ONE ITERATION METHOD FOR SOLVING CONVENTIONAL DIFFERENTIAL EQUATIONS

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1. Let us examine the following problem:

$$\frac{dx}{dt} = F(t, x, x), \quad x(0) = x_0, \tag{1}$$

where the function $F(t, x, y)$ is determined in $R = [0, T] \times [x_0 - r, x_0 + r] \times [x_0 - r, x_0 + r]$ and is continuous in terms of the combination of variables.

We shall assume that the problem

$$\frac{dx}{dt} = F[t, \eta(t), x], \quad x(0) = x_0 \tag{2}$$

has a single solution $\tilde{x}(t) (|\tilde{x} - x_0| \leqslant r)$ at any fixed continuous function $\eta(t) (|\eta - x_0| \leqslant r)$ and it may be found.

To solve problem (1), we shall construct the sequence of $x_n(t)$ functions in the following way:

$$\left.\begin{aligned} & x_0(t) = x_0, \quad x_n(0) = x_0, \\ & x_n'(t) = F\left[t, x_{n-1}(t), x_n(t)\right] \quad (n = 1, 2, \ldots). \end{aligned}\right\} \tag{3}$$

We shall prove the following theorems which give sufficient conditions for the convergence of the constructed approximations (3) to the solution of problem (1).

T h e o r e m 1. Let the function $F(t, x, y)$ be defined in R, continuous in terms of the combination of variables and satisfy the condition

$$|F(t, \bar{x}, \bar{y}) - F(t, x, y)| \leqslant \varphi(t, |\bar{x} - x|, |\bar{y} - y|),$$

where the function $\varphi(t, u, v)$ is defined at $0 \leqslant t \leqslant T;\ 0 \leqslant u, v \leqslant 2r$, is continuous in terms of the combination of variables and does not disappear in terms of u. Moreover, it is assumed that, at any fixed continuous function $\beta(t)$ $(0 \leqslant t \leqslant T; \quad 0 \leqslant \beta(t) \leqslant 2r)$, the problem

$$\frac{du}{dt} = \varphi\left[t, \beta(t), u\right], \quad u(0) = 0 \tag{4}$$

has a single solution from $[0, 2r]$ and the problem

$$\frac{du}{dt} = \varphi(t, u, u), \quad u(0) = 0 \tag{5}$$

has only a zero solution.

Then problem (1) has a single solution and this solution is the boundary (uniformly in t) of approximations (3). The rate of convergence of $x_n(t)$ to the solution $x(t)$ is determined from the inequality

$$|x_n(t) - x(t)| \leqslant \varepsilon_n(t), \tag{6}$$

where

$$\left.\begin{aligned} &\varepsilon_0(t) = 2r, \\ &\varepsilon_n'(t) = \varphi\left[t,\ \varepsilon_{n-1}(t),\ \varepsilon_n(t)\right], \\ &\varepsilon_n(0) = 0 \quad (n = 1,\ 2,\ \ldots). \end{aligned}\right\} \tag{7}$$

P r o o f. The existence and singleness of the solution to problem (1) are clear. Let $v_n(t) = |x_n(t) - x(t)|$, where $x_n(t)$ — is the sequence of functions defined by equalities (3) and $x(t)$ is the solution to problem (1). Then we have

$$D_* v_n(t) \leqslant |x_n'(t) - x'(t)| = \left|F\left[t,\ x_{n-1}(t),\ x_n(t)\right] - F\left[t,\ x(t), x(t)\right]\right|, \tag{8}$$
$$D_* v_n(t) \leqslant \varphi\left[t,\ v_{n-1}(t),\ v_n(t)\right],\quad v_n(0) = 0.$$

Let us assume that $n = 1$, Then

$$D_* v_1(t) \leqslant \varphi\left[t,\ v_0(t),\ v_1(t)\right] \leqslant \varphi\left[t,\ \varepsilon_0(t),\ v_1(t)\right],\ v_1(0) = 0.$$

By applying the theorem of differential inequalities and considering that $\varepsilon_1(t)$ is the only solution to problem

$$\frac{du}{dt} = \varphi\left[t,\ \varepsilon_0(t),\ u\right],\ u(0) = 0,$$

we obtain $v_1(t) \leqslant \varepsilon_1(t)$. Let us assume that $v_{n-1}(t) \leqslant \varepsilon_{n-1}(t)$; then from inequality (8),

$$D_* v_n(t) \leqslant \varphi\left[t,\ \varepsilon_{n-1}(t),\ v_n(t)\right],\quad v_n(0) = 0.$$

By again applying the theorem of differential inequalities and considering that $\varepsilon_n(t)$ is the only solution to the problem

$$\frac{du}{dt} = \varphi\left[t,\ \varepsilon_{n-1}(t),\ u\right],\ u(0) = 0,$$

we obtain $v_n(t) \leqslant \varepsilon_n(t)$.

We have thus proved that inequality (6) occurs at al $n = 1,\ 2,\ \ldots$

It is easy to check that $\{\varepsilon_n(t)\}$ does not increase and $\varepsilon_n(t) \geqslant 0$, therefore it converges. From (7) we have

$$\varepsilon_n(t) = \int_0^t \varphi\left[s,\ \varepsilon_{n-1}(s),\ \varepsilon_n(s)\right] ds.$$

By crossing to the limit in these inequalities, we obtain

$$\varepsilon(t) = \int_0^t \varphi\left[s,\ \varepsilon(s),\ \varepsilon(s)\right] ds,$$

where $\varepsilon(t) = \lim_{n\to\infty} \varepsilon_n(t)$. Consequently, $\varepsilon(t)$ is the solution to problem (5). As $\varepsilon(t) = 0$, therefore $\lim_{n\to\infty} \varepsilon_n(t) = 0$. According to the Dini theorem, this convergence will be uniform in terms of t. The theorem is completely proved.

We now give other sufficient conditions for the convergence of the successive approximations (3).

T h e o r e m 2. Let the function $F(t, x, y)$ be defined in R, continuous in the combination of variables and satisfy the conditions

$$\begin{aligned} &|F(t, \bar{x}, y) - F(t, x, y)| \leqslant \varphi_1(t, |\bar{x} - x|), \\ &\operatorname{sgn}(\bar{y} - y)\left[F(t, x, \bar{y}) - F(t, x, y)\right] \leqslant \varphi_2(t, |\bar{y} - y|), \end{aligned} \tag{9}$$

where the functions $\varphi_i(t, u)$ $(0 \leqslant t \leqslant T;\ 0 \leqslant u \leqslant 2r;\ i = 1, 2)$ are continuous in the combination of variables and $\varphi_1(t, u)$ does not disappear in terms of u.

Let the problem

$$\frac{du}{dt} = \varphi_2(t, u) + \alpha(t),\ u(0) = 0 \tag{10}$$

for any fixed continuous function $\alpha(t)$ $(0 \leqslant \alpha(t) \leqslant N,\ N = \max_{0<t<T} \varphi_1(t, 2r))$ have a single solution from $[0, 2r]$, and the problem

$$\frac{du}{dt} = \varphi_1(t, u) + \varphi_2(t, u),\ u(0) = 0 \tag{11}$$

has only a zero solution.

Then problem (1) has a single solution: this solution may be found by means of successive approximations (3) and the convergence rate is defined as follows:

$$|x_n(t) - x(t)| \leqslant \varepsilon_{n-1}(t),$$

where

$$\left.\begin{aligned} &\varepsilon_0'(t) = \varphi_2[t, \varepsilon_0(t)] + N, \quad \varepsilon_0(0) = 0, \\ &\varepsilon_n'(t) = \varphi_2[t, \varepsilon_n(t)] + \varphi_1[t, \varepsilon_{n-1}(t)], \\ &\varepsilon_n(0) = 0 \quad (n = 1, 2, \ldots). \end{aligned}\right\} \quad (12)$$

P r o o f. Let us introduce the notation $v_n(t) = |x_n(t) - x(t)|$, where $x_n(t)$ is the sequence of functions defined by equalities (3), $x(t)$ —is the solution to problem (1) (the existence and singleness of $x(t)$ are easily proven) and then we have

$$D_* v_n(t) \leqslant \operatorname{sgn}(x_n - x)(x_n' - x') = \operatorname{sgn}(x_n - x)[F(t, x_{n-1}, x_n) - F(t, x, x)] =$$
$$= \operatorname{sgn}(x_n - x)[F(t, x_{n-1}, x) - F(t, x, x)] + \operatorname{sgn}(x_n - x) \times$$
$$\times [F(t, x_{n-1}, x_n) - F(t, x_{n-1}, x)] \leqslant |F(t, x_{n-1}, x) - F(t, x, x)| +$$
$$+ \varphi_2(t, |x_n - x|) \leqslant \varphi_1(t, |x_{n-1} - x|) + \varphi_2(t, |x_n - x|),$$
$$D_* v_n(t) \leqslant \varphi_1[t, v_{n-1}(t)] + \varphi_2[t, v_n(t)], \quad v_n(0) = 0.$$

It is easy to prove that it follows from these inequalities that $v_n(t) \leqslant \varepsilon_{n-1}(t)$, i.e. $|x_n(t) - x(t)| \leqslant \varepsilon_{n-1}(t)$, where $\varepsilon_{n-1}(t)$ is determined from (12). As problem (11) has only a zero solution according to the condition of the theorem and $\lim\limits_{n\to\infty} \varepsilon_n(t) = \varepsilon(t)$ (this is because $\{\varepsilon_n(t)\}$ does not increase and $\varepsilon_n(t) \geqslant 0$), therefore, by crossing to the limit in the inequalities

$$\varepsilon_n(t) = \int_0^t \{\varphi_2[s, \varepsilon_n(s)] + \varphi_1[s, \varepsilon_{n-1}(s)]\}\, ds,$$

we find that $\varepsilon(t) = 0$, and consequently $\lim\limits_{n\to\infty} x_n(t) = x(t)$ is uniform in terms of $t \in [0, T]$.

2. Now let

$$MT \leqslant r, \quad M = \max_{(t,x,y)\in R} |F(t, x, y)| \quad (13)$$

and we define the successive approximations for the solution to problem (1) as follows:

$$\left.\begin{aligned} &x_0(t) = x_0 \ (0 \leqslant t \leqslant T),\ x_n(t) = x_0 \left(0 \leqslant t \leqslant \frac{T}{n}\right), \\ &x_n'(t) = F\left[t,\ x_{n-1}(t),\ x_n\left(t - \frac{T}{n}\right)\right] \left(\frac{T}{n} \leqslant t \leqslant T\right). \end{aligned}\right\} \quad (14)$$

From condition (13) it follows that $|x_n(t) - x_0| \leqslant r$, i.e. the inequalities (14) have a meaning.

T h e o r e m 3. Let all the conditions of theorem 2 be fulfilled, except that problem (10) has a single solution from $[0,\ 2r]$ for any fixed continuous function $\alpha(t)$ $(0 \leqslant \alpha(t) \leqslant N + \delta_1)$ and the sequence of functions $\varepsilon_n(t)$ is defined thus:

$$\left.\begin{aligned} &\varepsilon_0'(t) = \varphi_2[t,\ \varepsilon_0(t)] + N + \delta_1,\ \varepsilon_0(0) = 0, \\ &\varepsilon_n'(t) = \varphi_2[t,\ \varepsilon_n(t)] + \varphi_1[t,\ \varepsilon_{n-1}(t)] + \delta_n, \\ &\varepsilon_n(0) = 0 \quad (n = 1,\ 2,\ \ldots), \end{aligned}\right\} \quad (15)$$

where

$$\delta_n = \sup_{m > n} \max_{0 < t < T} \left| F[t,\ x_{m-1}(t),\ x_m(t)] - F\left[t,\ x_{m-1}(t),\ x_m\left(t - \frac{T}{m}\right)\right]\right|.$$

Then problem (1) has a single solution and this solution is the limit (uniform in t) of approximations (14) and

$$|x_n(t) - x(t)| \leqslant \varepsilon_{n-1}(t). \quad (16)$$

P r o o f. Let $v_n(t) = |x_n(t) - x(t)|$; then

$$\begin{aligned} D_* v_n(t) \leqslant \operatorname{sgn}[x_n(t) - x(t)] \Big\{ F\left[t,\ x_{n-1}(t),\ x_n\left(t - \frac{T}{n}\right)\right] - \\ - F[t,\ x(t),\ x(t)]\Big\} \leqslant \operatorname{sgn}[x_n(t) - x(t)] \times \\ \times \{F[t,\ x_{n-1}(t),\ x_n(t)] - F[t,\ x_{n-1}(t),\ x(t)]\} + \\ + |F[(t,\ x_{n-1}(t),\ x(t)] - F[t,\ x(t),\ x(t)]| + \\ + \left| F\left[t,\ x_{n-1}(t),\ x_n\left(t - \frac{T}{n}\right)\right] - F[t,\ x_{n-1}(t),\ x_n(t)]\right|, \\ D_* v_n(t) \leqslant \varphi_2[t,\ v_n(t)] + \varphi_1[t,\ v_{n-1}(t)] + \delta_n,\ v_n(0) = 0. \end{aligned} \quad (17)$$

Let us show that $v_n(t) \leqslant \varepsilon_{n-1}(t)$. Obviously, $v_1(t) \leqslant \varepsilon_0(t)$. In fact, from (17) we have

$$D_* v_1(t) \leqslant \varphi_2[t,\ v_1(t)] + \varphi_1[t,\ v_0(t)] + \delta_1,$$
$$D_* v_1(t) \leqslant \varphi_2[t,\ v_1(t)] + N + \delta_1,\ v_1(0) = 0.$$

By applying the theorem of differential inequalities, we obtain $v_1(t) \leqslant \varepsilon_0(t)$, as the problem

$$\frac{du}{dt} = \varphi_2(t, u) + N + \delta_1, \; u(0) = 0$$

has a single solution $\varepsilon_0(t)$.

Let $v_{n-1}(t) \leqslant \varepsilon_{n-2}(t)$. Then, considering that $\varphi_1(t, u)$ does not vanish in terms of u, we obtain from (17)

$$D_* v_n(t) \leqslant \varphi_2[t, v_n(t)] + \varphi_1[t, \varepsilon_{n-2}(t)] + \delta_{n-1}, \; v_n(0) = 0,$$

By applying once more the theorem of differential inequalities, we thus obtain $v_n(t) \leqslant \varepsilon_{n-1}(t)$. Inequalities (16) are thus proved.

It follows from the incontinuity $F(t, x, y)$ and from the equicontinuity of the series of functions $\{x_n(t)\}$ that $\lim\limits_{n\to\infty} \delta_n = 0$. Taking this into account, we obtain from (15) $\lim\limits_{n\to\infty} \varepsilon_n(t) = \varepsilon(t) = 0$. Crossing to the limit at $n \to \infty$ in (16), we confirm the theorem.

N o t e. We should point out that if condition (9) in theorem 3 is replaced by the condition

$$|F(t, x, \bar{y}) - F(t, x, y)| \leqslant \varphi_2(t, |\bar{y} - y|);$$

where $\varphi_2(t, u)$ does not vanish in terms of u, then the δ_n sequence may be determined in the following way:

$$\delta_n = \max_{0<t<T} \varphi_2\left(t, \frac{MT}{n}\right),$$

as

$$\left| x_n\left(t - \frac{T}{n}\right) - x_n(t) \right| \leqslant \frac{MT}{n}.$$

3. Let us now examine the problem

$$\frac{dx}{dt} = f(t, x), \; x(0) = x_0, \tag{18}$$

where the continuous function $f(t, x)$ is twice continuously differentiable in terms of x.

For an approximate solution to problem (18), let us construct successive approximations by the Newton-Kantorovich method:

$$\frac{dx_n}{dt} = f'_x(t, x_{n-1})(x_n - x_{n-1}) + f(t, x_{n-1}), \; x_n(0) = x_0. \tag{19}$$

We introduce the notation

$$F(t, x, y) = f'_x(t, x)(y - x) + f(t, x).$$

Then problem (18) and the successive approximations (19) for that problem may be written as follows:

$$\frac{dx}{dt} = F(t, x, x), \quad x(0) = x_0, \tag{20}$$

$$\frac{dx_n}{dt} = F(t, x_{n-1}, x_n), \; x_n(0) = x_0. \tag{21}$$

Thus, the problem of convergence of the Newton-Kantorovich approximations to the solution to problem (18) is reduced to a proof of the convergence of approximations (21) to the solution to problem (20). To do this, it is sufficient to prove that the function $F(t, x, y)$ meets all the conditions of one of the proven theorems 1 - 3.

We shall now construct the successive approximations for the solution of problem (18) as follows:

$$\frac{dx_n}{dt} = f'_x(t, x_0)(x_n - x_{n-1}) + f(t, x_{n-1}), \; x_n(0) = x_0.$$

This is a modification of the Newton-Kantorovich approximations for the solution to problem (18). In this case we must put

$$F(t, x, y) = f'_x(t, x_0)(y - x) + f(t, x).$$

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