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ANALYTICAL STUDY OF THE CONJECTURE RULE FOR THE COMBINATION OF MULTIPOLE EFFECTS IN LHC

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Abstract

This paper summarizes the analytical investigation done on the conjecture law found by tracking for the effect on the dynamic aperture of the combination of two multipoles of various order. A one-dimensional model leading to an integrable system has been used to find closed formulae for the dynamic aperture associated with a fully distributed multipole. The combination has then been studied and the resulting expression compared with the assumed conjecture law. For integrated multipoles small with respect to the focusing strength, the conjecture appears to hold, though with an exponent different from the one expected by crude reasoning.

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1 Introduction

In some past investigations of the dynamic aperture of the LHC (Ref. 1), short term calculations were performed with averages over different seeds for imperfection initialization, in order to deduce scaling laws with respect to various parameters such as the number of multipoles in the machine, their strength, and the multipolarity-order. In this context, the combined effect of different multipoles was studied and rules of thumb for the result on the dynamic aperture of such combinations with different order n were found. These rules were tested for several seeds and large number of perturbing elements and used to analyse tolerances in the LHC dipoles.

In constructing a heuristic model, the average dynamic aperture d was plotted as a function of the multipole strength (Ref. 1). Then, in the attempt to fit the obtained curve, the total d resulting from the combination of the separate d-values associated with two multipoles of order n and m was assumed to verify the following combination law:

$$\frac{1}{d^\nu} = \frac{1}{d_n^\nu} + \frac{1}{d_m^\nu} \quad ; \quad \nu \approx 4 \tag{1}$$

where ν is an arbitrary exponent. The best value found for the fit is $\nu = 4$ and could not be justified in a deterministic way, the more so that it happened later to apply to 4D-tracking simulations as well as to 6D-tracking (synchrotron motion included). This surprising rule of thumb is intriguing, in particular because it has been satisfied by subsequent results of simulations with various imperfections in the LHC elements.

Starting with these observations, we thougth that it would be interesting to understand somewhat the reasons for the existence of such a rule. In order to have a possible insight into the mechanisms involved and not to repeat numerical tracking which does not easily deliver them, we decided to try an analytical approach in spite of its limitations. Indeed, the basic idea consists in considering an integrable system for which it is possible to get a closed formula for the limit of stability. Therefore, the model retained deals with a one-dimensional motion, constant focusing and a flat distribution of the multipole field, the order and combination of the multipoles being arbitrary. The explicit solution for the dynamic aperture associated with this motion is given in Section 2. A particular case of combining distributed sextupole and decapole is treated in Section 3, where the results are then compared with the conjecture recalled hereabove. They show some interesting features which are discussed in the Conclusions.

2 Dynamical aperture for distributed single multipoles

The first step in the procedure is to compute the dynamical aperture of distributed single multipoles being superimposed to a constant quadrupole field. The equation of motion in the pure horizontal plane is

$$x'' + Q^2 x + K_n x^{n-1} = 0 \tag{2}$$

where n describes the order of the multipolare field (K_3 - sextupole , K_4 - octupole a.s.o.) and Q^2 stands for an overall linear focusing acting on the particle motion. As the **dynamical aperture** of this system we define the maximum initial value $x(0)$ when $\dot{x}(0) = 0$ that leads to bounded motion. In the case of Eq. (2) it is relatively easy to compute the dynamical aperture in closed expressions. The basic strategy is to write down a first integral and investigating the associated invariant curves in the phase space for its property to be closed (stable case) or open (unstable case) curves. Since Eq. (2) is derivable from a Hamiltonian function

$$H(x,p) = \frac{1}{2}(Q^2x^2 + p^2) + \frac{1}{n}K_n x^n = Const. \tag{3}$$

and $\partial H/\partial t = 0$ a first integral of motion is given by the Hamiltonian itself. The constant is evidently given by

$$Const. = \frac{1}{2}Q^2x_0^2 + \frac{1}{n}K_n x_0^n \tag{4}$$

Since the invariants (3) are symmetric w.r.t the canonical momentum p, we expect opening of the curves towards the x direction in the unstable case. We therefore look for a condition of transition between real and complex solutions of the equation

$$F(x) = H(x,0) - Const = 0 \tag{5}$$

with respect to x. A transition between real and complex solutions takes place if

$$F'(x) = 0 \Longrightarrow x = x_{Extr} \quad \text{and} \quad F(x_{Extr}) = 0 \tag{6}$$

For a sextupole $n = 3$ we obtain

$$F(x) = \frac{1}{2}Q^2x^2 + \frac{1}{3}K_3x^3 - \frac{1}{2}Q^2x_0^2 - \frac{1}{3}K_3x_0^3 = 0 \tag{7}$$

Using (6) we find

$$F'(x) = Q^2x + K_3x^2 = 0 \Longrightarrow x_{Extr} = -\frac{Q^2}{K_3} \tag{8}$$

$$F(x_{Extr}) = \frac{1}{6}\frac{Q^6}{K_3^2} - \frac{1}{2}Q^2x_0^2 - \frac{1}{3}K_3x_0^3 = 0 \tag{9}$$

The smallest (in absolute value) real solution of this equation is the dynamic aperture. The solution can easily be found by introducing

$$x_0 = \lambda\frac{Q^2}{K_3} \tag{10}$$

which gives

$$\frac{1}{3}\lambda^3 + \frac{1}{2}\lambda^2 - \frac{1}{6} = 0 \Longrightarrow \lambda = \frac{1}{2} \tag{11}$$

so that finally

$$x_{0\,Max} = \frac{Q^2}{2K_3} \tag{12}$$

Generalizing this procedure to distributed single multipoles of order n we find from (5) that

$$\frac{1}{2}Q^2 x_{Extr}^2 + \frac{1}{n}K_n x_{Extr}^n - \frac{1}{2}Q^2 x_0^2 - \frac{1}{n}K_n x_0^n = 0 \tag{13}$$

with

$$Q^2 + K_n x_{Extr}^{n-2} = 0 \tag{14}$$

Eq. (14) can be solved as

$$x_{Extr} = \pm \left(\frac{Q^2}{K_n}\right)^{1/(n-2)} \tag{15}$$

The positive sign relates to even multipoles (octupoles,dodecapoles..) while the negative sign is valid for odd multipoles (sextupoles,decapoles ...). Inserting x_{Extr} into (13) finally results in a definition equation for the dynamical aperture x_0 as

$$\mp K_n \frac{x_0^n}{n} - \frac{1}{2}Q^2 x_0^2 - \frac{Q^{2n/(n-2)}}{nK_n^{2/(n-2)}} + \frac{Q^{2n/(n-2)}}{2K_n^{2/(n-2)}} = 0 \tag{16}$$

Using the substitution

$$x_0 = \lambda \frac{Q^{2/(n-2)}}{K_n^{1/(n-2)}} \tag{17}$$

a new equation for λ not depending on Q and K_n can be established:

$$\left(\frac{1}{2} - \frac{1}{n}\right) - \frac{1}{2}\lambda^2 \mp \frac{1}{n}\lambda^n = 0 \tag{18}$$

Again the plus sign relates to even and the negative sign to odd multipoles. It turns out that for even multipoles $\lambda = 1$ is always the smallest positive solution,hence the general relation for the dynamical aperture in this case is

$$x_{0\,Max} = \frac{Q^{2/(n-2)}}{K_n^{1/(n-2)}} \quad ; \quad \text{n even} \tag{19}$$

For odd multipole however, the smallest positive solution differs from unity. In the case of a sextupole $\lambda = \lambda_3 = 1/2$. In the case of a decapole it is still possible to express λ in term of roots and we obtain

$$\lambda_5 = \frac{2 + 10^{1/3}}{3} = 0.72212... \tag{20}$$

This has been possible since one solution of the fifth order equation is equal to -1 and the problem is reducible to a fourth order polynomial. As n increases λ_n approaches $+1$ which indeed becomes solution of (18) as n tends to infinity. In Fig. 1 we plot the coefficient λ_n against n (n odd).

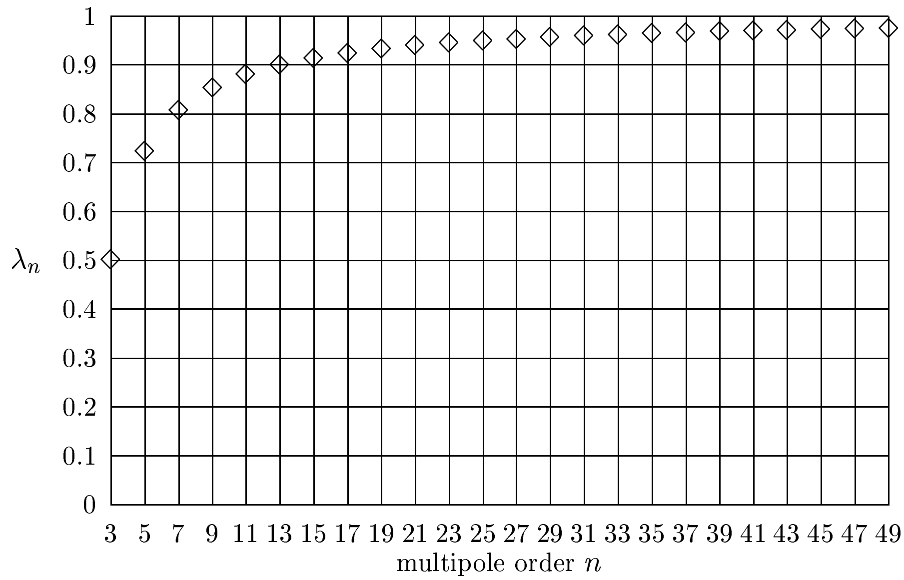

Fig.1 Coefficient λ_n for odd n

It should also be noted that the functional dependence of the single multipole aperture on Q and K_n as given in (17) agrees with the one given in (Ref. 1).

3 Test of a combined case

In the next step, we test the case of a horizontal motion with constant sextupole and decapole components (K_3 and K_5). The purpose is to compute analytically the single multipole apertures related to these multipole types as well as the aperture for the combined case. This should finally enable us to check the conjecture given in Ref. 1 that the inverse dynamical apertures due to single multipoles to some power simply add to give the inverse total aperture to the same power,

$$\frac{1}{d^\nu} = \frac{1}{d_3^\nu} + \frac{1}{d_5^\nu} \quad ; \quad \nu \approx 4 \tag{21}$$

The differential equation for this case is given by

$$\ddot{x} + Q^2 x + K_3 x^2 + K_5 x^4 = 0 \tag{22}$$

while its Hamiltonian reads as

$$H(x,p) = \frac{1}{2}(p^2 + Q^2 x^2) + \frac{1}{3}K_3 x^3 + \frac{1}{5}K_5 x^5 = Const. = \frac{1}{2}(p_0^2 + Q^2 x_0^2) + \frac{1}{3}K_3 x_0^3 + \frac{1}{5}K_5 x_0^5 \tag{23}$$

As before we restrict ourselves to the case $p_0 = 0$ and we look for the opening of the invariant curves in the x direction. Hence. eq. (23) reduces to

$$F(x) = \frac{1}{2}Q^2 x^2 + \frac{1}{3}K_3 x^3 + \frac{1}{5}K_5 x^5 = \frac{1}{2}Q^2 x_0^2 + \frac{1}{3}K_3 x_0^3 + \frac{1}{5}K_5 x_0^5 = C \tag{24}$$

Using again the conditions for unbounded motion,

$$F(x_{Extr}) = C \tag{25}$$
$$F'(x_{Extr}) = 0 \tag{26}$$

we find a fifth order equation for the limiting amplitude x_0 that is written as follows

$$\frac{1}{2}Q^2x_0^2 + \frac{1}{3}K_3x_0^3 + \frac{1}{5}K_5x_0^5 = F(x_{Extr}) \tag{27}$$

where x_{Extr} satisfies the following cubic equation:

$$K_5x_{Extr}^3 + K_3x_{Extr} + Q^2 = 0 \tag{28}$$

which has a real Cardanian solution

$$x_{Extr} = -2^{1/3}\frac{K_3}{\alpha} + \frac{\alpha}{3K_5 2^{1/3}} \tag{29}$$

where

$$\alpha = \left[U - 27(QK_5)^2\right]^{1/3} \tag{30}$$

and

$$U = 3\sqrt{3}\sqrt{4(K_3K_5)^3 + 27(QK_5)^4} \tag{31}$$

The dynamical aperture d for the combined case is then the smallest real solution of the fifth order equation (27).

3.1 Check of the law of superposition

We now use the results of the previous section in order to check the empirical law of superposition (21). In the assumption that this law is exactly valid, the following equation must hold:

$$\frac{1}{d_c^\nu} = \frac{1}{[Q^2/(2K_3)]^\nu} + \frac{1}{[(2+10^{1/3})Q^{2/3}/(3K_5^{1/3})]^\nu} \tag{32}$$

where d_c is the minumum-amplitude solution of the fifth order equation (27). At this stage already, we can conclude that (32) can **never hold exactly** for any number ν. The reason is that the combined aperture d_c is the solution of a general fifth order equation, the coefficients of which depend on the multipole strengths K_3 and K_5. It is indeed well known that the general solution of a fifth order polynomial equation cannot be expressed in terms of roots but only in terms of Jacobian elliptic functions. Hence, the left hand side of (32) will depend on elliptic functions in K_3 , K_5 and Q while the right hand side just depends on square and cubic roots in these variables. So from the strictly mathematical point of view, the proposed superposition law is already disproved.

However, we may look for the number ν (at some given Q) that minimizes the expression

$$S(K_3, K_5) = \frac{1}{d_c^\nu} - \frac{1}{[Q^2/(2K_3)]^\nu} - \frac{1}{[(2+10^{1/3})Q^{2/3}/(3K_5^{1/3})]^\nu} \tag{33}$$

within some interval for the multipole strengths K_3 and K_5. In order to do this systematically we choose the following procedure:

- We loop over the linear focusing Q in the interval $(5, 30)$ in steps of $\Delta Q = 1$
- For every value of Q we loop over the sextupole K_3 and decapole strength K_5 in the interval $(0.5, 5.0)$ in steps of 0.1.
- For each pair of K_3 and K_5, we compute the dynamical apertures for the single multipoles (d_3 and d_5) as well as the combined aperture d_c, by solving the fifth order equation (27).

- We loop over various exponents ν_n, and compute the quantities G_n and D_n for each ν_n

$$G_n = \sum_{k=1}^{k_{max}} \left[\frac{1}{d_c^{\nu_n}(k)} - \frac{1}{d_3^{\nu_n}(k)} - \frac{1}{d_5^{\nu_n}(k)} \right]^2 \tag{34}$$

$$D_n = \sum_{k=1}^{k_{max}} \left[\frac{1}{d_c^{\nu_n}(k)} \right]^2 \tag{35}$$

- We next introduce a quantity that characterizes the quality of the fitting of the exponent ν to the proposed superposition law, i.e.

$$\rho = \frac{Min_n(G_n)}{D_{n_{min}}} \tag{36}$$

A good fit is therefore associated with the condition $\|\rho\| << 1$.

- We eventually plot the best fitting value of ν against the focusing force Q.

In Fig.2 we see that the best fitting exponent ν varies by about a factor 2 within the chosen interval of Q indicating that the law of superposition does not work well if the linear focusing is strongly varied, keeping constant the interval for the strengths K_3 and K_5.

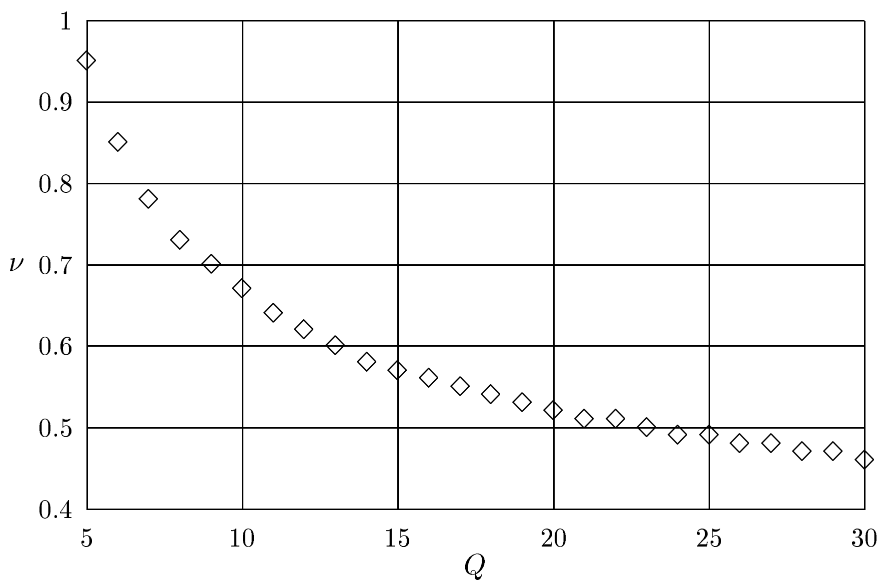

Fig.2 Best fitting exponent ν versus the linear focusing Q

In Fig. 3 finally we show the obtained value of ρ against Q and we realize that the fit obtained for every value of the linear focusing in the chosen interval is very good. Thus we conclude that for a given constant linear focusing (only the multipole strengths being varied), the proposed superposition law applies very well.

Next, we have to note that the increase of the best fitting ν for smaller Q corresponds to a similar rise of the quality factor ρ, associated in turn to the fact that the multipole strength K_n is almost comparable to Q, making the ratio K_n/Q^2 which enters the equation (32) as large as 1 to 10 %. Considering the systematic sextupole component in the dipoles of the LHC, we notice that this ratio is smaller than 0.1 %, and even smaller for the decapole component. Hence, for a more realistic description of the LHC case, we have to vary the focusing Q and the component K_n together, while maintaining the ratio defined above about constant and small.

We effectively did this exercise, still repeating the exact procedure described previously and minimizing the expression (33) within now a limited interval for K_3 and K_5 of $\pm 10\%$ around the values corresponding to a constant ratio. We have considered two values for the ratio equal to either 0.1 % or even 5 times smaller which corresponds to the two curves on Fig. 4. This figure gives the best fitting values of ν in these conditions for a Q-interval from 10 to 50. Now the quality factor ρ remains always smaller than 0.001 and the exponent almost constantly equal to 0.4. Note that the Q-value of our model that corresponds to the LHC should be the total wave-number coming from the 8 arcs, i.e. ≈ 46, and the multipole component should then be the integral over the arcs of the systematic dipole errors.

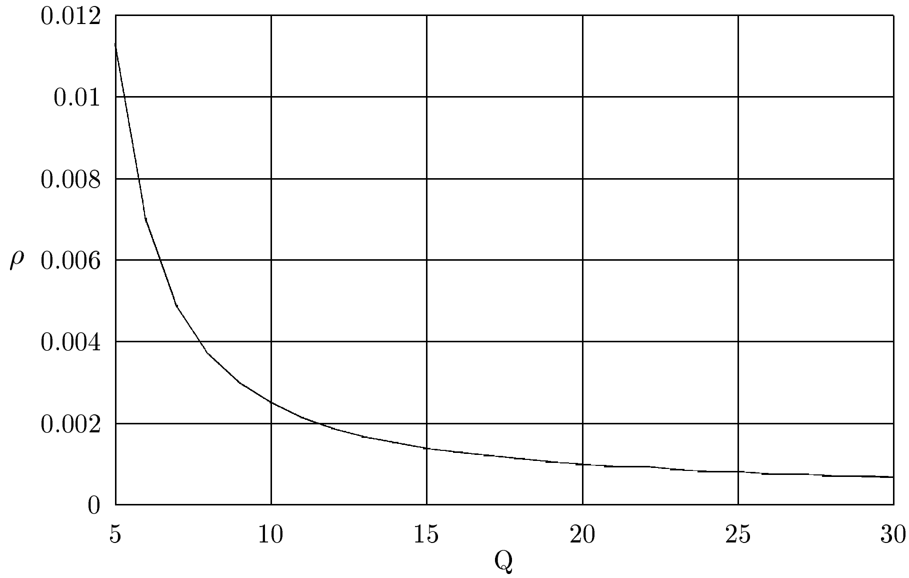

Fig.3 Quality factors ρ against linear focusing Q

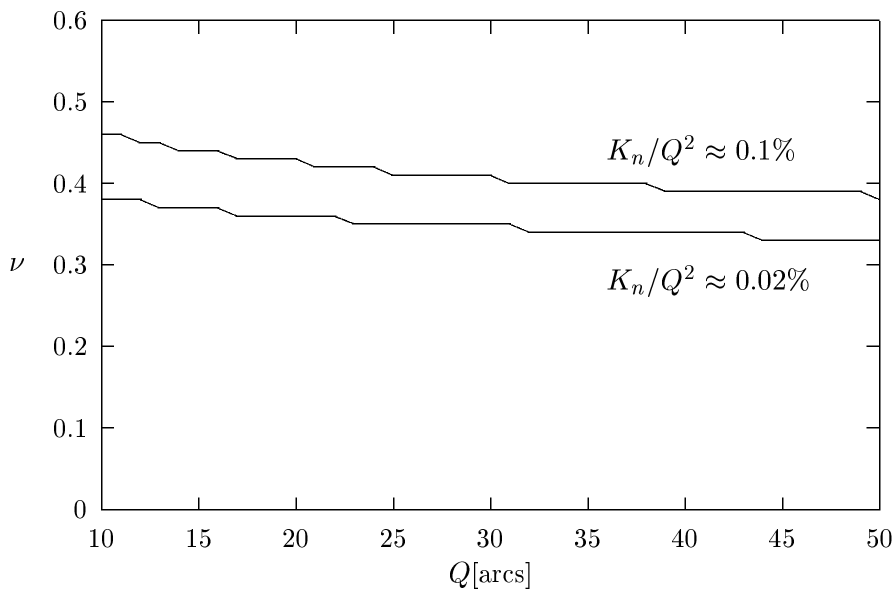

Fig.4 Exponent ν versus Q for a constant K_n/Q^2 ratio

For large variations of the focusing and multipole strength, the best found fitting-values of ν lie in the interval $0.4 < \nu < 0.9$ and remain close to 0.4 for a small ratio K_n/Q^2. These values do seriously disagree with the one given in Ref. 1, which is equal to 4. Following the argumentation in this reference, we could think that the exponent 4 is equal to the number of phase space dimensions for the coupled horizontal and vertical betatron motion. Consequently, in our one-dimensional model of a pure horizontal motion we should obtain an exponent $\nu = 2$, which is not the case in our results. We therefore conclude that the form of the superposition law given in Ref. 1 seems basically to hold as long as the linear beam focusing is held constant or the multipole strength is small with respect to the wave-number Q, but with an exponent ν not equal to the value expected from a crude reasoning.

4 Conclusions

As resulting from our analytical investigations with a simplified equation of motion, the following points are particularly interesting to underline. Though the conjecture cannot be proved in general, it holds remarkably well within some assumptions. At constant focusing strength Q, it is possible to find a unique value of the exponent ν which satisfies the conjecture for a range of amplitudes of the multipole components K_n relatively large and with a good quality of the fit conditions (i.e. small ρ values). This exponent ν however varies expectedly with the ratio K_n/Q^2 which enters the expression for the dynamical aperture. Both the exponent and the fit error ρ increase rapidly when K_n becomes comparable to Q, which is an irrealistic case. Nevertheless, if this ratio is smaller than a fraction of a permil as is the case in a collider like LHC, the fit is always good and ν remains close to an asymptotic value for any focusing wave-number Q. The fact that this value is approximately equal to 0.4 in a one-dimensional motion could not be explained.

5 Reference

1. V. Ziemann, Crude Scaling Laws for the Dynamic Aperture of LHC from Random Non-Linear Errors, *CERN SL/Note 95-20(AP), 1995.*