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Large-order Behaviour due to Ultraviolet Renormalons in QCD

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Abstract

Ultraviolet renormalons, contrary to their infrared counterparts, lead to a universal contribution to the large-order behaviour of perturbative expansions in QCD. In this letter, we determine nature of the leading ultraviolet renormalon singularity for the inclusive hadroproduction cross section in e^+e^- annihilation, for hadronic τ decays and the moments of deep-inelastic scattering structure functions. We comment on the relevance of ultraviolet renormalons to estimates in low orders of perturbation theory.

1. Perturbative expansions of physical quantities in QCD are divergent, and as such they are related to a measurement only through the additional assumption that the series is asymptotic. Of the two known mechanisms that cause divergence of the series, instantons [1, 2] are unimportant in large orders and the asymptotic behaviour of the series expansion is determined by renormalons [3, 4]. According to whether the divergence arises from regions of small or large momentum in internal integrations, renormalons are classified as infrared (IR) or ultraviolet (UV). The growth of perturbative coefficients due to IR renormalons depends on whether one considers, for example, deep-inelastic structure functions, hadronic event shape observables or the hadronic total cross section in e^+e^- annihilation and is closely related to power corrections to these observables. For this reason they have recently been studied intensely [5]. UV renormalons are often regarded to be more complicated and have mostly been left aside, perhaps because they are Borel summable. However, the structure of UV renormalons is universal in the sense that it depends only on the theory under consideration, in our case QCD. Moreover, for some quantities of interest, such as the hadronic total cross section in e^+e^- annihilation, the sign-alternating UV renormalon behaviour determines the actual large-order behaviour of the series expansion. In this letter we determine the universal ultraviolet renormalon asymptotics in QCD and apply it to a number of observables of phenomenological interest.

The universality of UV renormalons was recognised by Parisi [6], who noted that UV renormalons could be compensated by adding higher-dimension operators to the Lagrangian just as logarithmic UV divergences can be compensated by dimension-four counterterms. To state Parisi's hypothesis precisely, we consider a quantity $R(\alpha_s)$, expanded as

$$R(\alpha_s) = A\left(1+\sum_{n=0}^{\infty} r_n\alpha_s^{n+1}\right). \tag{1}$$

The dependence on external momenta is not indicated and α_s denotes the coupling renormalised at a scale μ. The UV renormalons produce poles at $t = m/\beta_0 < 0$ in the Borel transform $B[R](t) = \sum_{n=0} r_n t^n/n!$, where m is a positive integer and $\beta_0 = -b/(4\pi)$ $(b = 11 - 2N_f/3)$ the first coefficient of the β-function. It follows that the integral[1]

$$I[R](\alpha_s) = \int\limits_{0+i\epsilon}^{-t_c+i\epsilon} dt\, e^{-t/\alpha_s}\, B[R](t) \qquad t_c > 0 \tag{2}$$

is complex, for $t > -1/\beta_0$, and its imaginary part is unambiguously related to UV renormalon singularities, and therefore asymptotic behaviour, and vice versa. In the following, we will be concerned explicitly only with the leading UV renormalon singularity

[1] Since UV renormalons produce sign-alternating factorial behaviour, they do not lead to ambiguities in the usual Borel integral. Because the consideration of ambiguities will simplify the renormalisation group considerations below, we define the integral parallel to the negative axis. The imaginary parts created by UV renormalon poles are now exponentially large in α_s. This need not bother us, because α_s could be considered negative without any change in our derivation.

at $t = 1/\beta_0$. The hypothesis of Parisi [6], for this situation, states that

$$\mathrm{Im}\, I[R](\alpha_s, p_k) = \frac{1}{\mu^2} \sum_i e^{-1/(\beta_0 \alpha_s))} \alpha_s^{-\beta_1/\beta_0^2}\, C_i(\alpha_s)\, R_{\mathcal{O}_i}(\alpha_s, p_k). \tag{3}$$

Thus, the leading UV renormalon behaviour is determined by single zero-momentum insertions of dimension-six operators[2] $\mathcal{O}_i$ into the Green function from which the quantity R is derived. The important point here is that the coefficient function C_i is universal – it is independent of the external momenta p_k and the quantity R – which is related to the fact that UV renormalons arise from loop momentum regions much larger than any external scale. The dimension-six operators may be thought of as an additional term

$$\Delta\mathcal{L} = -\frac{i}{\mu^2} \sum_i e^{-1/(\beta_0 \alpha_s))} \alpha_s^{-\beta_1/\beta_0^2}\, C_i(\alpha_s)\, \mathcal{O}_i \tag{4}$$

in the QCD Lagrangian with coefficients such that for any R the imaginary part of $I[R]$ is compensated by the additional contribution to R from $\Delta\mathcal{L}$. By comparing the renormalisation group equations for $I[R]$ and $R_{\mathcal{O}_i}$ one derives that

$$\frac{d}{d\alpha_s}\, C_j(\alpha_s) = \frac{\gamma_{ij}(\alpha_s)}{2\beta(\alpha_s)}\, C_i(\alpha_s), \tag{5}$$

where $\gamma(\alpha_s)$ is the anomalous dimension matrix pertaining to the dimension-six operators $\mathcal{O}_i$. The solution to this equation determines completely the α_s-dependence of the C_i. This in turn allows us to compute the Borel transform $B[R](t)$ in the vicinity of the singular point $t = 1/\beta_0$ up to an over-all constant. Finally, the nature of the singularity determines the large-order behaviour. These manipulations can be summarised in the substitution rule

$$e^{-1/(\beta_0 \alpha_s)}\, (\beta_0 \alpha_s)^\lambda \longrightarrow \frac{1}{\pi} \sum_n \beta_0^n\, n!\, n^{-\lambda}\, \alpha_s^{n+1} \tag{6}$$

to obtain the leading asymptotic behaviour of R from the α_s-dependence of $\mathrm{Im}\, I[R]$.

The leading UV renormalon divergence has been subjected to detailed diagrammatic study in abelian models [7]–[10], which confirmed the general structure of (3). In [7] four-fermion operators were identified as sources of leading behaviour in QED. In [10] (3) was shown to be valid in QED to all orders in the $1/N_f$ expansion and it was indicated how one could understand the restoration of the full non-abelian β_0 starting from a $1/N_f$ expansion of the non-abelian theory. Both [7, 10] corroborated the earlier expectation [11]–[13] that while (5) determines the α_s-dependence of C_i, the integration constants, which are related to the constants that specify the over-all normalisations K_i of the UV renormalon asymptotic behaviour, could not be calculated systematically, except in expansions like $1/N_f$. Once this is realized, all exact obtainable information can be obtained solely by solving the renormalisation group equations for C_i above. This

[2]The fact that higher (than four) dimensional operators start at dimension six determines that the position of the leading UV singularity is indeed at $t = 1/\beta_0$.

problem has already been solved for Heavy Quark Effective Theory [9] and QED [7, 10]. In the following sections we treat the slightly more complicated case of QCD. That is, for

$$r_n \overset{n\to\infty}{=} \beta_0^n\, n!\, n^{\beta_1/\beta_0^2} \sum_i K_i\, n^{\delta_i}\, (1 + O(1/n)) \tag{7}$$

we determine the so far unknown constants δ_i. Note that the $1/n$ corrections to the asymptotic behaviour are in principle calculable as well. This would require the two-loop anomalous dimension matrix as well as the one-loop corrections to the Green functions with operator insertion, $R_{\mathcal{O}_i}$. The constants K_i remain unknown. However, because of the universality of coefficients functions C_i, the ratio of K_i's for different observables is calculable.

2. In this section we determine the leading UV renormalon behaviour of current correlation functions

$$i \int d^4x\, e^{iqx}\, \langle 0|T(j_\mu(x) j_\nu(0))|0\rangle = (q_\mu q_\nu - g_{\mu\nu} q^2)\, \Pi(q^2) \tag{8}$$

in massless QCD with N_f flavours. We will consider colour-singlet vector and axial-vector currents and let them also be flavour-singlets. In the following section, we generalise to the real case where flavour symmetry is broken by electric or axial charges or the current is flavour non-diagonal. Thus, in expressions like $(\bar\psi M \psi)$ a sum over flavour, colour and spinor indices is implied and M is a matrix in colour and spinor space, but unity in flavour space.

To account for the external currents, we introduce two $U(1)$ background fields that couple to the vector and axial-vector current and consider the Lagrangian

$$\mathcal{L}_{QCD} + j_V^\mu v_\mu + j_A^\mu a_\mu. \tag{9}$$

The corresponding field strength tensors are defined as usual and denoted by $F_{\mu\nu}$ and $H_{\mu\nu}$, respectively, such that $\partial_\mu F^{\mu\nu} = j_V^\nu$ and $\partial_\mu H^{\mu\nu} = j_A^\nu$. A basis of dimension-six operators is then given by

$$\begin{aligned}
\mathcal{O}_1 &= (\bar\psi\gamma_\mu\psi)(\bar\psi\gamma^\mu\psi) \\
\mathcal{O}_2 &= (\bar\psi\gamma_\mu\gamma_5\psi)(\bar\psi\gamma^\mu\gamma_5\psi) \\
\mathcal{O}_3 &= (\bar\psi\gamma_\mu T^A\psi)(\bar\psi\gamma^\mu T^A\psi) \\
\mathcal{O}_4 &= (\bar\psi\gamma_\mu\gamma_5 T^A\psi)(\bar\psi\gamma^\mu\gamma_5 T^A\psi)
\end{aligned} \tag{10}$$

$$\mathcal{O}_5 = \frac{1}{g}\, f_{ABC}\, G^A_{\mu\nu} G^{\nu\,B}_{\rho} G^{\rho\mu\,C} \tag{11}$$

$$\mathcal{O}_6 = \frac{1}{g^2}\, (\bar\psi\gamma_\mu\psi)\, \partial_\nu F^{\nu\mu} \qquad \mathcal{O}_7 = \frac{1}{g^2}\, (\bar\psi\gamma_\mu\gamma_5\psi)\, \partial_\nu H^{\nu\mu} \tag{12}$$

$$\mathcal{O}_8 = \frac{1}{g^4}\, \partial_\nu F^{\nu\mu}\, \partial^\rho F_{\rho\mu} \qquad \mathcal{O}_9 = \frac{1}{g^4}\, \partial_\nu H^{\nu\mu}\, \partial^\rho H_{\rho\mu}. \tag{13}$$

We need not consider gauge-variant operators and those that vanish by the equations of motion as their zero-momentum insertions into the above correlation functions do not contribute. The over-all factors $1/g^k$ have been inserted for convenience. Note that we did not include four-fermion operators of scalar, pseudo-scalar or tensor type. They can not be generated in massless QCD, because the number of Dirac matrices on any fermion line that connects to an external fermion in a four-point function is always odd. The coefficients C_i corresponding to these operators therefore vanish exactly. It is straightforward to compute the leading-order anomalous dimension matrix γ, defined such that the renormalised operators satisfy

$$\left(\delta_{ij}\,\mu\frac{d}{d\mu}+\gamma_{ij}\right)\mathcal{O}_j=0. \tag{14}$$

We find $\gamma=\gamma^{(1)}\alpha_s/(4\pi)$ with[3]

$$\gamma^{(1)}=\begin{pmatrix} A & 0 & B\\ 0 & \gamma_{55} & 0\\ 0 & 0 & C\end{pmatrix} \tag{15}$$

and ($C_F=(N_c^2-1)/(2N_c)$, N_c the number of colours)

$$A=\begin{pmatrix} 0 & 0 & \frac{8}{3} & 12\\ 0 & 0 & \frac{44}{3} & 0\\ 0 & \frac{6C_F}{N_c} & -\frac{9N_c^2+4}{3N_c}+\frac{8N_f}{3} & \frac{3(N_c^2-4)}{N_c}\\ \frac{6C_F}{N_c} & 0 & \frac{3(N_c^2-4)}{N_c}-\frac{4}{3N_c} & -3N_c\end{pmatrix}. \tag{16}$$

The non-zero entries of the 4×4 sub-matrices B, C are: $B_{11}=B_{22}=8(2N_cN_f+1)/3$, $B_{12}=B_{21}=8/3$, $B_{31}=B_{32}=B_{41}=B_{42}=8C_F/3$, $C_{11}=C_{22}=-2b$, $C_{33}=C_{44}=-4b$, $C_{13}=C_{24}=8N_cN_f/3$. The mixing of $\mathcal{O}_5$ into itself is given by $\gamma_{55}=-8(N_c-N_f)/3$ [14]. Note that $\mathcal{O}_5$ could mix into $\mathcal{O}_3$ at order α_s. We find that due to a cancellation of different diagrams the corresponding entry γ_{53} vanishes. As a consequence, $\mathcal{O}_5$ decouples from the mixing at leading order, in agreement with the first reference of [14].

To solve (5) with $\gamma(\alpha_s)$ and $\beta(\alpha_s)$ evaluated at leading order let $2b\lambda_i$, $i=1\ldots4$, be the eigenvalues of A and $\lambda_5=\gamma_{55}/(2b)$. Let U be the matrix that diagonalises A. Since the integration constants must be considered as non-perturbative, we need not keep track of factors multiplying these constants, unless they are exactly zero. Thus we only note that no element of U vanishes for values of N_f of interest. Since C, and therefore $\gamma^{(1)}$, is triangular, we readily obtain

$$C_i(\alpha_s)=\sum_{k=1}^{4}C_{ik}^{[1]}\alpha_s^{-\lambda_k}\qquad i=1\ldots4$$

[3]The renormalisation of dimension-six operators has been studied previously in various contexts, mainly QCD sum rules. We recalculated the entries of $\gamma^{(1)}$ in our basis, except for γ_{55}, which we take from [14].

N_f	λ_1	λ_2	λ_3	λ_4	λ_5
3	0.379	0.126	−0.332	−0.753	0
4	0.487	0.140	−0.302	−0.791	4/25
5	0.630	0.155	−0.275	−0.843	8/23
6	0.817	0.172	−0.254	−0.910	4/7

Table 1: Numerical values of λ_i ($N_c = 3$).

$$\begin{aligned}
C_5(\alpha_s) &= C_5^{[1]} \alpha_s^{-\lambda_5} \\
C_i(\alpha_s) &= C_i^{[2]} \alpha_s + \sum_{k=1}^{4} C_{ik}^{[1]} \alpha_s^{-\lambda_k} \qquad i = 6, 7 \\
C_i(\alpha_s) &= C_i^{[2]} \alpha_s + C_i^{[3]} \alpha_s^2 + \sum_{k=1}^{4} C_{ik}^{[1]} \alpha_s^{-\lambda_k} \qquad i = 8, 9
\end{aligned} \tag{17}$$

with α_s-independent non-vanishing constants $C^{[l]}$ that depend on nine integration constants and the elements of $\gamma^{(1)}$. The exponents λ_k are reported in Table 1. The values for λ_1 to λ_4 at $N_f = 3$ are in agreement with [15]. We emphasise again that the coefficient functions C_i are independent of the particular Green function or observable under consideration.

We now consider $R = q^2 d\Pi(q^2)/dq^2$, the 'Adler functions' derived from the correlation functions of two vector or axial-vector currents and write their perturbative expansions as in (1). Having found the α_s-dependence of the coefficient functions, we further require $R_{\mathcal{O}_i}(\alpha_s, q)$ at leading order. Up to constants, we have $R_{\mathcal{O}_i}(\alpha_s, q) \propto \alpha_s^0$, $i = 1 \ldots 4$, $R_{\mathcal{O}_5}(\alpha_s, q) \propto \alpha_s$, $R_{\mathcal{O}_i}(\alpha_s, q) \propto \alpha_s^{-1}$, $i = 6, 7$ and $R_{\mathcal{O}_i}(\alpha_s, q) \propto \alpha_s^{-2}$ for $i = 8, 9$. Combining all factors in (3) and taking into account the substitution rule (6) to obtain the large-order behaviour of R from the imaginary part of $I[R]$ our final result reads

$$r_n \overset{n\to\infty}{=} \beta_0^n \, n! \, n^{\beta_1/\beta_0^2} \left[\sum_{i=1}^{4} K_i \, n^{2+\lambda_i} + K_5 \, n^{-1+\lambda_5} + K_6 + K_8 \, n \right] (1 + O(1/n)) \tag{18}$$

for two vector currents.[4] The result for two axial currents is identical, except for different over-all constants K_i. Note that the leading behaviour is generated by the largest eigenvalue of the anomalous dimension matrix of four-fermion operators. The contribution from the three-gluon operator (proportional to K_5) is suppressed. The dominant contribution from this operator comes in fact from its mixing into $\mathcal{O}_{6/7}$ at next-to-leading order (not computed here) rather than the 'direct' contribution through $R_{\mathcal{O}_5}(\alpha_s)$ in (18) above. In general, such next-to-leading order contributions result in $1/n$-corrections to the asymptotic behaviour (18). Eq. (18) holds when the series is expressed in terms

[4] We display in (18) the general structure of the result, although it is not consistent to keep all terms without computing $1/n$-corrections to the leading contribution proportional to K_1.

of the $\overline{\mathrm{MS}}$ renormalised coupling α_s, the convention we assume throughout this note. If a different coupling is employed that is related to the $\overline{\mathrm{MS}}$ coupling by a factorially divergent series, the coefficients r_n change accordingly and (18) may not be valid.

3. The previous result for flavour-U(N_f) singlet currents applies practically unaltered to all observables of interest. We briefly discuss them case by case.

e^+e^- **annihilation into hadrons.** Consider first annihilation through virtual photons. The external current is $j_\mu = \bar{\psi}\gamma_\mu Q\psi$, where $Q_{ij} = \mathrm{diag}(e_u, e_d, \ldots)$ is a matrix in flavour space and flavour indices are summed over. Since flavour symmetry is broken only by the external current (all quarks are still considered as massless), the 'QCD operators' $\mathcal{O}_{1-5}$ remain unaltered. The basis of 'current operators' $\mathcal{O}_{6-9}$ has to be altered to include the operators $(\mathrm{tr}Q)\,\bar{\psi}\gamma^\mu\psi\,\partial_\nu F^{\nu\mu}$ and $\bar{\psi}\gamma^\mu Q\psi\,\partial_\nu F^{\nu\mu}$ instead of $\mathcal{O}_6$. (Similar modifications would apply for the axial-vector current.) This ensures that mixing of four-fermion operators into the current operators contributes proportional to $\mathrm{tr}Q^2 = \sum_f e_f^2$ and $(\mathrm{tr}Q)^2 = (\sum_f e_f)^2$, as required by the existence of 'flavour non-singlet' and 'light-by-light scattering' terms. The matrices B and C in (15) change, but their pattern of non-zero entries does not. Thus, as we are not interested in over-all constants, (18) carries over to the present case. The leading asymptotic behaviour of the Adler function, expressed as in (1), is

$$d_n \stackrel{n\to\infty}{=} K_d\,\beta_0^n\,n!\,n^{2+\beta_1/\beta_0^2+\lambda_1} = K_d\,\beta_0^n\,n!\,n^{1.97}, \tag{19}$$

where in the last line we have taken $N_f = 5$. This is to be compared with the large-N_f limit [16] which is recovered by setting $n^{1.97} \to n$. Eq. (19) holds separately for the 'flavour non-singlet' and 'light-by-light scattering' contributions. In large orders, the expansion of the e^+e^- hadronic cross section is related to the Adler function by

$$r_n^{e^+e^-} \stackrel{n\to\infty}{\propto} \frac{d_n}{n}. \tag{20}$$

The suppression by one power of n follows from the fact that the coefficient of the leading asymptotic behaviour of the d_n is polynomial in the external momentum and therefore does not contribute to the discontinuity.

If we now consider the hadronic width of the Z^0, the inclusion of axial currents proceeds along the same lines as above for the vector current and modifies over-all constants, but again the general pattern of mixing remains the same. The expansion coefficients of the Z^0 width therefore grow as $r_n^{e^+e^-}$ in large orders. Note that the fact that flavour-singlet terms arise at order α_s^2 for axial currents, but α_s^3 for vector currents does not affect large-order estimates.

Hadronic τ decay. For flavour non-singlet currents $j_\mu = \sum_{f=d,s} V_{uf}\,\bar{u}\gamma_\mu(1-\gamma_5)q_f$ the 'light-by-light scattering' diagrams are absent. For an appropriate basis of current operators this is reflected in a change of entries in the matrices B and C. However, their pattern of non-zero entries is not changed. The contour integral that relates the τ width to the Adler function of the currents suppresses the large-order behaviour just as in case of the e^+e^- cross section above. We therefore have

$$r_n^\tau \stackrel{n\to\infty}{\propto} r_n^{e^+e^-} \stackrel{n\to\infty}{\propto} \beta_0^n\, n!\, n^{0.59}. \tag{21}$$

In the present case we set $N_f = 3$.

Moments of deep-inelastic scattering (DIS) structure functions. The operator product expansion allows us to write (at leading twist)

$$\int_0^1 dx\, x^{N-1}\, F_k(x,Q) = \sum_i C^i_{k,N}\left(\frac{Q^2}{\mu^2}, \alpha_s\right)\, A^i_N(\mu). \tag{22}$$

Here the structure function F_k can be $2F_1$, F_2/x or F_3, $A^i_N(\mu)$ denotes (reduced) proton matrix elements of twist-2 operators and $C^i_{k,N}(\alpha_s)$ the coefficients functions. To compute the large-order behaviour of the perturbative expansion of coefficient functions, we need to compute the insertion of dimension-six operators into the quark matrix elements of current-current correlation functions.[5] Because we consider quark matrix elements (rather than vacuum matrix elements as above), insertions of current-current operators such as $\mathcal{O}_{8/9}$ vanish and the leading asymptotic behaviour arises from insertion of operators such as $\mathcal{O}_{6/7}$. Furthermore, collinear singularities have to be factorised by computing the analogues of $A^i_N(\mu)$ between quark states and dividing them out. We choose the $\overline{\rm MS}$ factorisation scheme, in which case the $A^i_N(\mu)$ are pure poles. Dividing them out then does not modify the large-order behaviour of the finite terms. Finally, taking moments results in a moment-dependent over-all constant, but the n-dependence is the same for all moments. Thus,

$$C^i_{k,N}(\alpha_s) \stackrel{n\to\infty}{=} \sum_n K^i_{k,N}\, \beta_0^n\, n!\, n^{1+\beta_1/\beta_0^2+\lambda_1}\, \alpha_s^{n+1}. \tag{23}$$

Compared to the Adler function in (19) one has one power of n less, because the operators $\mathcal{O}_{8/9}$ do not contribute. This reflects that at a given order in α_s the diagrams that contribute to the coefficient functions have one loop less compared to the diagrams that contribute to the Adler function. Eq. (23) applies in particular to the case of QCD corrections to the GLS and Bjorken sum rule.

In general, the contribution from IR renormalons to the asymptotic behaviour can compete with the UV renormalon asymptotics computed in this note. For the moments of DIS structure functions, one finds $r_n^{IR}/r_n^{UV} \sim (-1)^n$ and the nature of the IR renormalon singularity is determined by the anomalous dimensions of twist-4 operators [17]. For the special case of the GLS sum rule, using the anomalous dimension calculated in [18], we

[5]Gluon matrix elements are suppressed and do not contribute to the leading large-order behaviour.

obtain

$$C_{GLS}(\alpha_s) \stackrel{n\to\infty}{=} \sum_n \beta_0^n\, n! \left[K_{GLS}^{UV}\, n^{1+\beta_1/\beta_0^2+\lambda_1} + K_{GLS}^{IR}\,(-1)^n\, n^{-\beta_1/\beta_0^2-(4/3b)(N_c-1/N_c)}\right] \alpha_s^{n+1}. \tag{24}$$

For the interesting case of $N_f > 2$, the UV renormalon behaviour formally dominates at very large n. For e^+e^- annihilation, τ and Z^0 decay the first IR renormalon singularity occurs at $t = -2/\beta_0$ [19] and thus its contribution to the asymptotic behaviour is suppressed as $r_n^{IR}/r_n^{UV} \sim (-2)^{-n}$. Thus, UV renormalons determine the asymptotic behaviour in all cases considered here. However, IR renormalons tend to have large over-all normalisation factors in the $\overline{\mathrm{MS}}$ scheme as compared to UV renormalons [12, 20]. Thus, dominance of IR renormalons at intermediate n could be expected and is indeed observed in the fixed-sign behaviour of the known exact coefficients in the $\overline{\mathrm{MS}}$ scheme.

The knowledge of the nature of the UV singularity, which we provide in this note, could be used to optimise conformal mapping techniques to dispose of UV renormalon growth (for an application of conformal mappings in this context, see [21, 22]) or to improve Pade-type approximations for the Borel transform of the perturbative series by combining the information on singularities with the known low-order coefficients. The utility of this procedure may be limited since the leading UV asymptotic behaviour considered above is not relevant for orders as low as $n = 2$, the highest order known exactly. This is especially so, because the leading behaviour is related to four-fermion operators. Subgraphs that contribute to the coefficients of these operators appear first at order α_s^2. It takes several more orders to see the exponentiated effect of two-loop running of the coupling and the eigenvalues of the anomalous dimension matrix embodied in the factor $n^{\beta_1/\beta_0^2+\lambda_1}$.

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