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U-Invariants, Black-Hole Entropy and Fixed Scalars

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Abstract

The absolute (moduli-independent) U-invariants of all $N > 2$ extended supergravities at $D = 4$ are derived in terms of (moduli-dependent) central and matter charges.
These invariants give a general definition of the "topological" Bekenstein-Hawking entropy formula for extremal black-holes and reduce to the square of the black-hole ADM mass for "fixed scalars" which extremize the black-hole "potential" energy.
The Hessian matrix of the black-hole potential at "fixed scalars", in contrast to $N = 2$ theories, is shown to be degenerate, with rank $(N-2)(N-3)+2n$ (N being the number of supersymmetries and n the number of matter multiplets) and semipositive definite.

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1 Introduction

Recently, considerable progress has been made in the study of general properties of black holes arising in supersymmetric theories of gravity such as extended supergravities, string theory and M-theory [1]. Of particular interest are extremal black holes in four dimensions which correspond to BPS saturated states [2] and whose ADM mass depends, beyond the quantized values of electric and magnetic charges, on the asymptotic value of scalars at infinity. The latter describe the moduli space of the theory.

Another physical relevant quantity, which depends only on quantized electric and magnetic charges, is the black hole entropy, which can be defined macroscopically, through the Bekenstein-Hawking area-entropy relation or microscopically, through D-branes techniques [3] by counting of microstates [4]. It has been further realized that the scalar fields, independently of their values at infinity, flow towards the black hole horizon to a fixed value of pure topological nature given by a certain ratio of electric and magnetic charges [5].

These "fixed scalars" correspond to the extrema of the ADM mass in moduli space while the black-hole entropy is the actual value of the squared ADM mass at this point [6].

In theories with $N > 2$, extremal black-holes preserving one supersymmetry have the further property that all central charge eigenvalues other than the one equal to the BPS mass flow to zero for "fixed scalars". The black-hole entropy is still given by the square of the ADM mass for "fixed scalars"[7].

Recently [8], the nature of these extrema has been further studied and shown that they generically correspond to non degenerate minima for $N = 2$ theories whose relevant moduli space is the special geometry of $N = 2$ vector multiplets.

The entropy formula turns out to be in all cases a U-duality invariant expression (homogeneous of degree two) built out of electric and magnetic charges and as such can be in fact also computed through certain (moduli-independent) topological quantities which only depend on the nature of the U-duality groups and the appropriate representations of electric and magnetic charges. For example, in the $N = 8$ theory the entropy was shown to correspond to the unique quartic E_7 invariant built with its 56 dimensional representation [9].

In this paper we intend to make further progress in this subject by deriving, for all $N > 2$ theories, topological (moduli-independent) U-invariants constructed in terms of (moduli-dependent) central charges and matter charges, and show that, as expected, they coincide with the squared ADM mass at "fixed scalars".

We also show that the Hessian of the black-hole potential, as it arises from the geodesic action [10][8], is degenerate at the extremum for $N > 2$ theories, and we discuss the nature of this degeneracy. The Hessian is of physical relevance because it is related to a thermodynamical quantity named Weinhold metric [8].

The paper is organized as follows:
In section 2 we give the topological U-invariants for all $N > 2$ supergravities and show, in an appendix, how to derive them with a simple mathematical tool. In section 3 we discuss the degeneracy and the rank of the Hessian matrix. In section 4 we discuss these results in a string theory perspective.

2 Central charges, U-invariants and entropy

Extremal black-holes preserving one supersymmetry correspond to N-extended multiplets with

$$M_{ADM} = |Z_1| > |Z_2| \cdots > |Z_{[N/2]}| \tag{2.1}$$

where Z_α, $\alpha = 1, \cdots, [N/2]$, are the proper values of the central charge antisymmetric matrix written in normal form [11]. The central charges $Z_{AB} = -Z_{BA}$, $A, B = 1, \cdots, N$, and matter charges Z_I, $I = 1, \cdots, n$ are those (moduli-dependent) symplectic invariant combinations of field strenghts and their duals (integrated over a large two-sphere) which appear in the gravitino and gaugino supersymmetry variations respectively [12], [13], [14]. Note that the total number of vector fields is $n_v = N(N-1)/2 + n$ (with the exception of $N = 6$ in which case there is an extra singlet graviphoton)[15].
It was shown in ref. [7] that at the attractor point, where M_{ADM} is extremized, supersymmetry requires that Z_α, $\alpha > 1$, vanish toghether with the matter charges Z_I, $I = 1, \cdots, n$ (n is the number of matter multiplets, which can exist only for $N = 3, 4$)

This result can be used to show that for "fixed scalars", corresponding to the attractor point, the scalar "potential" of the geodesic action [10][8]

$$V = -\frac{1}{2} P^t \mathcal{M}(\mathcal{N}) P \tag{2.2}$$

is extremized in moduli space. Here P is the symplectic vector $P = (p^\Lambda, q_\Lambda)$ of quantized electric and magnetic charges and $\mathcal{M}(\mathcal{N})$ is a symplectic $2n_v \times 2n_v$ matrix whose $n_v \times n_v$ blocks are given in terms of the $n_v \times n_v$ vector kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ ($-Im\mathcal{N}, Re\mathcal{N}$ are the normalizations of the kinetic F^2 and the topological F^*F terms respectively) and

$$\mathcal{M}(\mathcal{N}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.3}$$

with:

$$\begin{aligned} A &= Im\mathcal{N} + Re\mathcal{N} Im\mathcal{N}^{-1} Re\mathcal{N} \\ B &= -Re\mathcal{N} Im\mathcal{N}^{-1} \\ C &= -Im\mathcal{N}^{-1} Re\mathcal{N} \\ D &= Im\mathcal{N}^{-1} \end{aligned} \tag{2.4}$$

The above assertion comes from the important identity, shown in ref. [13], [14] to be valid in all $N \geq 2$ theories:

$$-\frac{1}{2} P^t \mathcal{M}(\mathcal{N}) P = \frac{1}{2} Z_{AB} \overline{Z}^{AB} + Z_I \overline{Z}^I \tag{2.5}$$

The main purpose of this section is to provide particular expressions which give the entropy formula as a moduli–independent quantity in the entire moduli space and not just at the critical points. Namely, we are looking for quantities $S\left(Z_{AB}(\phi), \overline{Z}^{AB}(\phi), Z_I(\phi), \overline{Z}^I(\phi)\right)$ such that $\frac{\partial}{\partial \phi^i} S = 0$, ϕ^i being the moduli coordinates [1].

These formulae generalize the quartic $E_{7(-7)}$ invariant of $N = 8$ supergravity [9] to all other cases. We will show in the appendix how these invariants can be computed in an almost trivial fashion by using the (non compact) Cartan elements of G/H.

[1] The Bekenstein-Hawking entropy $S_{BH} = \frac{A}{4}$ is actually πS in our notation.

Let us first consider the theories $N = 3, 4$, where matter can be present [16], [17].

The U–duality groups [2] are, in these cases, $SU(3,n)$ and $SU(1,1) \times SO(6,n)$ respectively. The central and matter charges Z_{AB}, Z_I transform in an obvious way under the isotropy groups

$$H = SU(3) \times SU(n) \times U(1) \qquad (N = 3) \tag{2.6}$$

$$H = SU(4) \times O(n) \times U(1) \qquad (N = 4) \tag{2.7}$$

Under the action of the elements of G/H the charges get mixed with their complex conjugate. The infinitesimal transformation can be read from the differential relations satisfied by the charges [14]:

$$\nabla Z_{AB} = \frac{1}{2} P_{ABCD} \overline{Z}^{CD} + P_{ABI} \overline{Z}^I \tag{2.8}$$

$$\nabla Z_I = \frac{1}{2} P_{ABI} \overline{Z}^{AB} + P_{IJ} \overline{Z}^J \tag{2.9}$$

where the matrices P_{ABCD}, P_{ABI}, P_{IJ} are the subblocks of the vielbein of G/H [14]:

$$\mathcal{P} \equiv L^{-1} \nabla L = \begin{pmatrix} P_{ABCD} & P_{ABI} \\ P_{IAB} & P_{IJ} \end{pmatrix} \tag{2.10}$$

written in terms of the indices of $H = H_{Aut} \times H_{matter}$.

For $N = 3$:

$$P^{ABCD} = P_{IJ} = 0, \quad P_{ABI} \equiv \epsilon_{ABC} P_I^C \quad Z_{AB} \equiv \epsilon_{ABC} Z^C \tag{2.11}$$

Then the variations are:

$$\delta Z^A = \xi_I^A \overline{Z}^I \tag{2.12}$$

$$\delta Z_I = \xi_I^A \overline{Z}_A \tag{2.13}$$

where ξ_I^A are infinitesimal parameters of $K = G/H$.

So, the U-invariant expression is:

$$S = Z^A \overline{Z}_A - Z_I \overline{Z}^I \tag{2.14}$$

In other words, $\nabla_i S = \partial_i S = 0$, where the covariant derivative is defined in ref. [14].

Note that at the attractor point ($Z_I = 0$) it coincides with the moduli-dependent potential (2.2) computed at its extremum.

For $N = 4$

$$P_{ABCD} = \epsilon_{ABCD} P, \quad P_{IJ} = \eta_{IJ} P, \quad P_{ABI} = \frac{1}{2} \eta_{IJ} \epsilon_{ABCD} \overline{P}^{CDJ} \tag{2.15}$$

and the transformations of $K = \frac{SU(1,1)}{U(1)} \times \frac{O(6,n)}{O(6) \times O(n)}$ are:

$$\delta Z_{AB} = \frac{1}{2} \xi \epsilon_{ABCD} \overline{Z}^{CD} + \xi_{ABI} \overline{Z}^I \tag{2.16}$$

$$\delta Z_I = \xi \eta_{IJ} \overline{Z}^J + \frac{1}{2} \xi_{ABI} \overline{Z}^{AB} \tag{2.17}$$

[2] Here we denote by U-duality group the isometry group G acting on the scalars, although only a restriction of it to integers is the proper U-duality group [18].

with $\overline{\xi}^{ABI} = \frac{1}{2}\eta^{IJ}\epsilon^{ABCD}\xi_{CDJ}$.

There are three $O(6,n)$ invariants given by I_1, I_2, $\overline{I}_2$ where:

$$I_1 = \frac{1}{2}Z_{AB}\overline{Z}_{AB} - Z_I\overline{Z}^I \tag{2.18}$$

$$I_2 = \frac{1}{4}\epsilon^{ABCD}Z_{AB}Z_{CD} - \overline{Z}_I\overline{Z}^I \tag{2.19}$$

and the unique $SU(1,1)\times O(6,n)$ invariant S, $\nabla S = 0$, is given by:

$$S = \sqrt{(I_1)^2 - |I_2|^2} \tag{2.20}$$

At the attractor point $Z_I = 0$ and $\epsilon^{ABCD}Z_{AB}Z_{CD} = 0$ so that S reduces to the square of the BPS mass.

For $N = 5,6,8$ the U-duality invariant expression S is the square root of a unique invariant under the corresponding U-duality groups $SU(5,1)$, $O^*(12)$ and $E_{7(-7)}$. The strategy is to find a quartic expression S^2 in terms of Z_{AB} such that $\nabla S = 0$, i.e. S is moduli-independent.

As before, this quantity is a particular combination of the H quartic invariants.

For $SU(5,1)$ there are only two $U(5)$ quartic invariants. In terms of the matrix $A_A^{\ B} = Z_{AC}\overline{Z}^{CB}$ they are: $(TrA)^2$, $Tr(A^2)$, where

$$TrA = Z_{AB}\overline{Z}^{BA} \tag{2.21}$$

$$Tr(A^2) = Z_{AB}\overline{Z}^{BC}Z_{CD}\overline{Z}^{DA} \tag{2.22}$$

As before, the relative coefficient is fixed by the transformation properties of Z_{AB} under $\frac{SU(5,1)}{U(5)}$ elements of infinitesimal parameter ξ^C:

$$\delta Z_{AB} = \frac{1}{2}\xi^C\epsilon_{CABPQ}\overline{Z}^{PQ} \tag{2.23}$$

It then follows that the required invariant is:

$$S = \frac{1}{2}\sqrt{4Tr(A^2) - (TrA)^2} \tag{2.24}$$

For $N = 8$ the $SU(8)$ invariants are [3]:

$$I_1 = (TrA)^2 \tag{2.25}$$

$$I_2 = Tr(A^2) \tag{2.26}$$

$$I_3 = Pf\,Z = \frac{1}{2^4 4!}\epsilon^{ABCDEFGH}Z_{AB}Z_{CD}Z_{EF}Z_{GH} \tag{2.27}$$

The $\frac{E_{7(-7)}}{SU(8)}$ transformations are:

$$\delta Z_{AB} = \frac{1}{2}\xi_{ABCD}\overline{Z}^{CD} \tag{2.28}$$

[3] The Pfaffian of an $(n \times n)$ (n even) antisymmetric matrix is defined as $PfZ = \frac{1}{2^n n!}\epsilon^{A_1\cdots A_n}Z_{A_1A_2}\cdots Z_{A_{N-1}A_N}$, with the property: $|PfZ| = |detZ|^{1/2}$.

where ξ_{ABCD} satisfies the reality constraint:

$$\xi_{ABCD} = \frac{1}{24}\epsilon_{ABCDEFGH}\overline{\xi}^{EFGH} \tag{2.29}$$

One finds the following $E_{7(-7)}$ invariant [9]:

$$S = \frac{1}{2}\sqrt{4Tr(A^2) - (TrA)^2 + 32Re(Pf\ Z)} \tag{2.30}$$

The $N = 6$ case is the more complicated because under $U(6)$ the left-handed spinor of $O^*(12)$ splits into:

$$32_L \to (15, 1) + (\overline{15}, -1) + (1, -3) + (1, 3) \tag{2.31}$$

The transformations of $\frac{O^*(12)}{U(6)}$ are:

$$\begin{aligned} \delta Z_{AB} &= \frac{1}{4}\epsilon_{ABCDEF}\xi^{CD}\overline{Z}^{EF} + \xi_{AB}\overline{X} && (2.32) \\ \delta X &= \frac{1}{2}\xi_{AB}\overline{Z}^{AB} && (2.33) \end{aligned}$$

where we denote by X the $SU(6)$ singlet.

The quartic $U(6)$ invariants are:

$$\begin{aligned} I_1 &= (TrA)^2 && (2.34) \\ I_2 &= Tr(A^2) && (2.35) \\ I_3 &= Re(Pf\ ZX) = \frac{1}{2^3 3!}Re(\epsilon^{ABCDEF}Z_{AB}Z_{CD}Z_{EF}X) && (2.36) \\ I_4 &= (TrA)X\overline{X} && (2.37) \\ I_5 &= X^2\overline{X}^2 && (2.38) \end{aligned}$$

The unique $O^*(12)$ invariant is:

$$\begin{aligned} S &= \frac{1}{2}\sqrt{4I_2 - I_1 + 32I_3 + 4I_4 + 4I_5} && (2.39) \\ \nabla S &= 0 && (2.40) \end{aligned}$$

Note that at the attractor point $Pf\ Z = 0$, $X = 0$ and S reduces to the square of the BPS mass.

3 Extrema of the BPS mass and fixed scalars

In this section we would like to extend the analysis of the extrema of the black-hole induced potential

$$V = \frac{1}{2}Z_{AB}\overline{Z}^{AB} + Z_I\overline{Z}^I \tag{3.41}$$

which was performed in ref [8] for the $N = 2$ case to all $N > 2$ theories.

We recall that, in the case of $N = 2$ special geometry with metric $g_{i\bar{j}}$, at the fixed scalar critical point $\partial_i V = 0$ the Hessian matrix reduces to:

$$(\nabla_i \nabla_{\bar{j}} V)_{fixed} = (\partial_i \partial_{\bar{j}} V)_{fixed} = 2 g_{i\bar{j}} V_{fixed} \qquad (3.42)$$

$$(\nabla_i \nabla_j V)_{fixed} = 0 \qquad (3.43)$$

The Hessian matrix is strictly positive-definite if the critical point is not at the singular point of the vector multiplet moduli-space. This matrix was related to the Weinhold metric earlier introduced in the geometric approach to thermodynamics and used for the study of critical phenomena [8].

For N-extended supersymmetry, a form of this matrix was also given and shown to be equal to [4]:

$$V_{ij} = (\partial_i \partial_j V)_{fixed} = Z_{CD} Z^{AB} (\frac{1}{2} P^{CDPQ}_{\ \ ,j} P_{ABPQ,i} + P^{CD}_{I,j} P^{I}_{AB,j}). \qquad (3.44)$$

It is our purpose to further investigate properties of the Weinhold metric for fixed scalars.

Let us first observe that the extremum conditions $\nabla_i V = 0$, using the relation between the covariant derivatives of the central charges, reduce to the conditions:

$$\epsilon^{ABCDL_1 \cdots L_{N-4}} Z_{AB} Z_{CD} = 0, \quad Z_I = 0 \qquad (3.45)$$

These equations give the fixed scalars in terms of electric and magnetic charges and also show that the topological invariants of the previous section reduce to the extremum of the square of the ADM mass since, when the above conditions are fulfilled, $(TrA)^2 = 2Tr(A^2)$, where $A_A^B = Z_{AB} \overline{Z}^{BC}$.

On the other hand, when these conditions are fulfilled, it is easy to see that the Hessian matrix is degenerate. To see this, it is sufficient to go, making an H transformation, to the normal frame in which these conditions imply $Z_{12} \neq 0$ with the other charges vanishing. Then we have:

$$\partial_i \partial_j V|_{fixed} = 4|Z_{12}|^2 (\frac{1}{2} P^{12ab}_j P_{12ab,i} + P^{12I}_{,j} P_{12I,i}), \quad a, b \neq 1, 2 \qquad (3.46)$$

To understand the pattern of degeneracy for all N, we observe that when only one central charge in not vanishing the theory effectively reduces to an $N = 2$ theory. Then the actual degeneracy respects $N = 2$ multiplicity of the scalars degrees of freedom in the sense that the degenerate directions will correspond to the hypermultiplet content of $N > 2$ theories when decomposed with respect to $N = 2$ supersymmetry.

Note that for $N = 3$, $N = 4$, where P_{ABI} is present, the Hessian is block diagonal.

For $N = 3$, referring to eq. (2.11), since the scalar manifold is Kähler, P_{ABI} is a (1,0)-form while $P^{ABI} = \overline{P}_{ABI}$ is a (0,1)-form.

The scalars appearing in the $N = 2$ vector multiplet and hypermultiplet content of the vielbein are P_{3I} for the vector multiplets and P_{aI} $(a = 1, 2)$ for the hypermultiplets. From equation (3.46), which for the $N = 3$ case reads

$$\partial_{\bar{j}} \partial_i V|_{fixed} = 2|Z_{12}|^2 P_{3I,\bar{j}} P^{3I}_{,i} \qquad (3.47)$$

[4]Generically the indices i, j refer to real coordinates, unless the manifold is Kählerian, in which case we use holomorphic coordinates and formula (3.44) reduces to the hermitean $i\bar{j}$ entries of the Hessian matrix.

we see that the metric has $4n$ real directions corresponding to n hypermultiplets which are degenerate.

For $N=4$, referring to (2.15), P is the $SU(1,1)/U(1)$ vielbein which gives one matter vector multiplet scalar while P_{12I} gives n matter vector multiplets. The directions which are hypermultiplets correspond to P_{1aI}, P_{2aI} $(a=3,4)$. Therefore the "metric" V_{ij} is of rank $2n+2$.

For $N>4$, all the scalars are in the gravity multiplet and correspond to P_{ABCD}.

The splitting in vector and hypermultiplet scalars proceeds as before. Namely, in the $N=5$ case we set $P_{ABCD} = \epsilon_{ABCDL} P^L$ $(A,B,C,D,L=1,\cdots 5)$. In this case the vector multiplet scalars are P^a $(a=3,4,5)$ while the hypermultiplet scalars are P^1, P^2 ($n_V=3$, $n_h=1$).

For $N=6$, we set $P_{ABCD} = \frac{1}{2}\epsilon_{ABCDEF} P^{EF}$. The vector multiplet scalars are now described by P^{12}, P^{ab} $(A,B,...=1,...,6;\ a,b=3,\cdots 6)$, while the hypermultiplet scalars are given in terms of P^{1a}, P^{2a}. Therefore we get $n_V = 6+1 = 7$, $n_h=4$.

This case is different from the others because, besides the hypermultiplets P^{1a}, P^{2a}, also the vector multiplet direction P^{12} is degenerate.

Finally, for $N=8$ we have P_{1abc}, P_{2abc} as hypermultiplet scalars and P_{abcd} as vector multiplet scalars, which give $n_V=15$, $n_h=10$ (note that in this case the vielbein satisfies a reality condition: $P_{ABCD} = \frac{1}{4!}\epsilon_{ABCDPQRS}\overline{P}^{PQRS}$). We have in this case 40 degenerate directions.

In conclusion we see that the rank of the matrix V_{ij} is $(N-2)(N-3)+2n$ for all the four dimensional theories.

4 Relations to string theories

N-extended supergravities are related to strings compactified on six-manifolds M_N preserving N supersymmetries at $D=4$. Since we are presently considering $N>2$, the most common cases are $N=4$ and $N=8$. The first can be achieved in heterotic or Type II string, with $M_4 = T_6$ in heterotic and $M_4 = K_3 \times T_2$ in Type II theory. These theories are known to be dual at a non perturbative level [18], [19], [20]. $N=8$ corresponds to $M_8 = T_6$ in Type II.

Less familiar are the $N=3,5$ and 6 cases which were studied in ref. [21].

Interestingly enough, the latter cases can be obtained by compactification of Type II on asymmetric orbifolds with $3 = 2_L + 1_R$, $5 = 4_L + 1_R$ and $6 = 4_L + 2_R$ respectively.

BPS states considered in this paper should correspond to massive states in these theories for which only a subset of them is known in the perturbative framework.

In attemps to test non perturbative string properties it would be interesting to check the existence of the BPS states and their entropy by using microscopic considerations.

We finally observe that, unlike $N=8$, the moduli spaces of $N=3,5,6$ theories are locally Kählerian (as $N=2$) with coset spaces of rank 3 $(n \geq 3)$, 1 and 3 respectively.

For $N=5,6$ these spaces are also special Kähler (which is also the case for $N=3$ when $n=1,3$) [22] [23].

We can use the previous observations to construct U-invariants for some $N=2$ special geometries looking at the representation content of vectors and their duals with respect to U-dualities.

Let us first consider $N = 2$ theories with U-duality $SU(1, n)$ and $SU(3, 3)$. These groups emerge in discussing string compactifications on some $N = 2$ orbifolds (i.e. orbifold points of Calabi-Yau threefolds)[23][24].

The vector content is respectively given by the fundamental representation of $SU(1, n)$ and the twenty dimentional threefold antisymmetric rep. of $SU(3, 3)$ [25].

Amazingly, the first representation occurs as in $N = 3$ matter coupled theories, while the latter is the same as in $N = 5$ supergravity (note that $SU(1, n)$, $SU(3, n)$ and $SU(3, 3)$, $SU(5, 1)$ are just different non compact forms of the same $SU(m)$ groups).

From the results of the previous section we conclude that the special manifolds $\frac{SU(1,n)}{SU(n)\times U(1)}$ and $\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$ admit respectively a quadratic [12], [26] and a quartic topological invariant. The $N = 2$ special manifold $\frac{O^*(12)}{U(6)}$ has a vector content which is a left spinor of $O^*(12)$, as in the $N = 6$ theory, therefore it admits a quartic invariant.

Finally, the $N = 2$ special manifolds $\frac{SU(1,1)}{U(1)} \times \frac{O(2,n)}{O(2)\times O(n)}$, which emerge in $N = 2$ compactifications of both heterotic and Type II strings [24], admit a quartic invariant which can be read from the $N = 4$ quartic invariant in which the $\frac{SU(1,1)}{U(1)}$ matter charge is identified with the second eigenvalue of the $N = 4$ central charge.

All the above topological invariants can then be interpreted as entropy of a variety of $N = 2$ black-holes.

Appendix: A simple determination of the U-invariants

In order to determine the quartic U-invariant expressions S^2 , $\nabla S = 0$, of the $N > 4$ theories, it is useful to use, as a calculational tool, transformations of the coset which preserve the normal form of the Z_{AB} matrix. It turns out that these transformations are certain Cartan elements in $K = G/H$ [27], that is they belong to $O(1, 1)^p \in K$, with $p = 1$ for $N = 5$, $p = 3$ for $N = 6, 8$.

These elements act only on the Z_{AB} (in normal form), but they uniquely determine the U-invariants since they mix the eigenvalues e_i $(i = 1, \cdots, [N/2])$.

For $N = 5$, $SU(5, 1)/U(5)$ has rank one (see ref. [28]) and the element is:

$$\delta e_1 = \xi e_2; \quad \delta e_2 = \xi e_1 \tag{4.48}$$

which is indeed a $O(1, 1)$ transformation with unique invariant

$$|(e_1)^2 - (e_2)^2| = \frac{1}{2}\sqrt{8\left((e_1)^4 + (e_2)^4\right) - 4\left((e_1)^2 + (e_2)^2\right)^2} \tag{4.49}$$

For $N = 6$, we have $\xi_1 \equiv \xi_{12}; \xi_2 \equiv \xi_{34}; \xi_3 \equiv \xi_{56}$ and we obtain the 3 Cartan elements of $O^*(12)/U(6)$, which has rank 3, that is it is a $O(1, 1)^3$ in $O^*(12)/U(6)$. Denoting by e the singlet charge, we have the following $O(1, 1)^3$ transformations:

$$\begin{aligned}
\delta e_1 &= \xi_2 e_3 + \xi_3 e_2 + \xi_1 e && (4.50)\\
\delta e_2 &= \xi_1 e_3 + \xi_3 e_1 + \xi_2 e && (4.51)\\
\delta e_3 &= \xi_1 e_2 + \xi_2 e_1 + \xi_3 e && (4.52)\\
\delta e &= \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 && (4.53)
\end{aligned}$$

these transformations fix uniquely the $O^*(12)$ invariant constructed out of the five $U(6)$ invariants displayed in (2.34-2.38).

For $N = 8$ the infinitesimal parameter is ξ_{ABCD} and, using the reality condition, we get again a $O(1,1)^3$ in $E_{7(-7)}/SU(8)$. Setting $\xi_{1234} = \xi_{5678} \equiv \xi_{12}$, $\xi_{1256} = \xi_{3478} \equiv \xi_{13}$, $\xi_{1278} = \xi_{3456} \equiv \xi_{14}$, we have the following set of transformations:

$$\delta e_1 = \xi_{12}e_2 + \xi_{13}e_3 + \xi_{14}e_4 \tag{4.54}$$
$$\delta e_2 = \xi_{12}e_1 + \xi_{13}e_4 + \xi_{14}e_3 \tag{4.55}$$
$$\delta e_3 = \xi_{12}e_4 + \xi_{13}e_1 + \xi_{14}e_2 \tag{4.56}$$
$$\delta e_4 = \xi_{12}e_3 + \xi_{13}e_2 + \xi_{14}e_1 \tag{4.57}$$

These transformations fix uniquely the relative coefficients of the three $SU(8)$ invariants:

$$I_1 = e_1^4 + e_2^4 + e_3^4 + e_4^4 \tag{4.58}$$
$$I_2 = (e_1^2 + e_2^2 + e_3^2 + e_4^2)^2 \tag{4.59}$$
$$I_3 = e_1 e_2 e_3 e_4 \tag{4.60}$$
$$\tag{4.61}$$

It is easy to see that the transformations (4.50-4.53) and (4.54-4.57) correspond to three commuting matrices (with square equal to $\mathbb{1}$):

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.62}$$

which are proper non compact Cartan elements of K. The reason we get the same transformations for $N = 6$ and $N = 8$ is because the extra singlet e of $N = 6$ can be identified with the fourth eigenvalue of the central charge of $N = 8$.

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