

CERN-TH/97-53

hep-th/9703189

# Complex Matrix Models and Statistics of Branched Coverings of 2D Surfaces

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We present a complex matrix gauge model defined on an arbitrary two-dimensional orientable lattice. We rewrite the model's partition function in terms of a sum over representations of the group  $U(N)$ . The model solves the general combinatorial problem of counting branched covers of orientable Riemann surfaces with any given, fixed branch point structure. We then define an appropriate continuum limit allowing the branch points to freely float over the surface. The simplest such limit reproduces two-dimensional chiral  $U(N)$  Yang-Mills theory and its string description due to Gross and Taylor.

SPhT-97/022

CERN-TH/97-53

March 1997

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## 1. Introduction

Recently, two of the authors have considered a complex matrix model which describes the ensemble of branched coverings of a two-dimensional manifold [1]. The model has been interpreted as a string theory invariant with respect to area-preserving diffeomorphisms of the target space. In this paper, we continue the investigation of this model.

The geometrical problem we solve consists in the enumeration of the branched coverings of a two-dimensional manifold with given number of punctures. The target manifold is characterized by its topology, the number of punctures, and its total area. All structures we are considering are invariant under area-preserving diffeomorphisms of the target manifold. We assume that the covering surfaces can have branch points located at the punctures. Two covering surfaces related by an area-preserving diffeomorphism are considered identical.

The problem will be reformulated in terms of a lattice gauge theory of  $N \times N$  complex matrices defined on a lattice representing a cell decomposition of the target manifold. The vertices of the lattice are the punctures of the target surface. The  $\frac{1}{N}$  perturbative expansion of this model generates the covering surfaces with the corresponding combinatorial factors. These discrete surfaces can be interpreted as cell decompositions of continuum surfaces covering the target space. It should be intuitively clear that the combinatorics of these surfaces should not depend on the cellular decomposition of the target manifold, but only on global features like the topology of the manifold, the topology of the covering Riemann surface, and the number and types of branch points allowed. Our solution below will confirm this intuition.

The solution of the model is given in terms of a sum over polynomial representations of the group  $U(N)$ . This sum is, by construction, a generating function for the combinatorics of covering maps. The order of the representation gives the degree of the (connected and disconnected) coverings. The sum depends on the genus  $G$  of the target manifold, and a number of variables tied to geometrical data of the covering maps:  $N^{-2}$  is the genus expansion parameter of the covering surfaces, and we associate, for each cell corner  $p$ , a weight  $t_k^{(p)}$  corresponding to a branch point at  $p$  of order  $k$ .

For some applications, one would consider a slightly more general problem of counting coverings with branch points that can occur anywhere on the smooth target manifold. In order to allow for this possibility, we are led to take a *continuum limit*: We simply cover the target manifold by a microscopically small cell decomposition, with the weights of these branch points tuned correspondingly. We will find that the combinatorics of the resulting

statistics of movable branch points actually simplifies significantly over the general case of fixed branch points: Many special configurations of enhanced symmetry, corresponding to coalescing branch points, are scaled away. The weights associated with the punctures are not subdued to scaling; this makes the difference between the punctures and the rest of the points of the cell decomposition.

Apart from the obvious mathematical interest of our approach, we are able to connect our results to recent work on the QCD string in two dimensions. In the case of a target space with nonnegative global curvature, the chiral (i.e., orientation preserving) sector of the Gross and Taylor [2] string theory describing two dimensional Yang-Mills theory, is a particular case of the string theory defined by our matrix model. We discuss the problem of the “ $\Omega$  factors” in the string interpretation of the Yang-Mills theory and interpret the “ $\Omega$ -points” as punctures in the target space.

## 2. Definition of the model

Consider a smooth, two dimensional, compact, closed and orientable manifold  $\mathcal{M}_G$  of genus  $G$  with  $\mathcal{N}_0$  marked points (punctures), which we denote by  $p = 1, \dots, \mathcal{N}_0$ . The manifold is compact in the following sense: We assume that there is a volume form  $dA$  on  $\mathcal{M}_G$  and the total area  $A_T = \int dA$  is finite. We will consider the ensemble of nonfolding surfaces covering  $\mathcal{M}_G$  and allowed to have branch points at the punctures. These surfaces are smooth everywhere on  $\mathcal{M}_G$  and are given a volume form inherited from the embedding. In this way, the area of a surface covering  $n$  times the target manifold is equal to  $nA_T$ .

We will resolve the problem by discretizing the target manifold. Introduce a cell decomposition of the original target manifold  $\mathcal{M}_G$  such that each cell is a polyhedron homeomorphic to a disc. The vertices of the cell decomposition are by construction the  $\mathcal{N}_0$  punctures of  $\mathcal{M}_G$ . The resulting polyhedral surface (cellular complex)  $M_G$  is thus characterized by its genus  $G$ , and by its set of points  $p$ , links  $\ell$  and cells  $c$ . The numbers of points, links and cells which we denote correspondingly by  $\mathcal{N}_0$ ,  $\mathcal{N}_1$  and  $\mathcal{N}_2$ , are related by the Euler formula

$$\mathcal{N}_0 - \mathcal{N}_1 + \mathcal{N}_2 = 2 - 2G. \quad (2.1)$$

Each cell contributes a fraction  $A_c$  to the total area  $A_T$  of the target manifold, so that  $A_T = \sum_c A_c$ . A given manifold can be discretized in many different ways, but the choice of discretization is irrelevant for our problem.

A branched covering of  $M_G$  of degree  $n$  is a surface  $\Sigma$  obtained by taking  $n$  copies of each of the polygons of  $M_G$  and identifying pairwise the edges of the  $n$  polygons on either side of each link. The Riemann surface obtained in this way can have branch points of order  $k$  ( $k = 1, 2, \dots, n$ ) representing cyclic contractions of edges. They are located at the points  $p \in M_G$ . The discretized surface  $\Sigma$  has  $n\mathcal{N}_1$  links,  $n\mathcal{N}_2$  polygons and  $n\mathcal{N}_0 - \sum_p b_p$  points (with  $b_p$  the winding number minus one at point  $p$ ). Its total area is  $nA_T$ . Its genus  $g$  is given by the Riemann-Hurwitz formula:

$$2g - 2 = n(2G - 2) + \sum_p b_p. \quad (2.2)$$

The partition function is defined as the sum over all possible coverings  $\Sigma \rightarrow M_G$  conserving the orientation. A factor  $N^{2-2g}$  is assigned to the genus  $g$  of the covering surface. Furthermore, we introduce Boltzmann weights associated with its branch points. The weight of a branch point of order  $k$  is  $t_k^{(p)}$ , where  $k \geq 2$ . A regular (analytic) point gets a weight  $t_1^{(p)}$ .

The partition sum is now defined as a sum over all (not necessarily connected) coverings  $\Sigma \rightarrow M_G$ :

$$Z = \sum_{\Sigma \rightarrow M_G} e^{-nA_T} N^{2-2g} \prod_{p=1}^{\mathcal{N}_0} \prod_{k \geq 1} (t_k^{(p)})^{n_k}. \quad (2.3)$$

where, associated with the point  $p \in M_G$ ,  $n_k(p)$  (with  $k \geq 2$ ) is the number of the branch points of order  $k$  of  $\Sigma$  and  $n_1(p)$  the number of regular points. The symmetry factor of the map is understood in the sum.

Now we introduce a matrix model whose perturbative expansion coincides with (2.3). To each link  $\ell = \langle pp' \rangle$  we associate a field variable  $\Phi_\ell$  representing an  $N \times N$  matrix with complex elements. By definition  $\Phi_{\langle pp' \rangle} = \Phi_{\langle p'p \rangle}^\dagger$ . In order to be able to associate arbitrary weights to the branch points we will associate an external matrix field with the corners of the cells. Let us denote by  $(c, p >$  the corner of the cell  $c$  associated with the point  $p$ . The corresponding matrix will be denoted by  $B_{(c,p >}$ .

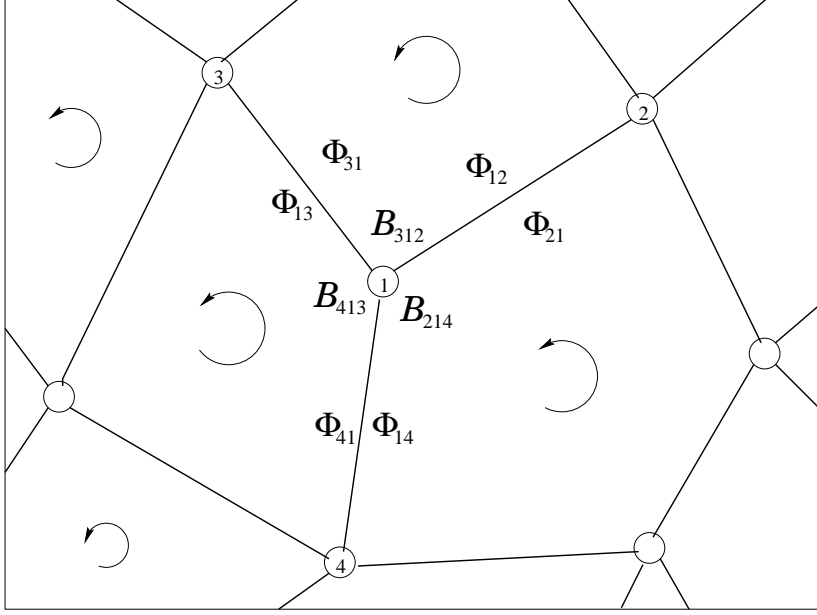


Fig.1: The matrices associated with the links  $\langle 12 \rangle, \langle 13 \rangle, \langle 14 \rangle$  and the corners  $\langle 214 \rangle, \langle 312 \rangle, \langle 413 \rangle$  associated with the point 1.

The partition function of the matrix model is defined as

$$Z = \int \prod_{\ell} [\mathcal{D}\Phi_{\ell}] \prod_c \exp(e^{-A_c} N \text{Tr}\Phi_c), \quad (2.4)$$

where  $\Phi_c$  denotes the ordered product of link and corner variables along the oriented boundary  $\partial c$  of the cell  $c$

$$\Phi_c = \prod_{\ell, p \in \partial c} \Phi_{\ell} B_{(c,p)}, \quad (2.5)$$

and the integration over the link variables is performed with the Gaussian measure

$$[\mathcal{D}\Phi_{\ell}] = (N/\pi)^{N^2} \prod_{i,j=1}^N d(\Phi_{\ell})_{ij} d(\Phi_{\ell})_{ij}^* e^{-N \text{Tr} \Phi_{\ell} \Phi_{\ell}^{\dagger}}. \quad (2.6)$$

The perturbative expansion of (2.4) gives exactly the partition function (2.3) of branched surfaces covering  $M_G$ . The weight  $t_k^{(p)}$  of a branch point of order  $k$  ( $k \geq 2$ ) or regular point ( $k = 1$ ) associated with the vertex  $p \in M_G$  equals

$$t_k^{(p)} = \frac{1}{N} \text{Tr}[B_p^k], \quad (2.7)$$

where we have defined the matrices  $B_p$  as the ordered product

$$B_p = \prod_c B_{(c,p)} \quad (2.8)$$

of the  $B$ -matrices around the vertex  $p$ .

### 3. Exact solution by the character expansion method

The method consists in replacing the integration over complex matrices by a sum over polynomial representations of  $U(N)$ . Applying to (2.4) the same strategy as in ref.[3], we expand the exponential of the action for each cell  $c$  as a sum over the Weyl characters  $\chi_h$  of these representations:

$$\exp(e^{-A_c} N \text{Tr}\Phi_c) = \sum_h \frac{\Delta_h}{\Omega_h} \chi_h(\Phi_c) e^{-A_c|h|}. \quad (3.1)$$

The representations are parametrized by the shifted weights  $h = \{h_1, h_2, \dots, h_N\}$ , where  $h_i$  are related to the lengths  $m_1, \dots, m_N$  of the rows of the Young tableau by  $h_i = N - i + m_i$  and are therefore subjected to the constraint  $h_1 > h_2 > \dots > h_N \geq 0$ . We will denote by  $|h| = \sum_i m_i$  the total number of boxes of the Young tableau. The dimension  $\Delta_h$  of the representation  $h$  is given by

$$\Delta_h = \prod_{i < j} \frac{h_i - h_j}{j - i}, \quad (3.2)$$

and the ‘‘Omega factor’’  $\Omega_h$  by

$$\Omega_h = N^{-|h|} \prod_{i=1}^N \frac{h_i!}{(N-i)!}. \quad (3.3)$$

An explicit representation of the Weyl characters  $\chi_h$  is only needed for the derivation of two essential integration formulas (fission and fusion rule, respectively):

$$\begin{aligned} \int [\mathcal{D}\Phi] \chi_h(\Phi_1 \Phi \Phi_2 \Phi^\dagger) &= \frac{\Omega_h}{\Delta_h} \chi_h(\Phi_1) \chi_h(\Phi_2), \\ \int [\mathcal{D}\Phi] \chi_h(\Phi_1 \Phi) \chi_{h'}(\Phi^\dagger \Phi_2) &= \delta_{h,h'} \frac{\Omega_h}{\Delta_h} \chi_h(\Phi_1 \Phi_2). \end{aligned} \quad (3.4)$$

For a simple proof of these identities, see [1]. It is now possible to exactly perform the integration with respect to the gaussian measure (2.6) at each link. Using the fusion rule one progressively eliminates links between adjoining cells until one is left with a single cell. The remaining links along that plaquette are integrated out by applying the fission rule. A similar procedure was first used in the exact solution of two-dimensional Yang-Mills theory [4],[5]. Employing the very useful relation

$$\chi_h(A_1) = \frac{\Delta_h}{\Omega_h} \quad \text{with} \quad \frac{1}{N} \text{Tr}A_1^k = \delta_{k,1}, \quad (3.5)$$

one finds the exact solution of the matrix model (2.4):

$$Z = \sum_h \left( \frac{\Delta_h}{\Omega_h} \right)^{2-2G} \prod_{p=1}^{\mathcal{N}_0} \left[ \frac{\chi_h(B_p)}{\chi_h(A_1)} \right] e^{-|h|A_T}. \quad (3.6)$$

The characters  $\chi_h(B_p)$  are related to the branch point weights  $t_k^{(p)}$  at the site  $p$  through the Frobenius formula (we omit the index  $p$  for clarity):

$$\chi_h(B) = \sum_{n_1+2n_2+3n_3+\dots=|h|} \text{ch}_h(1^{n_1}, 2^{n_2}, 3^{n_3}, \dots) \frac{|h|! N^{n_1+n_2+n_3+\dots}}{1^{n_1} n_1! 2^{n_2} n_2! 3^{n_3} n_3! \dots} t_1^{n_1} t_2^{n_2} t_3^{n_3} \dots \quad (3.7)$$

The sum is over all partitions of the covering number  $|h|$ , and the  $\text{ch}_h(1^{n_1}, 2^{n_2}, 3^{n_3}, \dots)$  are the characters of the symmetric group  $S_{|h|}$  corresponding to the representation  $h$  and the partition class with cycle structure  $(1^{n_1}, 2^{n_2}, 3^{n_3}, \dots)$ . Let us emphasize that (3.6), together with (3.7), is the complete solution of the combinatorial problem of counting branched covers (2.3).

In appendix A, we have given the first few terms of the expansion (3.6), in the special case where the branch point weights are identical on all  $\mathcal{N}_0$  sites. We have also given the first few terms of the free energy  $F = \log Z$ , corresponding to connected surfaces.

A special case of (3.6) is obtained when no branch points are allowed on the target manifold. This is done by choosing all matrices  $B_p = A_1$ . Then the solution (3.6) becomes completely independent of the cell decomposition of the manifold: The cell corners are indistinguishable from any other point on the surface. We obtain

$$Z = \sum_h \left( \frac{\Omega_h}{\Delta_h} \right)^{2G-2} e^{-|h|A_T}. \quad (3.8)$$

This is completely trivial for the sphere and very simple (see [1],[2]) for the torus, but highly non-trivial for higher target space genus  $G \geq 2$ :

$$\log Z = \begin{cases} N^2 e^{-A_T}, & \text{for } G = 0; \\ N^0 \sum_{k=1, m=1}^{\infty} \frac{1}{k} e^{-kmA_T} & \text{for } G = 1; \\ N^{2-2G} e^{-A_T} + (N^{2-2G})^2 \frac{1}{2} (4^G - 1) e^{-2A_T} + \dots & \text{for } G \geq 2. \end{cases} \quad (3.9)$$

For  $G \geq 2$ ,  $\log Z$  counts the number of smooth and locally invertible maps of a surface of genus  $g$  onto a surface of genus  $G$ . It would be interesting to find an analytic expression for  $\log Z$  for all  $G \geq 2$ .

#### 4. Continuum limit: statistics of movable branch points

In the preceding section we have presented the full solution to the problem of counting the branched covers of a smooth, closed and orientable target manifold of any genus  $G$  and  $\mathcal{N}_0$  punctures. By construction, the solution (3.6) depended, aside from  $G$  and the genus  $g$  of the branched cover, on the number and types of branch points at the punctures of  $\mathcal{M}_G$ : At each puncture  $p$  we are to specify a set of weights  $t_k^{(p)}$  associated with the winding numbers of the covering surface. It is natural to consider a related, but different combinatorial problem: Allow the covering surfaces to have branch points *anywhere* on the target manifold. In this case we have to sum over the positions of the branch points with an appropriate integral measure.

The solution of this problem is actually contained in the solution of the previous one. Since branching in our model is constrained to occur at the sites of the cell decomposition, we are led to consider a continuum limit: The cell decomposition has to densely cover the manifold. The simplest choice is to assign identical weights at each branching sites, that is, choose  $B_p = B$  everywhere, and assume that all cells have the same area  $A_c = \frac{A_T}{\mathcal{N}_0}$ . Set

$$\begin{aligned} t_1 &= \frac{1}{N} \text{Tr} B = 1, \\ t_k &= \frac{1}{N} \text{Tr} B^k = \frac{\tau_k}{\mathcal{N}_0} \quad \text{for } k \geq 2, \end{aligned} \tag{4.1}$$

and take  $\mathcal{N}_0 \rightarrow \infty$ , while holding  $A_T$  and the continuum couplings  $\tau_k$  fixed. Thus the probability for branching at a specific site  $p$  goes to zero in a prescribed way. In this continuum limit the configuration space of the branch points becomes infinite and their statistics drastically simplifies due to the fact that the probability to have more than one branch point associated with the same point of  $M_G$  tends to zero. Mathematically, this simplification is immediately seen from the partition function (3.6), and the Frobenius formula (3.7). The product of quotients of characters exponentiates, and (3.6) becomes:

$$\mathcal{Z} = \sum_h \left( \frac{\Delta_h}{\Omega_h} \right)^{2-2G} \exp \left[ \sum_{k=2}^{\infty} \tau_k N^{1-k} \xi_k^h \right] e^{-|h|A_T}. \tag{4.2}$$

Here, in view of (3.7), the tableau dependent numbers  $\xi_k^h$  are given by

$$\xi_k^h = \frac{|h|!}{k(|h| - k)!} \frac{\text{ch}_h(1^{|h|-k}, k^1)}{\text{ch}_h(1^{|h|})}. \tag{4.3}$$



An explicit formula is

$$\xi_k^h = \frac{1}{k} \sum_{i=1}^N h_i (h_i - 1) \dots (h_i - k + 1) \prod_{\substack{j=1 \\ j \neq i}}^N \left( 1 - \frac{k}{h_i - h_j} \right), \quad (4.4)$$

which is quickly derived from (3.5), (4.1), and the identity (coming from the Schur definition of the characters, see [6])

$$\frac{\partial}{\partial t_k} \log \chi_h(B) = \frac{N}{k} \sum_{i=1}^N \frac{\chi_{\tilde{h}^k}(B)}{\chi_h(B)} \quad \text{with} \quad \tilde{h}_i^k = h_i - \delta_{ki} k. \quad (4.5)$$

The  $\xi_k^h$  are actually symmetric polynomials of degree  $k$  in the weights  $h_i$  (see appendix B, where we have also listed a few examples). They are  $N$  independent as long as  $|h| < N$ .

In appendix B, we have given the first few terms of the expansion (4.2), as well as the first few terms of the free energy  $\mathcal{F} = \log \mathcal{Z}$ , corresponding to connected surfaces.

Finally, let us mention that we can keep the weights associated with a given number of points unscaled. Then we obtain the partition function of the ensemble of branched coverings of a target manifold of area  $A_T$ , genus  $G$  and  $\mathcal{N}_0$  punctures. The evident generalization of eqs. (3.6) and (4.2) is

$$Z = \sum_h \left( \frac{\Delta_h}{\Omega_h} \right)^{2-2G} \prod_{p=1}^{\mathcal{N}_0} \left[ \frac{\chi_h(B_p)}{\chi_h(A_1)} \right] \exp \left[ \sum_{k=2}^{\infty} \tau_k N^{1-k} \xi_k^h \right] e^{-|h|A_T}. \quad (4.6)$$

## 5. Relation to chiral 2D Yang-Mills theory

The solution of the map counting problem with movable branch points considered in the last section has an interesting physical interpretation. Let us restrict ourselves to the case of only simple branch points:  $\tau_k = 0$  for  $k \geq 3$ . Then, using the explicit result for  $\xi_2^h$  (see appendix B), and choosing

$$\frac{1}{2} \tau_2 = -A_T, \quad (5.1)$$

we can rewrite the partition function (4.2) as

$$\mathcal{Z} = \sum_h \left( \frac{\Delta_h}{\Omega_h} \right)^{2-2G} e^{-\frac{A_T}{N} C_2^h}. \quad (5.2)$$

where  $C_2^h$  is the second Casimir of the group  $U(N)$ . This is very nearly the partition function of two-dimensional  $U(N)$  Yang-Mills theory. This theory was shown to be solvable

on any target manifold in [4],[5]. Recently, it was demonstrated by Gross and Taylor [2], that  $YM_2$  can be interpreted as a string theory. This was achieved by the *tour de force* approach of expanding the exact solution in  $\frac{1}{N}$  and interpreting the terms as string maps. Here we are inverting the Gross-Taylor program: We start with a theory whose interpretation as a string theory generating covering maps from a worldsheet to the target manifold is manifest, and aim to derive  $YM_2$ . There remain, however, two subtle differences between (5.2) and  $YM_2$ :

(1) In  $U(N)$   $YM_2$  the sum over polynomial representations  $h$  is extended to a sum over coupled (non-polynomial) representations. This difference is easy to understand: The missing representations clearly correspond to orientation reversing maps, which have been eliminated from the start by our chiral matrix model. Indeed it became evident from the work of [2], that  $YM_2$  factorizes into two rather weakly interacting chiral sectors. Our class of models furnishes a precise realization of one such sector.

(2) Our partition function contains, for genus  $G \neq 1$ , the extra factor  $\Omega_h^{2G-2}$ , absent in  $YM_2$ . This factor *eliminates* the so-called “ $\Omega$  points” and “ $\Omega^{-1}$  points” introduced originally by Gross and Taylor in order to sustain the string picture for  $G \neq 1$ . For  $G = 0$ , we can add two punctures and associate weight 1 with the windings around them (i.e. take a matrix source  $B = 1$ ). The answer is given by (4.6) with  $\mathcal{N}_0 = 2$  and  $B_1 = B_2 = 1$  and coincide with the partition function of the Yang-Mills theory on the sphere. The two punctures are the two “ $\Omega$  points”. In summary, *in the case of a target space representing a sphere with two punctures our theory is exactly equivalent<sup>1</sup> to chiral  $YM_2$* . For  $G \geq 2$  however, it is not possible to eliminate the extra factor ( the “ $\Omega^{-1}$  points”) by adding punctures. On the other hand, the “ $\Omega^{-1}$  factors” have been given an interesting re-interpretation in the work of Cordes, Moore and Ramgoolam [7]. These authors showed that the  $YM_2$  partition function could be rewritten as a sum over covering maps *without* non-movable, special singularities, but weighted instead with special topological invariants (the “Euler character of Hurwitz moduli space”). It would be interesting if this result could be reproduced by a simple matrix model, similar in spirit to (2.4) and possessing an equally clear surface interpretation.

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<sup>1</sup> Adding a puncture to the target space of the Yang-Mills theory does not change anything. For example, the gauge theory defined on a cylinder with Dirichlet boundaries is identical to the gauge theory on the sphere.

## 6. Sphere-to-sphere maps: saddle point analysis

In the case of a target space of spherical topology  $G = 0$  we can apply a saddle point technique to the sum over representations to extract the contribution from coverings with spherical topology (i.e.  $g = 0$ ). This was explained in detail in [3],[6], and leads in the case of the model with fixed branch points (3.6) to the general Riemann-Hilbert problem discussed in [6]. Fortunately, the case of movable branch points (4.2) is much simpler. Introducing a continuous coordinate  $h = \frac{1}{N}h_i$  and a density  $\rho(h) = \partial i / \partial h_i$ , one finds, with the help of (4.4),

$$\int_b^a dh' \frac{\rho(h')}{h-h'} = \log(h-b) + \frac{1}{2}A_T + \frac{1}{2}\tilde{V}'(h), \quad (6.1)$$

where the effective potential  $\tilde{V}'(h)$  is, for a finite number of non-zero  $\tau_n$ 's, a polynomial in  $h$  whose coefficients are selfconsistently dependent on the first few moments  $H_n = N^{-1-n} \sum h_i^n$  of the resolvent  $H(h) = \int_0^a dh' \frac{\rho(h')}{h-h'}$ . It is found as follows. Define the potential

$$V(h) = \sum_{k=2}^{\infty} \frac{1}{k} \tau_k h^k. \quad (6.2)$$

Then

$$\tilde{V}'(h) = \frac{1}{2\pi i} \oint \frac{ds}{(h-s)^2} V(s e^{-H(s)}). \quad (6.3)$$

Here the contour surrounds the cut  $[b, a]$  of  $H(h)$ . The simplest non-trivial example consists of taking  $\tau_k = 0$  for  $k \geq 3$ . The analysis then becomes very similar to the one for chiral  $YM_2$ , obtained in [8]. Eq.(6.1) then reads:

$$\int_b^a \frac{dh' \rho(h')}{h-h'} = \ln(h-b) + \frac{A_T}{2} + \frac{\tau_2}{2}(1-h). \quad (6.4)$$

The term  $\log(h-b)$  is a consequence<sup>2</sup> of the fact that the  $h_i$  are a set of ordered integers. The density is therefore constrained to have a maximum value of 1 (see [6]). This equation is solved in the standard way by a contour integral

$$\begin{aligned} H(h) &= \int_0^a \frac{dh' \rho(h')}{h-h'} \\ &= \ln\left(\frac{h}{h-b}\right) + \sqrt{(h-a)(h-b)} \oint \frac{ds}{2\pi i} \frac{\ln(s-b) + \frac{A_T}{2} + \frac{\tau_2}{2}(1-h)}{(h-s)\sqrt{(s-a)(s-b)}}. \end{aligned} \quad (6.5)$$

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<sup>2</sup> A brief argument shows that for the phase corresponding to the sum over surfaces, a part of the density, starting at the origin and finishing at a point  $b$ , is saturated at its maximum value. The sum over coverings is an expansion in powers of  $e^{-A_T}$  and  $\tau_2$ . For small  $\tau_2$  and small  $e^{-A_T}$  i.e. large  $A_T$  the  $e^{-A_T} \sum_i h_i$  term in (4.2) attracts all the  $h_i$  towards the origin saturating the constraint  $h_{i+1} < h_i$ ,  $\rho(h) \leq 1$ .

Expanding the contour out to infinity we catch a pole at  $s = h$ , the discontinuity across the cut of the logarithm, and a contribution from the contour at infinity. The final result is

$$H(h) = \ln h + \frac{A_T}{2} - \frac{\tau_2}{2} + \frac{\tau_2}{2}(h - \sqrt{(h-a)(h-b)}) - 2 \ln \left[ \frac{\sqrt{h-a} + \sqrt{h-b}}{\sqrt{a-b}} \right]. \quad (6.6)$$

The density  $\rho(h)$  is as given by the discontinuity of  $H(h)$  across its cut: <sup>3</sup>

$$\rho(h) = \frac{2}{\pi} \cos^{-1} \left( \sqrt{\frac{h-b}{a-b}} \right) - \frac{\tau_2}{2} \sqrt{(h-a)(h-b)}, \quad (6.7)$$

and the cut points  $a$  and  $b$  are determined from the behaviour of  $H(h)$  for large  $h$ :

$$H(h) = \frac{1}{h} + \mathcal{O}\left(\frac{1}{h^2}\right). \quad (6.8)$$

Imposing this asymptotic behaviour on (6.6) leads to the two equations

$$\chi = \tau_2^2 e^{-A_T} e^\chi \quad \text{and} \quad \eta - \tau_2 e^{-A_T + \chi} = 1. \quad (6.9)$$

where we have defined  $\chi = A_T + 2 \log((a-b)/4)$  and  $\eta = (a+b)/2$ . Differentiating (4.2) w.r.t. to  $A_T$  leads to  $\frac{\partial}{\partial A} \mathcal{F} = - \langle h \rangle + \frac{1}{2}$  where the free energy  $\mathcal{F}$  is defined here by  $\mathcal{F} = 1/N^2 \ln \mathcal{Z}$ . The expectation value  $\langle h \rangle = \int_0^a dh \rho(h) h$  can be calculated from the expansion for  $H(h)$  (6.6). Using (6.9) it is possible to integrate up to obtain an explicit expression for the spherical contribution to  $\mathcal{F}$

$$\mathcal{F} = e^{-A_T + \chi} \left( 1 - \frac{3}{4} \chi + \frac{1}{6} \chi^2 \right). \quad (6.10)$$

From (6.9) it is clear that  $\chi$  is a power series in  $\tau_2^2 e^{-A_T}$ . We can thus perform a standard Lagrange inversion on (6.9),(6.10), and obtain after a short calculation:

$$\mathcal{F} = \sum_{n=1}^{\infty} \frac{n^{n-3}}{n!} \tau_2^{2n-2} e^{-nA_T}. \quad (6.11)$$

This result was first obtained in [9],[8]. To discuss the convergence properties of (6.11), it is natural to take the continuum coupling proportional to  $A_T$ , since the branch points can be located anywhere on the manifold (see also (5.1)):  $\tau_2 = tA_T$ . Note that the series is only convergent for  $t^2 A_T^2 e^{-A_T} < e^{-1}$ . Beyond this point the boundary conditions (6.9)

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<sup>3</sup> Specifically  $\rho(h) = \frac{i}{2\pi} (H(h+i\epsilon) - H(h-i\epsilon))$  where  $\epsilon$  is a small positive real number.

lead to a non-physical complex value for  $\chi$ . We see that the sum over branched coverings is convergent for both large and small areas. For large enough  $t$  and intermediate values of the area  $A_T$ , however, the entropy of the branch points is sufficient to cause the sum to diverge. It is interesting to understand this divergence in terms of the Young tableau density  $\rho(h)$ . Along the critical line  $(\tau_2, A_T)$  and  $\tau_2 > 0$  the density becomes flat at its upper end point  $a$ , i.e. the singularity at the end point changes from  $\rho(h) \sim (a - h)^{1/2}$  to  $\rho(h) \sim (a - h)^{3/2}$ . Along the critical line  $(\tau_2, A_T)$  and  $\tau_2 < 0$  it is the singularity of the density at the point  $b$  that changes from  $1/2$  to  $3/2$ . This is just as occurs in matrix models of pure  $2D$  gravity, and indeed for  $t^2 A^2 e^{-A_T} \sim e^{-1}$  the free energy (6.11) behaves as

$$\mathcal{F} \sim (e^{-1} - t^2 A_T^2 e^{-A_T})^{5/2}. \quad (6.12)$$

So far in our analysis we have ignored the constraint that we are summing over positive Young tableaux i.e. that  $b > 0$ . Since  $b = 0$  is not a singular point in the boundary conditions (6.9) it does not correspond to a singularity in the sum over surfaces. If, however, we take the sum over representations (4.2) as fundamental (as would be the case for QCD<sub>2</sub>) then one should take this constraint into account.

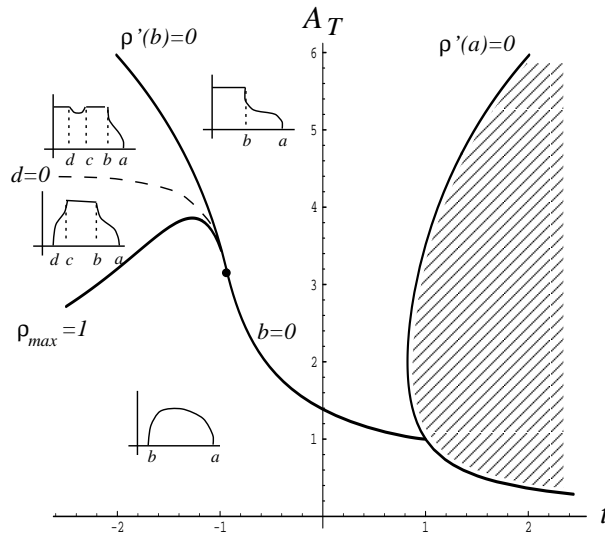


Fig 2. Phase diagram in the  $t, A_T$  plane.  
A typical density,  $\rho(h)$ , is sketched in each phase.

Setting  $b = 0$  in (6.9) leads to the pair of equations

$$t^2 A^2 e^{-A_T} = \chi e^{-\chi} \quad \text{and} \quad A_T = \chi + 2 \ln(2 - \sqrt{\chi}). \quad (6.13)$$

which determine  $t$  and  $A_T$  parametrically in terms of  $\chi$ . This curve is plotted in Fig. 2. It connects the dot on the left to the shaded area on the right. Immediately below this line the support of the density is entirely positive, i.e. it starts at a positive value of  $h$ . In addition it is less than one for the full range of its support, see Fig. 2. The phase transition across this line is exactly analogous to the large  $N$  phase transition that occurs in QCD (see [10][11]). There there is a phase transition separating a strong coupling regime from a weak coupling phase, and as is the case here, there is no singularity in either phase indicating the the transition point. Indeed it is possible to formulate the Gross-Witten [10] and Brézin-Gross [11] models in terms of a sum over representations [12] and the large  $N$  phase transition is precisely the point at which the density just begins to touch (or pull away from) the origin. Further analysis shows that there are two further phases for the model, as first observed in [8]. There is one phase where the density has an entirely positive support but attains its maximal value  $\rho(h) = 1$  over a single finite interval, and another phase where the density starts at the origin and has two separated intervals where  $\rho(h) = 1$ . Both of these phases involve two separated nontrivial cuts and can be calculated in terms of elliptic functions. We do not present here explicit results for any of these extra phases. The complete phase diagram for the partition function is shown in Fig. 2. In each phase is sketched a typical density. Along the transition lines are indicated the corresponding critical behaviours of the density. The exact position of the line along which the point  $d$  equals zero has not been calculated. We indicate this by the using a dotted line. The sum over representations is thus seen to have a much richer phase structure than the perturbative sum over surfaces it generates.

## 7. Concluding remarks

We have introduced a new class of matrix gauge models which solve the general problem of counting branched covers of orientable two-dimensional manifolds with specified branch point structure. The result is given as a weighted sum over polynomial representations of  $U(N)$ . We conjecture that a similar methodology could be applied to non-orientable surfaces by replacing the complex matrices by real matrices and the group  $U(N)$  by  $O(N)$ .

Our approach actually treats also the case of manifolds with boundaries. Indeed, due to the invariance with respect to area-preserving diffeomorphisms, the problem does not depend on the length of the boundaries and the latter can be considered as punctures.

As an interesting by-product of our investigation, we have derived in a constructive and transparent fashion important features of the original Gross-Taylor interpretation of  $YM_2$  as a string theory.

An interesting problem which we did not consider in this paper is the calculation of Wilson loops. The Wilson loop functional defined as the sum of all nonfolding branched surfaces spanning a closed contour  $\mathcal{C} \in \mathcal{M}_G$  should satisfy a set of loop equations with a contact term similar to those considered in [13].

## Appendix A. Free energies for $\mathcal{N}_0$ fixed, identical branch points

In this appendix we give some examples of how to extract the combinatorics of connected coverings of a smooth manifold with  $\mathcal{N}_0$  fixed, identical branch points. The partition sum (3.6) for this case becomes

$$Z = \sum_h \left( \frac{\Delta_h}{\Omega_h} \right)^{2-2G} \left[ \frac{\chi_h(B)}{\chi_h(A_1)} \right]^{\mathcal{N}_0} e^{-|h|A_T}. \quad (\text{A.1})$$

Define

$$Z = 1 + \sum_{n=1}^{\infty} Z_n e^{-nA_T}. \quad (\text{A.2})$$

Using the Frobenius formula for characters, the first few  $Z_n$ 's, up to order four, are found to be:

$$\begin{aligned} Z_1 &= N^{2-2G} t_1^{\mathcal{N}_0} \\ Z_2 &= \left(\frac{1}{2}N^2\right)^{2-2G} \left[ (t_1^2 + \frac{1}{N}t_2)^{\mathcal{N}_0} + (t_1^2 - \frac{1}{N}t_2)^{\mathcal{N}_0} \right] \\ Z_3 &= \left(\frac{1}{6}N^3\right)^{2-2G} \left[ (t_1^3 + \frac{3}{N}t_1t_2 + \frac{2}{N^2}t_3)^{\mathcal{N}_0} + (t_1^3 - \frac{3}{N}t_1t_2 + \frac{2}{N^2}t_3)^{\mathcal{N}_0} \right] + \\ &\quad + \left(\frac{1}{3}N^3\right)^{2-2G} \left[ (t_1^3 - \frac{1}{N^2}t_3)^{\mathcal{N}_0} \right] \\ Z_4 &= \left(\frac{1}{24}N^4\right)^{2-2G} \left[ (t_1^4 + \frac{6}{N}t_1^2t_2 + \frac{8}{N^2}t_1t_3 + \frac{3}{N^2}t_2^2 + \frac{6}{N^3}t_4)^{\mathcal{N}_0} + \right. \\ &\quad \left. + (t_1^4 - \frac{6}{N}t_1^2t_2 + \frac{8}{N^2}t_1t_3 + \frac{3}{N^2}t_2^2 - \frac{6}{N^3}t_4)^{\mathcal{N}_0} \right] + \\ &\quad + \left(\frac{1}{8}N^4\right)^{2-2G} \left[ (t_1^4 + \frac{2}{N}t_1^2t_2 - \frac{1}{N^2}t_2^2 - \frac{2}{N^3}t_4)^{\mathcal{N}_0} + (t_1^4 - \frac{2}{N}t_1^2t_2 - \frac{1}{N^2}t_2^2 + \frac{2}{N^3}t_4)^{\mathcal{N}_0} \right] + \\ &\quad + \left(\frac{1}{12}N^4\right)^{2-2G} \left[ (t_1^4 - \frac{4}{N^2}t_1t_3 + \frac{3}{N^2}t_2^2)^{\mathcal{N}_0} \right]. \end{aligned} \quad (\text{A.3})$$

These expressions allow us to obtain explicit results for the map counting problem. The free energy counting connected surfaces is

$$F = \log Z = \sum_{n=1}^{\infty} F_n e^{-nA_T}. \quad (\text{A.4})$$

Let us present some explicit low order results:



$G = 0$ :

$$\begin{aligned}
F_1 &= N^2 t_1^{\mathcal{N}_0} \\
F_2 &= \sum_{g=0}^{[\frac{1}{2}\mathcal{N}_0]-1} N^{2-2g} \frac{1}{2} \binom{\mathcal{N}_0}{2g+2} (t_1^2)^{\mathcal{N}_0-2g-2} t_2^{2g+2} \\
F_3 &= N^2 \left[ 4 \binom{\mathcal{N}_0}{4} t_1^{3\mathcal{N}_0-8} t_2^4 + \binom{\mathcal{N}_0}{(2,1)} t_1^{3\mathcal{N}_0-7} t_2^2 t_3 + \frac{1}{3} \binom{\mathcal{N}_0}{2} t_1^{3\mathcal{N}_0-6} t_3^2 \right] + \\
&\quad + N^0 \left[ 40 \binom{\mathcal{N}_0}{6} t_1^{3\mathcal{N}_0-12} t_2^6 + \frac{3}{2} \binom{\mathcal{N}_0}{(2,2,1)} t_1^{3\mathcal{N}_0-11} t_2^4 t_3 + \right. \\
&\quad \left. + 2 \binom{\mathcal{N}_0}{(2,2)} t_1^{3\mathcal{N}_0-10} t_2^2 t_3^2 + \frac{1}{3} \binom{\mathcal{N}_0}{3} t_1^{3\mathcal{N}_0-9} t_3^3 \right] + \mathcal{O}(N^{-2}) \\
F_4 &= N^2 \left[ \left( 124 \binom{\mathcal{N}_0}{6} + 13 \binom{\mathcal{N}_0}{(1,4)} + \frac{5}{4} \binom{\mathcal{N}_0}{(2,2)} + \frac{5}{16} \binom{\mathcal{N}_0}{3} - \frac{1}{16} \binom{2\mathcal{N}_0}{6} \right) t_1^{4\mathcal{N}_0-12} t_2^6 + \right. \\
&\quad + \left( 27 \binom{\mathcal{N}_0}{(1,4)} + 3 \binom{\mathcal{N}_0}{(1,1,2)} \right) t_1^{4\mathcal{N}_0-11} t_2^4 t_3 + \\
&\quad + \left( 6 \binom{\mathcal{N}_0}{(2,2)} + \binom{\mathcal{N}_0}{(2,1)} \right) t_1^{4\mathcal{N}_0-10} t_2^2 t_3^2 + \\
&\quad + \binom{\mathcal{N}_0}{(1,1,1)} t_1^{4\mathcal{N}_0-9} t_2 t_3 t_4 + \binom{\mathcal{N}_0}{3} t_1^{4\mathcal{N}_0-9} t_3^3 + \frac{1}{4} \binom{\mathcal{N}_0}{2} t_1^{4\mathcal{N}_0-8} t_4^2 + \\
&\quad \left. + \left( 4 \binom{\mathcal{N}_0}{(3,1)} + \frac{1}{2} \binom{\mathcal{N}_0}{(1,1,1)} \right) t_1^{4\mathcal{N}_0-10} t_2^3 t_4 \right] + \mathcal{O}(N^0).
\end{aligned} \tag{A.5}$$

$G = 1$ :

$$\begin{aligned}
F_1 &= t_1^{\mathcal{N}_0} \\
F_2 &= \frac{3}{2} t_1^{2\mathcal{N}_0} + \sum_{g=2}^{[\frac{1}{2}\mathcal{N}_0]+1} N^{2-2g} 2 \binom{\mathcal{N}_0}{2g-2} (t_1^2)^{\mathcal{N}_0+2-2g} t_2^{2g-2} \\
F_3 &= \frac{4}{3} t_1^{3\mathcal{N}_0} + N^{-2} \left[ 16 \binom{\mathcal{N}_0}{2} t_1^{3\mathcal{N}_0-4} t_2^2 + 3\mathcal{N}_0 t_1^{3\mathcal{N}_0-3} t_3 \right] + \mathcal{O}(N^{-4}) \\
F_4 &= \frac{7}{4} t_1^{4\mathcal{N}_0} + N^{-2} \left[ \left( 7\mathcal{N}_0 + 60 \binom{\mathcal{N}_0}{2} \right) t_1^{4\mathcal{N}_0-4} t_2^2 + 9\mathcal{N}_0 t_1^{4\mathcal{N}_0-3} t_3 \right] + \mathcal{O}(N^{-4}).
\end{aligned} \tag{A.6}$$

$G = 2$ :

$$\begin{aligned}
F_1 &= N^{-2} t_1^{\mathcal{N}_0} \\
F_2 &= N^{-4} \frac{15}{2} t_1^{2\mathcal{N}_0} + \sum_{g=4}^{[\frac{1}{2}\mathcal{N}_0]+3} N^{2-2g} 8 \binom{\mathcal{N}_0}{2g-6} (t_1^2)^{\mathcal{N}_0+6-2g} t_2^{2g-6} \\
F_3 &= N^{-6} \frac{220}{3} t_1^{3\mathcal{N}_0} + N^{-8} \left[ 640 \binom{\mathcal{N}_0}{2} t_1^{3\mathcal{N}_0-4} t_2^2 + 135 \mathcal{N}_0 t_1^{3\mathcal{N}_0-3} t_3 \right] + \mathcal{O}(N^{-10}) \\
F_4 &= N^{-8} \frac{5275}{4} t_1^{4\mathcal{N}_0} + N^{-10} \left[ \left( 3760 \mathcal{N}_0 + 41280 \binom{\mathcal{N}_0}{2} \right) t_1^{4\mathcal{N}_0-4} t_2^2 + 8505 \mathcal{N}_0 t_1^{4\mathcal{N}_0-3} t_3 \right] + \\
&\quad + \mathcal{O}(N^{-12}).
\end{aligned} \tag{A.7}$$

Here we have employed the standard notation for binomial and multinomial coefficients. It is instructive to draw the Riemann surfaces corresponding to the various terms in (A.5), (A.6), (A.7). As will be seen, the combinatorics involved increases rapidly in complexity.

The above examples are easily checked to be in agreement with the Riemann-Hurwitz formula. Of course, this formula only gives a *necessary* condition for the existence of a Riemann surface. The above results can be used to decide whether a Riemann surface of given  $G, g$  and branch point structure  $t_2^{\mathcal{N}_2} t_3^{\mathcal{N}_3} \dots$  actually *exists*. For example, we see from  $F_4$  in (A.5) that there exists a fourfold cover of the sphere by a sphere with exactly six simple branch points, where two branchpoints are located at each of three locations of the target manifold (the associated symmetry factor is  $\frac{5}{16}$ ). As a second example, we see from  $F_3$  in (A.6) that there exists a triple cover of the torus by a double torus with exactly one branch point of order 2 (the associated symmetry factor is 3).

## Appendix B. Free energies for an arbitrary number of movable branch points

In this appendix we give some concrete examples of how the maps are counted in the continuum limit we defined above. Define

$$\mathcal{Z} = 1 + \sum_{n=1}^{\infty} \mathcal{Z}_n e^{-nA_T}. \tag{B.1}$$

The first few auxiliary  $\xi_k^h$  (see (4.4)) are:

$$\begin{aligned}
\xi_1^h &= \sum_i h_i - \frac{1}{2}N(N-1) \\
\xi_2^h &= \frac{1}{2} \sum_i h_i^2 - \frac{2N-1}{2} \sum_i h_i + \frac{N(N-1)(2N-1)}{6} \\
\xi_3^h &= \frac{1}{3} \sum_i h_i^3 - \frac{1}{2} \sum_i h_i \sum_j h_j - \frac{2N-1}{2} \sum_i h_i^2 + \\
&\quad + \frac{1}{3} \left( \frac{9}{2}N^2 - \frac{9}{2}N + 2 \right) \sum_i h_i - \frac{N(N-1)(3N^2 - 3N + 2)}{8}
\end{aligned} \tag{B.2}$$

They are easily computed from the following representation of eq.(4.4):

$$\xi_n^h = \frac{1}{n^2} \oint \frac{dh}{2\pi i} h(h-1)\dots(h-n+1) \exp \left[ \sum_{p=1}^{\infty} \frac{(-n)^p}{p!} \left( \frac{\partial}{\partial h} \right)^{p-1} H(h) \right], \tag{B.3}$$

where we have introduced  $H(h) = \sum_{i=1}^N \frac{1}{h-h_i}$ . The first few  $\mathcal{Z}_n$ 's, up to order four, are:

$$\begin{aligned}
\mathcal{Z}_1 &= N^{2-2G} \\
\mathcal{Z}_2 &= \left( \frac{1}{2}N^2 \right)^{2-2G} \left( e^{\frac{1}{N}\tau_2} + e^{-\frac{1}{N}\tau_2} \right) \\
\mathcal{Z}_3 &= \left( \frac{1}{6}N^3 \right)^{2-2G} \left( e^{\frac{3}{N}\tau_2 + \frac{2}{N^2}\tau_3} + e^{-\frac{3}{N}\tau_2 + \frac{2}{N^2}\tau_3} \right) + \\
&\quad + \left( \frac{1}{3}N^3 \right)^{2-2G} e^{-\frac{1}{N^2}\tau_3} \\
\mathcal{Z}_4 &= \left( \frac{1}{24}N^4 \right)^{2-2G} \left( e^{\frac{6}{N}\tau_2 + \frac{8}{N^2}\tau_3 + \frac{6}{N^3}\tau_4} + e^{-\frac{6}{N}\tau_2 + \frac{8}{N^2}\tau_3 - \frac{6}{N^3}\tau_4} \right) + \\
&\quad + \left( \frac{1}{8}N^4 \right)^{2-2G} \left( e^{\frac{2}{N}\tau_2 - \frac{2}{N^3}\tau_4} + e^{-\frac{2}{N}\tau_2 + \frac{2}{N^3}\tau_4} \right) + \\
&\quad + \left( \frac{1}{12}N^4 \right)^{2-2G} e^{-\frac{4}{N^2}\tau_3}.
\end{aligned} \tag{B.4}$$

This gives the following continuum free energies (as can be checked easily from the results of appendix A):

$$\begin{aligned}
G &= 0: \\
\mathcal{F}_1 &= N^2 \\
\mathcal{F}_2 &= \sum_{g=0}^{\infty} N^{2-2g} \frac{1}{2(2g+2)!} \tau_2^{2g+2} \\
\mathcal{F}_3 &= N^2 \left[ \frac{1}{6}\tau_2^4 + \frac{1}{2}\tau_2^2\tau_3 + \frac{1}{6}\tau_3^2 \right] + \\
&\quad + \left[ \frac{1}{18}\tau_2^6 + \frac{3}{8}\tau_2^4\tau_3 + \frac{1}{2}\tau_2^2\tau_3^2 + \frac{1}{18}\tau_3^3 \right] + \mathcal{O}(N^0) \\
\mathcal{F}_4 &= N^2 \left[ \frac{1}{6}\tau_2^6 + \frac{27}{24}\tau_2^4\tau_3 + \frac{3}{2}\tau_2^2\tau_3^2 + \tau_2\tau_3\tau_4 + \frac{1}{6}\tau_3^3 + \frac{2}{3}\tau_2^3\tau_4 + \frac{1}{8}\tau_4^2 \right] + \mathcal{O}(N^0).
\end{aligned} \tag{B.5}$$

$G = 1:$

$$\begin{aligned}
\mathcal{F}_1 &= 1 \\
\mathcal{F}_2 &= \frac{3}{2} + \sum_{g=2}^{\infty} N^{2-2g} \frac{2}{(2g-2)!} \tau_2^{2g-2} \\
\mathcal{F}_3 &= \frac{4}{3} + N^{-2} [8\tau_2^2 + 3\tau_3] + \mathcal{O}(N^{-4}) \\
\mathcal{F}_4 &= \frac{7}{4} + N^{-2} [30\tau_2^2 + 9\tau_3] + \mathcal{O}(N^{-4}).
\end{aligned} \tag{B.6}$$

$G = 2:$

$$\begin{aligned}
\mathcal{F}_1 &= N^{-2} \\
\mathcal{F}_2 &= N^{-4} \frac{15}{2} + \sum_{g=4}^{\infty} N^{2-2g} \frac{8}{(2g-6)!} \tau_2^{2g-6} \\
\mathcal{F}_3 &= N^{-6} \frac{220}{3} + N^{-8} [320\tau_2^2 + 135\tau_3] + \mathcal{O}(N^{-10}) \\
\mathcal{F}_4 &= N^{-8} \frac{5275}{4} + N^{-10} [20640\tau_2^2 + 8505\tau_3] + \mathcal{O}(N^{-12}).
\end{aligned} \tag{B.7}$$

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