  $2N_f$  quarks in the fundamental representation, and a symmetric tensor  $X$  with a superpotential (9.1). Duality leads to a magnetic  $SO(2\bar{N}_c)$  gauge theory, with

$$\bar{N}_c = k(N_f + 2) - N_c \quad (9.7)$$

in agreement with the field theory results [25].

## 10. Comments

### 10.1. Quantum brane interactions

Gauge dynamics and its realization in terms of configurations of branes leads one to deduce that branes that stretch between various other branes in general interact with each other quantum mechanically. In section 4.4 we have given a qualitative description of these interactions, and noted that they are repulsive for branes ending on the same

side of an NS' fivebrane and attractive otherwise. One can think of branes ending from the left (right) on an NS' fivebrane as corresponding to positive (negative) charges. It would be interesting to understand the precise form of the interaction between branes by generalizing the results of [36] to systems with four supercharges, perhaps as a kind of long range Coulomb interaction.

### 10.2. Seiberg's duality

In gauge theory one distinguishes between two notions of  $N = 1$  duality. The weaker version is the statement that members of a dual pair share the same quantum chiral ring and moduli space of vacua, as a function of all possible deformations. In Seiberg's original work [21] this statement has been proven for supersymmetric QCD.

In generalizations of Seiberg's work (*e.g.* [22-27]) it is often difficult to prove directly in gauge theory that the full chiral rings and moduli spaces of proposed duals agree, since to do that it is necessary to understand the structure of the classical moduli space, as well as the full non-perturbative superpotential on it, both which are in general difficult tasks.

The stronger, conjectured, version of duality asserts that the full infrared limits of the electric and magnetic theories coincide. The status of the strong duality conjecture in field theory is not clear. In particular, in general the chiral ring does not specify the full infrared conformal field theory.

On a more qualitative level, in field theory it is not clear *why* the moduli spaces and chiral rings of members of a dual pair agree and, consequently, constructing duals of complicated theories is generally prohibitively complicated. The general structure seems quite mysterious.

Embedding the problem in string theory sheds light on some of the above issues. Duality is translated into a statement in brane mechanics. It is likely that string theory and brane dynamics serve as a guiding principle for all gauge theory dualities.

First, it leads to a derivation and better understanding [13] of "classical  $N = 1$  duality", which is a consequence of quantum duality in situations where the non-abelian gauge symmetry is broken and quantum effects are weak in the infrared.

One finds that the classical moduli spaces of both the electric and magnetic theories are embedded in (and provide different parametrizations of) a single moduli space of string vacua. This makes the relation between the two moduli spaces and chiral rings manifest. Furthermore, since both electric and magnetic gauge groups are broken, there is no strong infrared dynamics, and the above equivalence is not corrected quantum mechanically.

Near the origin of moduli space, the classical analysis of the space of brane configurations (or, equivalently, moduli space of vacua of the low energy gauge theories) leads to different structures on the electric and magnetic sides. However, in this situation strong quantum infrared dynamics has to be taken into account. In gauge theory it leads to a dynamically generated superpotential which lifts part of the classical moduli space. In string theory, quantum interactions between branes achieve the same. After these quantum effects have been taken into account, the electric and magnetic moduli spaces, and hence also the quantum chiral rings, coincide.

The study of field theory duality in terms of branes is often more tractable than the corresponding gauge theory analysis. The calculation of quantum superpotentials is replaced by a few universal rules describing brane interactions. This provides a powerful tool for analyzing the quantum moduli space of vacua.

The brane construction provides a proof of the fact that the quantum electric and magnetic moduli spaces and chiral rings agree. It is natural to ask whether it also shows the equivalence of the full infrared CFT's at the origin of moduli space. The answer to this question is not known; the issue is the following.

In theories without exactly marginal deformations of the infrared conformal field theory (in which the IR fixed point is an isolated CFT) the leading effect of  $L_6$ , the distance between the NS and NS' fivebranes, on the infrared CFT is through its influence on the QCD scale  $\Lambda$ , which appears in the low energy effective action in the gauge field kinetic term  $(\log \Lambda) \int d^2\theta W_\alpha^2$ . This term is irrelevant at long distances and, therefore, long distance physics does not depend on  $L_6$ . However, as the NS fivebrane passes the NS' fivebrane through the strong coupling region at  $L_6 = 0$ , it is a priori possible that there is a discontinuous jump in the physics. It might be that one can rule out such a jump by using the fact that one can go around the singularity by turning on a D-term.

In theories with exactly marginal operators, in which there is a line of infrared fixed points, changes of  $L_6$  will in general couple to the moduli, and duality provides a map of the electric line of fixed points to the magnetic one. Apart from that, the previous discussion of full infrared equivalence is similar.

### *10.3. Other brane configurations*

Among the many possible brane configurations preserving  $N = 1$  SUSY in four dimensions that were not discussed in this paper, an interesting class of examples is obtained by considering fourbranes stretched between NS and NS' fivebranes in the presence of D

sixbranes parallel to the NS' fivebranes and D' sixbranes which are parallel to the NS fivebranes.

The brane approach allows the enumeration of the dimension of the moduli space of vacua for all possible orderings of D and D' sixbranes along the  $x^6$  direction. As an example, consider the case of a single fourbrane stretched between an NS fivebrane and an NS' fivebrane in the presence of  $N_f$  D sixbranes and  $N'_f$  D' sixbranes between the two fivebranes. The dimension of moduli space depends on the ordering of the sixbranes, and ranges from  $2(N_f + N'_f) - 1$  when all the D' sixbranes are to the left of all the D sixbranes, to  $2(N'_f) - 1$  when  $N'_f > N_f$  and  $N_f$  D sixbranes and D' sixbranes alternate along the  $x^6$  direction.

It would be interesting to find the corresponding field theoretic effective potential accounting for this behavior.

#### 10.4. Other theories

The discussion in terms of branes is clearly general, and can be applied to other theories.

##### 1) $N=2$ in $d=4$

In  $N = 2$  supersymmetric gauge theories in four dimensions it can be used to prove the equivalence of Higgs branches of theories with different rank gauge groups, *e.g.*  $U(N_c)$  and  $U(N_f - N_c)$  gauge theories with  $N_f$  flavors of quarks [42,9]. However, as one approaches the root of the Higgs branch, it is clear from both the classical and quantum brane constructions that (for  $N_f \neq 2N_c$ ) there is a jump in the moduli space and chiral ring (defined by choosing an  $N = 1$  subalgebra). Thus, the brane picture predicts that the theories at the origin of moduli space are not equivalent (unless  $N_f = 2N_c$ ).

##### 2) $N=2$ in $d=3$

One can repeat the discussion of sections 5 – 9 for  $N = 2$  supersymmetric gauge theories in three dimensions, using branes in type IIB string theory. For vanishing (or in general equal) real masses, the brane construction predicts that the dimensions of the electric and magnetic moduli spaces agree, but to achieve a more detailed understanding one needs to find the geometric realization of the fields  $W_{\pm}$  introduced in equation (4.8). The gauge theory analysis of this problem was recently performed in [43] (see also [44]), where it was conjectured that the magnetic theory described by (4.8) is equivalent to the electric one. If one turns on different real masses, the electric and magnetic theories cannot agree, since

they have different Coulomb branches (see [39] and section 6). Therefore, it is unlikely that the full electric and magnetic infrared CFT's agree. The status of duality in these models remains to be understood.

3)  $N=(2,2)$  in  $d=2$

It would be interesting to study the implications of the brane construction in two dimensional gauge theories, which can be obtained from our four dimensional construction by T-duality in (say)  $x^2$  and  $x^3$ . This may provide a connection to the construction of [17].

**Note added:** Related recent work appears in [45-47].

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## Appendix A. A gauge theory analysis of the case $k' > 1$

Consider an  $SU(N_c)$  gauge theory with a field  $X$  transforming in the adjoint representation and  $N_f$  flavors of quarks  $Q^i, \tilde{Q}_i$ , coupled via the superpotential<sup>20</sup>:

$$W = \frac{s_0}{k+1} \text{Tr} X^{k+1} + \sum_{i=1}^{N_f} \lambda_i \tilde{Q}_i X Q^i \quad (\text{A.1})$$

In the presence of the superpotential (A.1) the nonanomalous global symmetry of the model is  $[U(1)]^{N_f}$ , all the symmetries being vectorlike. When all the  $\lambda_i$  are equal, the global symmetry is promoted to  $U(N_f)$ .

In the absence of the second term in the superpotential, the theory does not have a vacuum for  $N_f < N_c/k$  [22]. The Yukawa coupling stabilizes the model, ensuring the existence of a vacuum for any  $N_f$ . To see that, deform the  $X$  dependent part of the superpotential (A.1) to:

$$W_x = \sum_{i=0}^{k-1} \frac{s_i}{k+1-i} \text{Tr} X^{k+1-i} \quad (\text{A.2})$$

---

<sup>20</sup> Throughout this appendix we will replace  $k'$  by  $k$  to simplify the notation.

and study the theory for small, non-vanishing  $\{s_j\}$ , using the fact [22] that there is a vacuum for small non vanishing  $\{s_j\}$  if and only if there is a vacuum for  $s_j = s_0\delta_{j,0}$ . Generically, the bosonic potential  $V = |W'|^2$  has  $k$  distinct minima  $\{a_i\}$ ,  $i = 1, \dots, k$ . If  $r_i$  eigenvalues of  $X$  are placed in the minimum at  $a_i$ , the gauge symmetry is broken:

$$SU(N_c) \rightarrow SU(r_1) \times SU(r_2) \times \dots \times SU(r_k) \times [U(1)]^{k-1} \quad (\text{A.3})$$

with  $\sum r_i = N_c$ . From (A.1) it is clear that generically all quarks become massive; the mass of the  $i$ 'th flavor in the  $SU(r_j)$  colour group is  $m_i^{(j)} = \lambda_i a_j$ . The theory reduces in the infrared to a direct product of SQCD's with massive quarks. Such vacua exist for any number of flavors.

In order to understand the infrared behaviour of the model with the superpotential (A.1) we can use duality. The model with  $\lambda_j = 0$  has a dual description based on the gauge group  $SU(kN_f - N_c)$  [22]. We can consider the second term as a deformation of the electric theory and find its dual by adding the dual perturbation to the magnetic superpotential. This leads [22] to the deformed magnetic superpotential:

$$\bar{W} = \frac{\bar{s}_0}{k+1} \text{Tr} Y^{k+1} + \frac{s_0}{\mu^2} \sum_{j=1}^k M_j \tilde{q} Y^{k-j} q + \sum_{i=1}^{N_f} \lambda_i (M_2)_i^i \quad (\text{A.4})$$

where  $\bar{s}_0 = -s_0$ ,  $\mu$  is a scale parameter, and  $M_j$  are gauge singlet mesons representing  $\tilde{Q} X^{j-1} Q$  in the magnetic theory. In the presence of the perturbation  $\lambda_i$  some of the fields gain a mass and we can integrate them out. To find the vacua of the deformed theory we solve the  $F$ -term conditions:

$$\begin{aligned} \tilde{q}_i Y^p q^j &= 0; \quad (p \neq k-2) \\ \frac{s_0}{\mu^2} q_i Y^{k-2} q^j &= -\lambda_i \delta_i^j \\ \text{Tr} Y^k &= 0 \end{aligned} \quad (\text{A.5})$$

In addition one has to satisfy the  $D$ -term conditions.

Consider first the case when only one of the  $\{\lambda_i\}$ , say  $\lambda_1$ , is different from 0. The solution of (A.5) is then:

$$\begin{aligned} \tilde{q}_\alpha^1 &= b \delta_{\alpha,1} \\ q_1^\alpha &= b \delta^{\alpha, k-1} \\ Y_\beta^\alpha &= b \delta_{\beta-1}^\alpha; \quad \beta = 2, \dots, k-1 \\ (M_j)_1^1 &= 0 \end{aligned} \quad (\text{A.6})$$

where  $b = [\frac{\lambda_1 \mu^2}{s_0}]^{\frac{1}{k}}$ . The relative normalizations in (A.6) are fixed by the D-term conditions. Thus, the magnetic gauge group is broken to  $SU(kN_f - N_c - k + 1)$ . Expanding the superpotential (A.4) around the solution (A.6) we find that the magnetic meson fields  $(M_j)_1^a, (M_j)_a^1$  with  $a = 1, \dots, N_f$ , as well as the fields  $\tilde{q}_\alpha^1, q_1^\alpha, Y_m^\alpha$  for  $m = 2, \dots, k - 1$  and  $Y_\alpha^n$  for  $n = 1, 2, \dots, k - 2$  become massive due to the Higgs mechanism.

On the other hand  $Y_\alpha^{k-1}$  and  $Y_1^\alpha$  play a special role: expanding the  $Y^{k+1}$  term in (A.4) around (A.6) gives rise to a term in the superpotential of the form:

$$\frac{3\bar{s}_0}{k+1} b^{k-2} Y_\alpha^{k-1} Y_\beta^\alpha Y_1^\beta$$

which makes it clear that in this gauge,  $Y_\alpha^{k-1}$  and  $Y_1^\beta$  represent a massless magnetic quark with a superpotential of the type (A.1) with a magnetic Yukawa coupling,

$$\bar{\lambda}_1 = \frac{3\bar{s}_0}{k+1} b^{k-2} \quad (\text{A.7})$$

Repeating the procedure for all  $N_f$  flavors we obtain a magnetic theory with gauge group  $SU(N_f - N_c)$ ,  $N_f$  flavors of magnetic quarks  $q, \tilde{q}$ , and a magnetic superpotential  $\bar{W}$ :

$$\bar{W} = \frac{\bar{s}_0}{k+1} \text{Tr} Y^{k+1} + \sum_{i=1}^{N_f} \bar{\lambda}_i \tilde{q}_i Y q^i \quad (\text{A.8})$$

where  $\bar{\lambda}_i = \frac{3\bar{s}_0}{k+1} [\frac{\lambda_i \mu^2}{s_0}]^{k-2}$

Since the leading coefficients of the beta function in the electric and magnetic theories are  $2N_c - N_f$  with opposite signs, it is clear that the electric theory will have the following three regimes:

- a)  $N_f \leq N_c$ : a supersymmetric vacuum of the theory exists but the IR is trivial all the states being massive.
- b)  $N_c < N_f < 2N_c$ : the electric theory is dual to an infrared free magnetic theory. Therefore, the massless spectrum is given by the fields appearing in the magnetic lagrangian.
- c)  $N_f \geq 2N_c$ : the electric theory is infrared free and, therefore, the spectrum is given by the fields appearing in the electric lagrangian.

We see that in the process of turning on the Yukawa couplings  $\lambda_i$  in the electric theory, all the magnetic mesons of [22] become massive. At first sight this seems in contradiction with the fact that in the case  $k = 1$  (A.1) reduces to SQCD, where it is known [21] that a singlet magnetic meson  $M$  is necessary for duality. The resolution is that  $s_0$  in (A.1) is in that



case a mass, so the electric theory is in fact SQCD perturbed by a quartic superpotential of the form (7.2), generated in the process of integrating out the massive field  $X$ . On the magnetic side this translates to a mass term for the magnetic meson  $M$ , which can therefore be integrated out, giving rise to a quartic superpotential in terms of the magnetic quarks, in agreement with the direct analysis of (A.8).

Thus, the fact that the equivalence between (A.1) and (A.8) does not require singlet mesons, relies crucially on the fact that the parameter  $s_0$  is finite. For infinite  $s_0$  in the case  $k = 1$ , the coefficient of the quartic electric superpotential (7.2) vanishes and the magnetic meson of SQCD,  $M$  becomes massless again.

For  $k > 1$  one can repeat the same analysis; it is easy to see by deforming the  $X$  dependent part of the superpotential as in (A.2) and using the previous result for SCQD that in the limit  $s_0 \rightarrow \infty$  one still needs only one magnetic meson, in agreement with the result we got from brane dynamics in the text.

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