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Small Q balls

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Abstract

We develop an adequate description of non-topological solitons with a small charge, for which the thin-wall approximation is not valid. There is no classical lower limit on the charge of a stable Q-ball. We examine the parameters of these small-charge solitons and discuss the limits of applicability of the semilcassical approximation.

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Non-topological solitons [1], in particular Q-balls [2], are known to exist in the limit of a large charge. Modern theories often envision the physics beyond the Standard Model (SM) as associated with a number of scalar fields with various charges. All supersymmetric generalizations of the SM, in particular the minimal model, MSSM, allow for a variety of baryonic and leptonic balls built of squarks and sleptons, and the Higgs scalars [3]. These objects can be produced in the early Universe and can lead to interesting cosmological consequences. Q-balls with a small charge can be produced more easily at high temperatures. It is of great interest, therefore, to understand how small the charge of a stable Q-ball can be, and what are the parameters of such solitons in the limit of small Q.

The usual description of Q-balls [2] relies on the so-called thin-wall approximation and is only valid for a very large charge Q. A naive extrapolation of these results beyond their domain of validity would lead one to conclude that no stable low-Q solitons can exist. We will prove this not to be true, at least as long as the semiclassical treatment of Q-balls remains appropriate. There is no classical lower limit on the charge of a stable Q-ball. We will show that very small Q-balls (Q-beads) with charges $Q \gtrsim 1$ can exist and we will develop a formalism that yields an adequate description of these solitons.

1 Abelian Q-balls

Let us consider a field theory with a scalar potential $U(\varphi)$ which has a global minimum $U(0) = 0$ at $\varphi = 0$. Let $U(\varphi)$ have an unbroken global[1] U(1) symmetry at the origin, $\varphi = 0$. And let the scalar field φ have a unit charge with respect to this $U(1)$.

The charge (taken to be positive for definiteness) of some field configuration $\varphi(x,t)$ is

$$Q = \frac{1}{2i}\int \varphi^* \overset{\leftrightarrow}{\partial}_t \varphi \, d^3x \tag{1}$$

Since a trivial configuration $\varphi(x) \equiv 0$ has zero charge, the solution that minimizes the energy

$$E = \int d^3x \ \left[\frac{1}{2}|\dot{\varphi}|^2 + \frac{1}{2}|\nabla\varphi|^2 + U(\varphi)\right] \tag{2}$$

[1]Q-balls associated with a local symmetry have been constructed [4]. An important qualitative difference is that, in the case of a local symmetry, there is an upper limit on the charge of a stable Q ball.

and has a given charge $Q > 0$ must differ from zero in some (finite) domain. We will use the method of Lagrange multipliers to look for the minimum of E at fixed Q. We want to minimize

$$\mathcal{E}_\omega = E + \omega \left[Q - \frac{1}{2i} \int \varphi^* \partial_t \varphi \, d^3x \right], \tag{3}$$

where ω is a Lagrange multiplier. Variations of $\varphi(x,t)$ and those of ω can now be treated independently, the usual advantage of the Lagrange method.

One can re-write equation (3) as

$$\mathcal{E}_\omega = \int d^3x \, \frac{1}{2} \left| \partial_t \varphi - i\omega\varphi \right|^2 \; + \; \int d^3x \, \left[\frac{1}{2} |\nabla\varphi|^2 + \hat{U}_\omega(\varphi) \right] + \omega Q, \tag{4}$$

where

$$\hat{U}_\omega(\varphi) = U(\varphi) \; - \; \frac{1}{2} \, \omega^2 \, \varphi^2. \tag{5}$$

We are looking for a solution that extremizes $\mathcal{E}_\omega$, while all the physical quantities, including the energy, E, are time-independent. Only the first term in equation (4) appears to depend on time explicitly, but it vanishes at the minimum. To minimize this contribution to the energy, one must choose, therefore,

$$\varphi(x,t) = e^{i\omega t} \varphi(x), \tag{6}$$

where $\varphi(x)$ is real and independent of time. For this solution, equation (1) yields

$$Q = \omega \int \varphi^2(x) \; d^3x \tag{7}$$

It remains to find an extremum of the functional

$$\mathcal{E}_\omega = \int d^3x \, \left[\frac{1}{2} |\nabla\varphi(x)|^2 + \hat{U}_\omega(\varphi(x)) \right] + \omega Q, \tag{8}$$

with respect to ω and the variations of $\varphi(x)$ independently. We can first minimize $\mathcal{E}_\omega$ for a fixed ω, while varying the shape of $\varphi(x)$. This, however, is identical to the problem of finding the bounce $\bar{\varphi}_\omega(x)$ for tunneling in $d = 3$ Euclidean dimensions [5, 6, 9] in the potential $\hat{U}_\omega(\varphi)$ (Fig. 1). The first term in equation (8) is then nothing but the three-dimensional Euclidean

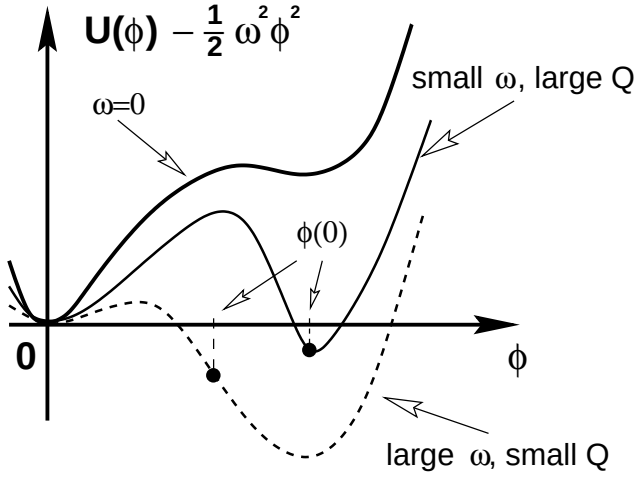

Figure 1: Finding a Q-ball is equivalent to finding a bounce that describes tunneling in the potential $\hat{U}_\omega(\varphi) = U(\varphi) - (1/2)\omega^2\varphi^2$. Thin-wall approximation is good for large Q (thin solid line), but breaks down when Q is small and, therefore, ω is almost as large as the mass term at the origin. In the latter case (dashed line), the "escape point", $\varphi(0)$ is close to the zero of the potential and is far from the global minimum. This allows for an alternative approximation that is good outside the thin-wall regime.

action $S_3[\bar{\varphi}_\omega(x)]$ of this bounce solution. In what follows we will use this analogy extensively. Over the years, the problem of tunneling has been studied intensely, and many properties of $\bar{\varphi}_\omega(x)$ and $S_3[\bar{\varphi}_\omega(x)]$ are well-known.

If the following condition [2] is satisfied,

$$U(\varphi) \Big/ \varphi^2 = \min, \quad \text{for } \varphi = \varphi_0 > 0 \tag{9}$$

the corresponding effective potential $\hat{U}_{\omega_0}(\varphi)$, where $\omega_0 = \sqrt{2U(\varphi_0^2)/\varphi^2}$, will have two degenerate minima, at $\varphi = 0$ and $\varphi = \varphi_0$.

The existence of the bounce solution $\bar{\varphi}_\omega(x)$ for $\omega_0 < \omega < U''(0)$ follows [6, 7] from the fact that $\hat{U}_\omega(\varphi)$ has a negative global minimum in addition to the local minimum at the origin. From Ref. [7] we know that the solution is spherically symmetric: $\bar{\varphi}(x) = \bar{\varphi}(r),\ r = \sqrt{\vec{x}^2}$.

The soliton we want to construct is precisely this bounce for the right choice of ω, namely that which minimizes $\mathcal{E}_\omega$. The last step is to find an extremum of

$$\mathcal{E}_\omega = S_3[\bar{\varphi}_\omega(x)] + \omega Q \tag{10}$$

with respect to ω. The conditions for the existence of such an extremum will be discussed

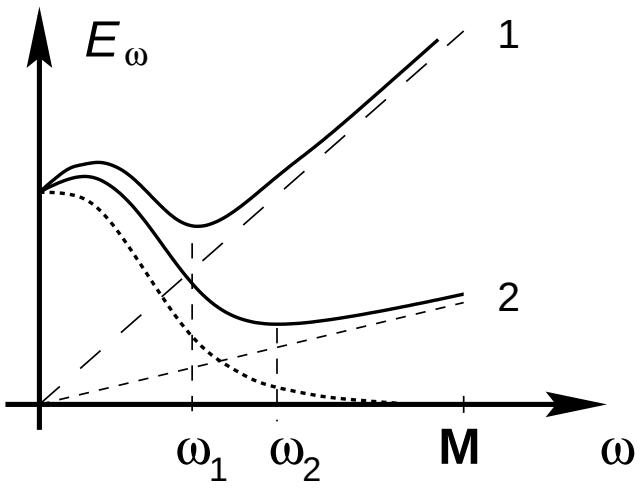

Figure 2: Solitons with a smaller charge Q have larger values of ω. The curved dotted line shows $S_3[\bar{\varphi}_\omega(x)]$ as a function of ω. The punctured straight lines 1 and 2 correspond to ωQ_1 and ωQ_2 for $Q_1 > Q_2$. The minimum of $S_3[\bar{\varphi}_\omega(x)] + \omega Q$ is reached at $\omega = \omega_1$, or ω_2, respectively. If $Q_1 > Q_2$, then $\omega_1 < \omega_2$.

below. The values of ω range from ω_0, the minimal value for which $\hat{U}_\omega(\varphi)$ is not everywhere positive and, therefore, the bounce solution can exist, to $\omega = U''(0)$, at which point $\hat{U}_\omega(\varphi)$ has no barrier. The latter does not automatically mean that a non-trivial solution of the equations of motion does not exist [8]. However, for our purposes, we will only have to consider ω in the range $\omega_0 < \omega < U''(0)$.

In the limit $\omega \to \omega_0 + 0$, the bounce (and, therefore, a Q-ball) solution can be analyzed in the thin-wall approximation [2]. However, for larger values of ω, the thin-wall approximation breaks down and so does the existence proof of Ref. [2]. In our analysis, we will use a different approximation [9], that which is valid in the limit of large ω.

Small, near-critical values of $\omega \approx \omega_0$, correspond to the large values of charge Q. There is a simple way to understand this. The first term in equation (10), $S_3[\bar{\varphi}_\omega(x)]$, is a monotone decreasing function of ω. (The smaller the barrier, the more probable the tunneling is. Thus S_3 must decrease with ω.) However, the last term in (8), ωQ, increases with ω. Therefore, if there is a minimum, it will be achieved for a smaller value of ω if the Q is larger, as illustrated in Fig. 2.

If, however, Q is sufficiently small, then the value of ω that minimizes $\mathcal{E}_\omega$ is large enough to destroy the near degeneracy of the two minima (Fig. 1), and the thin-wall approximation

breaks down.

2 Thick-wall approximation

There is, however, a powerful analytical approximation[2] [9] that can be used for calculating $S_3[\bar{\varphi}_\omega(x)]$ in the limit of very non-degenerate minima. It is based on the observation that for large ω (this case is analogous to tunneling into a very deep minimum), the "escape point" $\bar{\varphi}(0)$ (which is also the maximal value of φ inside the Q-ball) is close to a zero of $\hat{U}_\omega(\varphi)$, and is far from its minimum. As the barrier becomes smaller, the escape point $\bar{\varphi}(0)$ moves closer to the origin (Fig. 1). In this limit, one can neglect the dynamics at large φ and retain only the quadratic and cubic terms in the potential.

We will apply this strategy to minimize of $\mathcal{E}_\omega$ in equation (8). Let us consider a potential[3] $U(\varphi) = \frac{1}{2}M^2\varphi^2 - A\varphi^3 + \lambda_4\varphi^4$, where φ has a unit charge with respect to some global $U(1)$ symmetry unbroken at $\varphi = 0$. Then

$$\hat{U}_\omega(\varphi) = \frac{1}{2}(M^2 - \omega^2)\varphi^2 - A\varphi^3 + \lambda_4\varphi^4 \tag{11}$$

The bounce for the potential $\hat{U}_\omega(\varphi)$ for large ω (that is $0 < (M-\omega)/(M+\omega) \ll 1$) is the same as that for $\lambda_4 \to 0$. As we show below, the condition for the quartic terms to be negligible, $\lambda_4\varphi^4 \ll A\varphi^3$ when $(1/2)(M^2-\omega^2)\varphi^2 \approx A\varphi^3$, is satisfied whenever the charge is small enough. One can, therefore, neglect the quartic term in equation (11) and introduce some new dimensionless variables for the space-time coordinates, $\xi_i = (M^2-\omega^2)^{1/2}x_i$, and for the dynamical field, $\psi = \varphi A/(M^2-\omega^2)$. In terms of these variables, the expression for $\mathcal{E}_\omega$ becomes

$$\mathcal{E}_\omega = \frac{(M^2-\omega^2)^{3/2}}{A^2} \int d^3\xi \left[(\nabla_\xi\psi(\xi))^2 + \frac{1}{2}\psi^2(\xi) - \psi^3(\xi)\right] + wQ. \tag{12}$$

The first term in equation (12) is a dimensionful coefficient times the action S_ψ of the bounce in the potential $\Upsilon(\psi) = (1/2)\psi - \psi^3$. It has been computed [9, 12] numerically:

[2]Another approach to calculating $S_3[\bar{\varphi}_\omega(x)]$ analytically outside the thin-wall limit was proposed in Ref. [10]. To calculate the bounce action numerically, one can use the Improved Action method [11], which is particularly useful for systems with many scalar degrees of freedom.

[3]The φ^3 term represents some $U(1)$-symmetric cubic interaction, *e. g.*, $(\varphi^\dagger\varphi)^{(3/2)}$. In the MSSM, the requisite cubic interactions arise from the tri-linear couplings of the Higgs field to squarks and sleptons [3]. For clarity, we ignore the "flavor structure" of these cubic terms for now.

$S_\psi \approx 4.85$. Its precise value is not important, we will only need the fact that S_ψ is independent of M, ω, A, and Q. The corresponding bounce $\bar{\psi}(\xi)$ has a radius ~ 1 in dimensionless units [9]. We have, therefore, extremized $\mathcal{E}_\omega$ with respect to the variations of the $\psi(\xi)$, or, equivalently, those of $\varphi(x)$. What remains is to find a minimum of

$$\mathcal{E}_\omega = S_\psi \frac{(M^2-\omega^2)^{3/2}}{A^2} + wQ \tag{13}$$

with respect to ω, $0 < \omega < M$. This is possible as long as

$$\epsilon \equiv \frac{QA^2}{3S_\psi M^2} < \frac{1}{2} \tag{14}$$

Minimum is achieved at $\omega = \omega_{min} = M\left[(1+\sqrt{1-4\epsilon^2})/2\right]^{1/2}$. The resulting emergy (mass) of the soliton can be expanded in powers of ϵ:

$$E = QM\ \left[1 - \frac{1}{6}\epsilon^2 - \frac{1}{8}\epsilon^4\ -\ O(\epsilon^6)\right] \tag{15}$$

Since the mass of the soliton (15) is less than QM, it is stable with respect to decay into the φ quanta. The size R of the soliton is ~ 1 in dimensionless units $\sqrt{M^2-\omega^2}R$, that is

$$R^{-1} \sim (M^2-\omega^2)^{1/2} \approx \epsilon\, M\ \left(1 + \frac{1}{2}\epsilon^2 + \frac{7}{8}\epsilon^4 + O(\epsilon^6)\right) \tag{16}$$

Two conditions must be satisfied in order for the approximation we just used to be valid and self-consistent: (i) the quartic terms must be much smaller than the cubic and quadratic terms of the $\hat{U}$ in the vicinity of $\varphi(0)$ (and, therefore, for every $\varphi(x)$, because $\varphi(0) = max[\varphi(x)]$); and (ii) $\mathcal{E}_\omega$ in equation (13) must have a minimum for $0 < \omega < M$. These two requirements set the following respective constraints on the charge Q.

$$\begin{cases} Q \ \ll \ 3S_\psi M/\sqrt{\lambda_4}A \approx 14.6\, M/\sqrt{\lambda_4}A \\ Q \ < \ 3S_\psi M^2/2A^2 \approx 7.28\, M^2/A^2 \end{cases} \tag{17}$$

When these constraints are strongly violated, the thin-wall approximation can be used.

We conclude that in the limit of small Q, the thin wall approximation must be replaced by the approximation (12). The existence of the solition was proven in two steps. First, we noted that $\mathcal{E}_\omega$ has an extremum with respect to the variations of $\varphi(x)$ for fixed ω. This relies

on the existence proof in Ref. [6]. Second, using the "thick-wall" approximation, we showed that $\mathcal{E}_\omega$ has a minimum for some $\omega < M$.

The expressions for the energy (15) and the size (16) of the soliton are accurate when the cubic (tri-linear) couplings in the potential are not too large, and when the charge is small (equatiuons (14) and (17)).

There is no classical lower limit on charge: no matter how small Q is, there is a value of ω close to M, for which the energy of the configuration is minimal. However, for reasons of quantum stability, Q must be an integer. Therefore, $Q \geq 1$. Also, as we show below, in the limit $Q \to 1$, the quantum corrections can be significant.

3 Classical stability and quantum corrections

We would like to examine the second variation of energy and prove the stability of the soliton with respect to small variations that conserve charge. In the sector of a given charge Q (the only subspace of the functional space in which we are interested), E and $\mathcal{E}_\omega$ coincide, and so do their variations. The time derivatives enter only in the first term of equation (4), which is a non-negative function minimized by our solution (6). What remains is to consider $(\delta^2 E)_Q$ or $(\delta^2 \mathcal{E}_\omega)_Q$ with respect to the variations of the time-independent part of (6), $\delta\varphi(x)$. For arbitrary $\delta\varphi$, the variation of ω is induced, so that the charge conservation constraint (7) is satisfied. Starting with either of the expressions (2) or (8), one finds that

$$(\delta^2 E)_Q = \int \delta\varphi[-\Delta + U''(\bar{\varphi}(x)) + 3\omega^2]\delta\varphi \; d^3x \tag{18}$$

where $3\omega^2$ is the contribution of the second variation of the $(1/2)\omega^2\varphi^2$ term under the constant charge constraint: $\delta^2\{(1/2)\omega^2\varphi^2\}_Q = 3\omega^2\delta\varphi^2$. If we expand $\delta\varphi = \sum c_i\psi_i$ in terms of the orthonormalized eigenvectors of the differential operator in square brackets, ψ_i, corresponding to the eigenvalues λ_i, then $(\delta^2 E)_Q = \sum c_i^2\lambda_i$. Therefore, if the operator in square brackets in equation (18) has only positive eignevalues, then $(\delta^2 E)_Q$ is positive definite and the soliton is stable with respect to small perturbations. We, therefore, consider the following eigenvalue problem:

$$[-\Delta + U''(\bar{\varphi_\omega}(x)) + 3\omega^2]\,\psi_i \;=\; \lambda_i\psi_i \tag{19}$$

with the boundary condition $\psi(\infty) = 0$ and the normalization condition $\int \psi_i \psi_j = \delta_{ij}$.

It is easy to see that for large enough ω operator (19) has positive eigenvalues only. Indeed, equation (19) is just a Schrödinger equation for the potential $U''(\bar{\varphi}_\omega(x)) + 3\omega^2$. Since the potential $U(\varphi)$ has a minimum at the origin, there exists a value φ_{con}, such that $U''(\varphi) > 0$ for $0 < \varphi < \varphi_{con}$. For large ω, $\varphi(0)$ is small (Fig. 1) and, for a large enough ω (while still $\omega < U''(0)$), $\varphi(0) < \varphi_{con}$. Since $\forall x,\ \varphi(x) < \varphi(0)$, equation (19) describes a quantum-mechanical bound state of energy λ_i in the potential that is everywhere positive. Clearly, λ_i is then also positive (and, in fact, $\lambda_i > 3\omega^2$).

We have proven the stability of the Q-ball in the limit of large ω (small Q) with respect to small perturbations. Coleman proved [2] that Q-balls are stable with respect to all, not only small, deformations in the limit of large Q. We do not have a rigorous proof of the soliton stability in the intermediate region, although it seems plausible.

In order to evaluate the validity of the semiclassical approximation, one has to examine the magnitude of the quantum corrections to the mass of the soliton. Semiclassical results are reliable if the quantum fluctuations around the soliton are not large in comparison to its energy. The spectrum of such fluctuations is given by the same operator, $\delta^2 E/\delta\varphi^2$, in the soliton background, renormalized with respect to the oscillations around the trivial vacuum solution. The high-frequency modes cancel out, but the low-frequency spectrum of $\delta^2 E/\delta\varphi^2$ around the soliton will contain discrete levels which we would like to estimate.

Fortunately, we know something about the low-energy modes of a different operator, which differs from (19) by a constant. Indeed, for the three-dimensional bounce solution in the potential $\hat{U}_\omega(\varphi)$,

$$\frac{\delta^2 S_3}{\delta^2 \varphi} = -\Delta + \hat{U}''_\omega(\bar{\varphi}_\omega(x)) = -\Delta + U''(\bar{\varphi}_\omega(x)) - \omega^2. \tag{20}$$

The spectrum of the operator (20) was studied in Ref. [6]. We know that it has one negative eigenvalue $\tilde{\lambda}_1 < 0$. However, we just proved that $\lambda_1 = \tilde{\lambda}_1 + 4\omega^2 > 0$. Therefore, the lowest eigenvalue of the operator (19), λ_1, is in the range $0 < \lambda_1 < 4\omega^2$. One can, therefore, expect the corrections to the soliton mass squared to be (at the most) of order ω^2, which, in the limit of small Q, is $\sim U''(0)$. These corrections will be small in comparison to the soliton mass squared (15) if $Q^2 \gg 1$. However, the semiclassical approximation can become unreliable

for calculating the masses of the solitons with charge $Q \sim 1$. (At least, we don't have a proof to the contrary.) Nevertheless, even in this limit the size $R \sim \epsilon^{-1} M$ of the soliton remains large in comaprison to its De Broglie wavelength, which is an indication that semiclassical treatment may otherwise be appropriate for $Q \sim 1$.

4 Virial theorem

We will now prove a useful virial theorem which is valid for any Q, and requires no approximation. (In doing so, we will also illustrate that the soliton is a global minimum with respect to the size variations.) Let's consider a one-parameter family of functions obtained from the solution $\bar{\varphi}$ by expanding (contracting) it by a factor α:

$$\varphi_\alpha(x) = \bar{\varphi}(\alpha x) \tag{21}$$

The energy of φ_α is

$$E_\alpha = \frac{1}{\alpha^3} \frac{Q^2}{2 \int \bar{\varphi}^2 d^3x} \; + \; \alpha\, T + \alpha^3\, V, \tag{22}$$

where $T = \int \frac{1}{2} (\nabla \bar{\varphi})^2 \; d^3x$ is the gradient energy, and $V = \int U(\bar{\varphi}) d^3x$ is the potential energy of the Q-ball, both positive.

Theorem:

$$T + 3V = \frac{3\, Q^2}{2 \int \bar{\varphi}^2 \, d^3x} \tag{23}$$

Proof. Since $\bar{\varphi}$ is a stationary point of E with respect to all variations, it must, in particular, be an extremum with respect to scaling (21). Therefore, $dE_\alpha/d\alpha = 0$ at $\alpha = 1$. This yields relation (23). □

If $Q = 0$, there is no non-trivial solution, since $T \geq 0$ and $V \geq 0$ would imply $T = V = 0$. Equation (23) shows, in particular, how the non-topological solitons evade the Derrick's theorem [13].

We note that, since the coefficients of $1/\alpha^3$, α, and α^3 in equation (22) are all positive, the only exteremum of E_α with respect to α (the size of the soliton) is a global minimum.

In summary, Q-balls with a small charge exist and are classically stable. They can be treated semiclassically at least as long as $Q^2 \gg 1$. For $Q \approx 1$ the quantum corrections to the

soliton mass are not necessarily small, but the Q-ball remains an extended object, whose size is large in comparison to its De Broglie wavelength.

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